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Подгруппы, порожденные парой 2-торов в $\mathrm{GL}(4, K)$, III

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Аннотация

В данной статье мы завершаем описание подгрупп, порожденных парой 2-торов в $\mathrm{GL}(n, K)$. Напомним, что 2-торами в $\mathrm{GL}(n, K)$ называются подгруппы сопряженные диагональной подгруппе вида $\mathrm{diag}(\varepsilon, \varepsilon, 1, \ldots, 1)$. В работе [2] была доказана теорема редукции для пары m-торов. Из неё следует, что любая пара 2-торов может быть вложена в $\mathrm{GL}(6, K)$ одновременным сопряжением. Орбита пары 2-торов (X, Y) называется орбитой в $\mathrm{GL}(n, K)$, если пара (X, Y) вкладывается в $\mathrm{GL}(n, K)$ одновременным сопряжением и не вкладывается $\mathrm{GL}(n-1, K)$. Ясно, что n может принимать значения 3, 4, 5 и 6. В той же работе были описаны орбиты и порождения парами 2-торов в $\mathrm{GL}(6, K)$. В последующих работах были описаны пары 2-торов в $\mathrm{GL}(5, K)$, орбиты пары 2-торов в $\mathrm{GL}(4, K)$ и порождения в $\mathrm{GL}(4, K)$, соответствующие вырожденным случаям (редуктивная часть группы не более, чем $\mathrm{GL}(2, K)$). В этой работе мы описываем невырожденные случаи пар 2-торов в $\mathrm{GL}(4, K)$ и, таким образом, завершаем описание. Наиболее сложно устроенными подгруппами оказываются группы, у которых редуктивная часть совпадает с $\mathrm{SL}(2, K) \times \mathrm{SL}(2, K)$ или $\mathrm{SL}(2, L)$, где $[L : K] = 2$.

Ключевые слова: полная линейная группа, унипотентная корневая подгруппа, полупростая корневая подгруппа, m-торы, диагональная подгруппа.

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Subgroups generated by a pair of 2-tori in $\mathrm{GL}(4, K)$, III

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Abstract

In the present paper we complete the description of the subgroups generated by a pair of 2-tori in $\mathrm{GL}(n, K)$. Recall that 2-tori in $\mathrm{GL}(n, K)$ are the subgroups conjugate to the diagonal subgroup of the following form $\mathrm{diag}(\varepsilon, \varepsilon, 1, \dots, 1)$. In work [2] the reduction theorem for the pairs of m-tori was proved. It follows from it that any pair of 2-tori can be embedded in $\mathrm{GL}(6, K)$ by simultaneous conjugation. The orbit of a pair of 2-tori (X, Y) is called the orbit in $\mathrm{GL}(n, K)$, if the pair (X, Y) is embedded in $\mathrm{GL}(n, K)$ by simultaneous conjugation and it can not be embedded in $\mathrm{GL}(n-1, K)$. It is clear that n can take values 3, 4, 5 and 6. In the same work the orbits and spans of 2-tori in $\mathrm{GL}(6, K)$ were described. In the subsequent papers we described the pairs of 2-tori in $\mathrm{GL}(5, K)$, the orbits of pairs of 2-tori in $\mathrm{GL}(4, K)$ and the spans in $\mathrm{GL}(4, K)$ corresponding to degenerate cases (the reductive part of the group is not larger than $\mathrm{GL}(2, K)$). In this paper we describe undegenerate cases of pairs of 2-tori in $\mathrm{GL}(4, K)$. Thus we complete our description. The most difficult subgroups turns out the groups with a reductive part $\mathrm{SL}(2, K) \times \mathrm{SL}(2, K)$ or $\mathrm{SL}(2, L)$, where $[L : K] = 2$.

Keywords: general linear group, unipotent root subgroups, semisimple root subgroups, m-tori, diagonal subgroup.

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In memory of S. V. Vostokov

1. Introduction

The present paper is one of works of major cycle of ones dedicated to the geometry of microweight tori and long root tori in Chevalley groups that was announced in [8]. The same time this paper is final work in which we complete the description of subgroups generated by a pair of 2-tori in $\mathrm{GL}(n, K)$.

From the general viewpoint 2-*tori* in $\mathrm{GL}(n, K)$ are microweight tori of type ϖ_2 in the extended Chevalley group of type A_ℓ. Recall that such m-torus is the subgroup conjugate to the diagonal subgroup of the following form $\mathrm{diag}(\varepsilon, \dots, \varepsilon, 1, \dots, 1)$, where ε is taken m times.

N.A. Vavilov, Cohen, Cuypers and Sterk [6, 1] completely described orbits of microweight tori of case ϖ_1 (*reflection tori*) in $\mathrm{GL}(n, K)$, and the corresponding spans. The case of 2-tori also called *bireflection tori* is the next in difficulty.

In work [2] the reduction theorem for the pairs of m-tori was proved. It follows from it that any pair of 2-tori can be embedded in $\mathrm{GL}(6,K)$ by simultaneous conjugation. The orbit of a pair of 2-tori (X,Y) is called the orbit in $\mathrm{GL}(n,K)$, if the pair (X,Y) is embedded in $\mathrm{GL}(n,K)$ by simultaneous conjugation and it can not be embedded in $\mathrm{GL}(n-1,K)$.

In paper [2] we also described the orbits and spans of 2-tori in $\mathrm{GL}(6,K)$. In the next paper [9] the orbits and spans of 2-tori in $\mathrm{GL}(5,K)$ were considered. The case of $\mathrm{GL}(4,K)$ turned out the most difficult and required very cumbersome calculations. So these calculations were divided into three papers. In paper [3] we described the orbits of 2-tori in $\mathrm{GL}(4,K)$. In paper [4] we considered the degenerate cases of pairs of 2-tori in $\mathrm{GL}(4,K)$. Namely, the cases when the reductive part of the group generated by such a pair is not larger than $\mathrm{GL}(2,K)$. Finally, here we treat the undegenerate cases when a reductive part of span can be larger than $\mathrm{GL}(2,K)$.

The context of this problem and many references reader can find in the surveys [8], [5] and [7] and in the detailed introduction of [9]. The papers [10, 11] contains the known results concerning long root tori.

As a corollary of whole description of the spans of a pair of 2-tori we get the extremely important result. It allows to reduce completely the studying of the subgroups containing small semisimple elements to the studying of the subgroups containing small unipotent elements. At the same time the first author received it without classification of given pairs but under restrictions upon the ground field that should contain at least 19 elements (see [13]). Summarize this result and Theorem 2 below we have the next theorem.

Theorem 1. *Let X and Y be two non-commutative 2-tori in $\mathrm{GL}(n,K)$. Suppose that the field K is different from $\mathbb{F}_3$ and $\mathbb{F}_{2^n}$, where $n=1$, 2, 3 and 4. Then the subgroup $\langle X,Y\rangle$ generated by the groups X and Y contains an one-parameter subgroup, which is conjugated to the subgroup*

$$\{t_{12}(\varepsilon)t_{34}(\zeta\varepsilon),\ \varepsilon\in K\}\quad \textit{for some } \zeta\in K.$$

2. Notation

This paper is a direct sequel of [3] and [4] and we use the same notation as before. The authors highly recommend the reader to get acquainted with these works. Briefly we recall some of notations.

Let $G=\mathrm{GL}(n,K)$ be the general linear group of degree n over K. By $D=\mathrm{D}(n,K)$ we denote the subgroup of diagonal matrices in G, and $N=\mathrm{N}(n,K)$ denotes the subgroup of monomial matrices in G. The quotient group N/D is isomorphic to S_n, the symmetric group on n letters. Denote by $W=\mathrm{W}_n$ the group of permutation matrices in G. We identify S_n and W_n. Finally, $\mathrm{B}(n,K)$ denotes a group of lower triangular matrices (opposite Borel subgroup).

By $e_1,\dots,e_n$ we denote the standard base of $V=K^n$. Here e_i is the column, whose i-th component equals 1, whereas all other components are equal to 0. By $f_1,\dots,f_n$ we denote the standard base of ${}^nK$. It is dual to $e_1,\dots,e_n$ with respect to the standard pairing.

Denote by e_{ij} a standard matrix unit, i.e. the matrix whose entry in the position (i,j) is 1 and all the remaining entries are zeroes. Next, $x_{ij}(\xi)=e+\xi e_{ij}$ for $\xi\in K$ and $1\leqslant i\neq j\leqslant n$ denotes *elementary transvection*. For given $i\neq j$ we consider the corresponding unipotent *root subgroup* $X_{ij}=\{x_{ij}(\xi),\xi\in K\}$.

Similarly, by $d_i(\varepsilon)=e+(\varepsilon-1)e_{ii}$ we denote an *elementary pseudo-reflection*. For a given i we consider the corresponding **1-torus** $Q_i=\{d_i(\varepsilon),\ \varepsilon\in K^*\}$.

For the description of spans we put

$$X^{\gamma_1}_{ij,km} = \{x_{ij}(\gamma_1\varepsilon)x_{km}(\varepsilon),\ \varepsilon \in K\},$$

$$X^{\gamma_1,\gamma_2}_{ij,km,pq} = \{x_{ij}(\gamma_1\varepsilon)x_{km}(\gamma_2\varepsilon)x_{pq}(\varepsilon),\ \varepsilon \in K\},$$

$$Q_{ij} = \{d_i(\varepsilon)d_j(\varepsilon),\ \varepsilon \in K\},$$

where $\gamma_t \in K^*$, $t = 1, 2$. Note that we miss $\gamma_1 = 1$.

Also denote by $\mathrm{B}'(2, K)$ the double subgroup of lower triangular matrices, i.e. the subgroup generated by $X_{21,43}$, Q_{13}, Q_{24}. It is clear, that $\mathrm{B}'(2, K)$ is isomorphic to $\mathrm{B}(2, K)$.

Let $g \in G$. The largest subspace $W \leqslant V$ such that $g|_W = \mathrm{id}$ is called the *axis* of g. Similarly, the subspace $U = \{gv - v \mid v \in K^n\}$ is called the *centre* of g. Clearly, $\dim U = m$ and $\dim W = n - m$. Many useful properties of it can be found in [15].

The elementary 2-torus $Q = Q_{U_0,W_0} = \{\mathrm{diag}(\varepsilon, \varepsilon, 1, \dots, 1),\ \varepsilon \in K^*\}$ is defined by the subspaces $U_0 = \langle e_1, e_2\rangle$ and $W_0 = \langle f_1, f_2\rangle$. It means, that elements of it are

$$d_0(\varepsilon) = e + e_1(\varepsilon - 1)f_1 + e_2(\varepsilon - 1)f_2, \qquad \varepsilon \in K^*.$$

Therefore any 2-torus (see [2]) is conjugated to the elementary 2-torus Q. The elements of an arbitrary 2-torus are the elements of the following form

$$d(\varepsilon) = e + u_1(\varepsilon - 1)v_1 + u_2(\varepsilon - 1)v_2, \qquad \varepsilon \in K^*,$$

where $u_i = ge_i$, $v_i = f_ig^{-1}$, $1 \leqslant i \leqslant 2$, for some matrix $g \in \mathrm{GL}(n, K)$. Thus each 2-torus is completely determined by the subspaces $U = \langle u_1, u_2\rangle$ and $W = \langle v_1, v_2\rangle$.

The subspace U is precisely the *centre* of Q_{UW}, in the sense of being the centre of every $d(\varepsilon) \in Q_{UW}$, $\varepsilon \neq 1$. Similarly, the subspace $W^\perp$ orthogonal to $W \leqslant {}^nK$ with respect to the canonical pairing ${}^nK \times K^n \longrightarrow K$, is precisely the *axis* of Q_{UW}, in the above sense. Oftentimes we loosely refer to W itself as the axis of Q_{UW}.

Consider a pair of 2-tori X and Y with centers U_1 and U_2 and with axes W_1 and W_2, respictively. In paper [2] we introduce the following invariants for a pair of m-tori.

- $r = r(X, Y) = \dim(U_1 + U_2)$, • $s = s(X, Y) = \dim(W_1 + W_2)$.
- $p = p(X, Y) = \dim(U_1 \cap W_2^\perp)$, • $q = q(X, Y) = \dim(U_2 \cap W_1^\perp)$.

If a pair of 2-tori (X, Y) in $\mathrm{GL}(4, K)$ has invariants $r = s = 4$ we refer to it as *undegenerate* case and if at least one of invariants r or s less than 4 we refer to this pair as *degenerate* case. In this paper we calculate spans of undegenerate cases. This completes our description.

For the quadratic equation $t^2 + bt + c = 0$, b, $c \in K$, if it has two distinct roots t_1 and t_2, we define the matrix z_1 by

$$z_1 = \begin{pmatrix} -1 & 0 & -1 & 0 \\ t_1 & 0 & t_2 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & t_1 & 0 & t_2 \end{pmatrix}.$$

3. Undegenerate cases

Let X, Y be 2-tori in $\mathrm{GL}(4, K)$ with centers U_1, U_2 and axes W_1, W_2, respectively. Choose some bases in these subspaces

$$U_1 = \langle u_1, u_2\rangle, \quad U_2 = \langle u_3, u_4\rangle. \quad W_1 = \langle w_1, w_2\rangle, \quad W_2 = \langle w_3, w_4\rangle.$$

In the previous work [3] we described all orbits (X,Y) under action by simultaneous conjugation: (gXg^{-1}, gYg^{-1}), $g\in G$. For this purpose we listed all bases (u_1,u_2,u_3,u_4) and (w_1,w_2,w_3,w_4) corresponding to different orbits. Summarize the orbits corresponding to undegenerate cases in the next lemma.

Lemma 1. *Let X and Y be 2-tori in* GL$(4,K)$. *Assume that $r=s=4$, then the orbit (X,Y) is determined by one of the following bases.*

For $p=0$, we have

$$\begin{aligned}&u_1=e_1,\ u_2=e_2,\ w_1=f_1+\alpha f_3+f_4,\ w_2=f_2+\gamma f_3+\delta f_4,\\ &u_3=e_1+e_3,\ u_4=e_2+e_4,\ w_3=f_1,\ w_4=f_2,\end{aligned}\tag{r4s4a}$$

where $\alpha,\ \gamma,\ \delta\in K$ and $\gamma\neq\alpha\delta$.

$$\begin{aligned}&u_1=e_1,\ u_2=e_2,\ w_1=f_1+\alpha f_3,\ w_2=f_2+\delta f_4,\\ &u_3=e_1+e_3,\ u_4=e_2+e_4,\ w_3=f_1,\ w_4=f_2,\end{aligned}\tag{r4s4b}$$

where $\alpha,\ \delta\in K^*$.

For $p=q=1$, we have

$$\begin{aligned}&u_1=e_1,\ u_2=e_2,\ w_1=f_1-f_4,\ w_2=f_2,\\ &u_3=e_1+e_4,\ u_4=e_3,\ w_3=f_1,\ w_4=f_3.\end{aligned}\tag{p1q1c}$$

For $p=2$, we have

$$\begin{aligned}&u_1=e_1,\ u_2=e_2,\ w_1=f_1+\alpha f_3+f_4,\ w_2=f_2+\gamma f_3+\delta f_4,\\ &u_3=e_3,\ u_4=e_4,\ w_3=f_3,\ w_4=f_4,\end{aligned}\tag{p2q0a}$$

where $\alpha,\ \gamma,\ \delta\in K$ and $\alpha\delta\neq\gamma$.

$$\begin{aligned}&u_1=e_1,\ u_2=e_2,\ w_1=f_1+\alpha f_3+f_4,\ w_2=f_2+\alpha\delta f_3+\delta f_4,\\ &u_3=e_3,\ u_4=e_4,\ w_3=f_3,\ w_4=f_4,\end{aligned}\tag{p2q1a}$$

where $\alpha,\ \delta\in K$.

$$\begin{aligned}&u_1=e_1,\ u_2=e_2,\ w_1=f_1,\ w_2=f_2,\\ &u_3=e_3,\ u_4=e_4,\ w_3=f_3,\ w_4=f_4.\end{aligned}\tag{p2q2a}$$

In all calculations we have deal with the generators of 2-tori $X=\langle x(\varepsilon),\ \varepsilon\in K\rangle$ and $Y=\langle y(\eta),\ \eta\in K\rangle$. The generators $x(\varepsilon)$ and $y(\eta)$ are constructed with the help of bases listed in Lemma 1. Sometimes we consider the generators conjugated to them. By H we denote the subgroup generated by $x(\varepsilon)$ and $y(\eta)$ $H=\langle x(\varepsilon),y(\eta),\ \varepsilon,\eta\in K^*\rangle$.

As usually $[g_1,g_2]=g_1g_2g_1^{-1}g_2^{-1}$ denotes a commutator of two elements g_1 and g_2. Also put

$$z(\varepsilon,\eta)=[x(\varepsilon),y(\eta)]\,,\quad z_x(\varepsilon,\eta,\theta)=[z(\varepsilon,\eta),x(\theta)],\quad z_y(\varepsilon,\eta,\theta)=[z(\varepsilon,\eta),y(\theta)].$$

To simplify the proof of the main theorem, we also prove the following lemma.

Lemma 2. *Let X, Y be a pair of 2-tori in* GL$(4,K)$, $K\neq\mathbb{F}_2,\mathbb{F}_3$. *Suppose that their generators are as follows.*

$$\begin{aligned}x(\varepsilon)=&d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{\delta(\varepsilon-1)}{\varepsilon}\right),\\ y(\eta)=&d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1),\end{aligned}$$

where $\alpha\neq0$, $\delta\neq0$. Then X and Y generate one of the following subgroups H, listed in Table 1.

Таблица 1

	parameters	**H**
1	$\alpha \neq \delta$, $\alpha \neq -1$, $\delta \neq -1$	$(\mathrm{SL}(2,K) \times \mathrm{SL}(2,K)) \rtimes K^*$
2	$\alpha = -1$, $\delta \neq -1$; $\alpha \neq -1$, $\delta = -1$	$(\mathrm{U}(2,K) \times \mathrm{SL}(2,K)) \rtimes (K^* \times K^*)$
3	$\alpha = \delta$, $\alpha \neq -1$	$\mathrm{GL}(2,K)$
4	$\alpha = \delta = -1$	$\mathrm{B}(2,K)$

Proof. (1) Let $\alpha \neq \delta$, $\alpha \neq -1$, $\delta \neq -1$. Suppose $\alpha + \delta + 1 \neq 0$. For $\varepsilon, \eta \in K^*$, we put

$$k_1(\eta) = z_x\left(\frac{\alpha(\alpha+1)}{\alpha^2+\alpha-\delta^2-\delta}, \frac{\delta+1}{\delta}, \eta\right)^2, \quad k_2(\eta) = z_y\left(\frac{\delta}{\delta+1}, \frac{\alpha(\alpha+1)}{\alpha^2+\alpha-\delta^2-\delta}, \eta\right)^2,$$

$$k_3(\eta) = z_y\left(\frac{\alpha}{\alpha+1}, \frac{\delta(\delta+1)}{-\alpha^2-\alpha+\delta^2+\delta}, \eta\right)^2, \quad k_4(\eta) = z_x\left(\frac{\delta(\delta+1)}{-\alpha^2-\alpha+\delta^2+\delta}, \frac{\alpha+1}{\alpha}, \eta\right)^2.$$

Calculating the commutators of $k_i(\eta)$, $i = 1,2,3,4$, we get the subgroups X_{12}, X_{21}, X_{43} and X_{34}. In fact, for $\varepsilon, \eta \in K^*$, $\varepsilon^3 - \varepsilon^2 + \varepsilon - 1 \neq 0$, $\eta^3 - \eta^2 + \eta - 1 \neq 0$, we have

$$\begin{aligned} [k_1(\varepsilon), k_1(\eta)] &= x_{12}(u_1(\varepsilon,\eta)), \quad [k_2(\varepsilon), k_2(\eta)] = x_{21}(u_2(\varepsilon,\eta)), \\ [k_3(\varepsilon), k_3(\eta)] &= x_{43}(u_3(\varepsilon,\eta)), \quad [k_4(\varepsilon), k_4(\eta)] = x_{34}(u_4(\varepsilon,\eta)). \end{aligned}$$

where

$$\begin{aligned} u_1(\varepsilon,\eta) &= \frac{\alpha(\alpha+1)\left(\varepsilon^3-\varepsilon^2+\varepsilon-1\right)\left(\eta^3-\eta^2+\eta-1\right)\left(2\alpha^2+2\alpha-\delta(\delta+1)\right)(\varepsilon-\eta)}{\varepsilon^4\eta^4(\alpha-\delta)^2(\alpha+\delta+1)}, \\ u_2(\varepsilon,\eta) &= \frac{\left(\varepsilon^3-\varepsilon^2+\varepsilon-1\right)\left(\eta^3-\eta^2+\eta-1\right)\left(2\alpha^2+2\alpha-\delta(\delta+1)\right)(\varepsilon-\eta)}{\alpha(\alpha-\delta)}, \\ u_3(\varepsilon,\eta) &= \frac{\left(\varepsilon^3-\varepsilon^2+\varepsilon-1\right)\left(\eta^3-\eta^2+\eta-1\right)\left(\alpha^2+\alpha-2\delta(\delta+1)\right)(\varepsilon-\eta)}{\delta(\alpha-\delta)}, \\ u_4(\varepsilon,\eta) &= \frac{\delta(\delta+1)\left(\varepsilon^3-\varepsilon^2+\varepsilon-1\right)\left(\eta^3-\eta^2+\eta-1\right)\left(\alpha^2+\alpha-2\delta(\delta+1)\right)(\eta-\varepsilon)}{\varepsilon^4\eta^4(\alpha-\delta)^2(\alpha+\delta+1)}. \end{aligned}$$

Now it follows from the decomposition of $x(\varepsilon)$ and $y(\eta)$ that $Q_{13} \leqslant H$.

Suppose that $\alpha + \delta + 1 = 0$ and $\alpha \neq -1/2$. The generators have the following form.

$$\begin{aligned} x_1(\varepsilon) =& d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{(\alpha+1)(1-\varepsilon)}{\varepsilon}\right), \\ y_1(\eta) =& d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1). \end{aligned}$$

We put

$$\begin{aligned} r_1(\varepsilon,\eta) &= z_{x_1}\left(\varepsilon, \frac{\alpha+1}{\alpha}, \eta\right), \quad r_2(\varepsilon,\eta) = z_{x_1}\left(\varepsilon, \frac{\alpha}{\alpha+1}, \eta\right), \\ r_3(\varepsilon,\eta) &= z_{y_1}\left(\frac{\alpha}{\alpha+1}, \varepsilon, \eta\right), \quad r_4(\varepsilon,\eta) = z_{y_1}\left(\frac{\alpha+1}{\alpha}, \varepsilon, \eta\right). \end{aligned}$$

Let $\operatorname{char} K=2$. For $\varepsilon,\eta\neq 1$, $\varepsilon'=\varepsilon^2+\varepsilon+1\neq 0$, we have

$$\begin{aligned}
r_1(\varepsilon,\eta)r_1\left(\varepsilon+1,\frac{\varepsilon^3\eta}{\varepsilon^3+\varepsilon'(\eta+1)}\right)&=x_{12}\left(\varepsilon(\varepsilon+1)\left(\frac{\varepsilon^3\eta}{\varepsilon^3+\varepsilon'(\eta+1)}+\eta\right)\right),\\
r_2(\varepsilon,\eta)r_2\left(\varepsilon+1,\frac{\varepsilon^3\eta}{\varepsilon^3+\varepsilon'(\eta+1)}\right)&=x_{34}\left(\varepsilon(\varepsilon+1)\left(\frac{\varepsilon^3\eta}{\varepsilon^3+\varepsilon'(\eta+1)}+\eta\right)\right),\\
r_3(\varepsilon,\eta)r_3\left(\varepsilon+1,\frac{\varepsilon^2\eta+\eta+1}{\varepsilon^2}\right)&=x_{21}\left(\frac{\frac{\varepsilon^3}{(\varepsilon^2+1)\eta+1}+\frac{(\varepsilon+1)(\eta+1)}{\eta}+\varepsilon}{\alpha}\right),\\
r_4(\varepsilon,\eta)r_4\left(\varepsilon+1,\frac{\varepsilon^2\eta+\eta+1}{\varepsilon^2}\right)&=x_{43}\left(\frac{\frac{\varepsilon^3}{(\varepsilon^2+1)\eta+1}+\frac{(\varepsilon+1)(\eta+1)}{\eta}+\varepsilon}{\alpha+1}\right).
\end{aligned}$$

Now let $\operatorname{char} K\neq 2$. Suppose that $\alpha\neq -1/2$. For $\eta\neq 2,1/2$ we calculate the products,

$$\begin{aligned}
r_1(\varepsilon,\eta)r_1\left(\varepsilon,\frac{(\varepsilon-1)\eta}{2\varepsilon\eta-\varepsilon-2\eta+1}\right)&=x_{12}\left(\frac{2(\varepsilon-1)\varepsilon(\eta-1)^2(2\alpha+1)}{2\eta-1}\right),\\
r_2(\varepsilon,\eta)r_2\left(\varepsilon,\frac{(\varepsilon-1)\eta}{2\varepsilon\eta-\varepsilon-2\eta+1}\right)&=x_{34}\left(\frac{2(\varepsilon-1)\varepsilon(\eta-1)^2(2\alpha+1)}{2\eta-1}\right),\\
r_3(\varepsilon,\eta)r_3(\varepsilon,2-\eta)&=x_{21}\left(\frac{2(\varepsilon-1)(\eta-1)^2(2\alpha+1)}{(\eta-2)\eta\alpha}\right),\\
r_4(\varepsilon,\eta)r_4(\varepsilon,2-\eta)&=x_{43}\left(\frac{2(\varepsilon-1)(\eta-1)^2(2\alpha+1)}{(\eta-2)\eta(\alpha+1)}\right).
\end{aligned}$$

Hence we extract the subgroups X_{12}, X_{21}, X_{43} and X_{34}. It follows from the decomposition of $x_1(\varepsilon)$ and $y_1(\eta)$ that $Q_{13}\leqslant H$. It follows that $H\leqslant \mathrm{GL}(2,K)\times\mathrm{GL}(2,K)$. For any $\operatorname{diag}(x,y)\in H$, we have $\det(x)=\det(y)$. Thus

$$H\leqslant\{\operatorname{diag}(x,y)\in\mathrm{GL}(2,K)\times\mathrm{GL}(2,K)\mid \det(x)=\det(y)\}=H_1.$$

We claim that $H=H_1$. In fact, let $(x,y)\in H_1$. Write $x=d_1(\theta_1)g_1$, $y=d_3(\theta_2)g_2$, where $g_1\in\mathrm{SL}(2,K)=\langle X_{12},X_{21}\rangle$, $g_2\in\mathrm{SL}(2,K)=\langle X_{34},X_{43}\rangle$. Then $\det(x)=\det(d_1(\theta_1))$ and $\det(y)=\det(d_3(\theta_2))$. It follows that $\theta_1=\theta_2$ and $\operatorname{diag}(x,y)\in H$.

Define the determinant projection

$$\pi:\ H_1\to K^*,\quad \pi(\operatorname{diag}(x,y))=\det(x)(=\det(y)).$$

This is a group homomorphism and $\ker\pi=\{\operatorname{diag}(x,y)\in H_1\mid\det(x)=1\}=\mathrm{SL}(2,K)\times\mathrm{SL}(2,K)$, so we obtain the short exact sequence,

$$1\longrightarrow\mathrm{SL}(2,K)\times\mathrm{SL}(2,K)\xrightarrow{i}H_1\xrightarrow{\pi}K^*\longrightarrow 1.$$

For each $\lambda\in K^*$, set $\sigma(\lambda)=(\operatorname{diag}(\lambda,1),\operatorname{diag}(\lambda,1))\in H_1$. Then $\pi(\sigma(\lambda))=\det(\operatorname{diag}(\lambda,1))=$ $=\lambda$, hence $\pi\circ\sigma=\mathrm{id}_{K^*}$ and the sequence splits. The splitting determines an action of K^* on the kernel by conjugation inside H_1; explicitly for $(x,y)\in\mathrm{SL}(2,K)\times\mathrm{SL}(2,K)$,

$$\sigma(\lambda)(x,y)\sigma(\lambda)^{-1}=(\operatorname{diag}(\lambda,1)x\operatorname{diag}(\lambda,1)^{-1},\ \operatorname{diag}(\lambda,1)y\operatorname{diag}(\lambda,1)^{-1}),$$

which produces the action of $\varphi_1(\lambda)$. Thus it follows that $H_1\simeq(\mathrm{SL}(2,K)\times\mathrm{SL}(2,K))\rtimes_{\varphi_1}K^*$. The isomorphism ψ is defined as follows,

$$\psi:\ (\mathrm{SL}(2,K)\times\mathrm{SL}(2,K))\rtimes_{\varphi_1}K^*\to H_1,\quad \psi\big((x,y),\lambda\big)=(x\ \operatorname{diag}(\lambda,1),\ y\ \operatorname{diag}(\lambda,1)).$$

Finally we conclude that $H \simeq (\mathrm{SL}(2,K)\times \mathrm{SL}(2,K)) \rtimes K^*$ when $\alpha \neq \delta$, $\alpha \neq -1$, $\delta \neq -1$.
(2) Suppose that $\alpha \neq \delta$, $\alpha = -1$, $\delta \neq -1$. The generators have the following form.

$$\begin{aligned} x_2(\varepsilon) =& d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{\delta(\varepsilon-1)}{\varepsilon}\right),\\ y_2(\eta) =& d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1). \end{aligned}$$

Let $\operatorname{char} K = 2$. For $\eta \in K^*/\{1\}$, $\varepsilon^2+\eta+1 \neq 0$, $\varepsilon\eta+\eta+1 \neq 0$, we have

$$\begin{aligned} z_{x_2}\left(\varepsilon, \frac{\delta+1}{\delta}, \eta\right) z_{x_2}\left(\varepsilon+1, \frac{\varepsilon^2\eta}{\varepsilon^2+\eta+1}\right) &= x_{34}\left(\varepsilon(\varepsilon+1)\left(\frac{\varepsilon^2\eta}{\varepsilon^2+\eta+1}+\eta\right)\right),\\ z_{y_2}\left(\frac{\delta}{\delta+1}, \varepsilon, \eta\right) z_{y_2}\left(\frac{\delta}{\delta+1}, \varepsilon+1, \frac{\varepsilon\eta+\eta+1}{\varepsilon}\right) &= x_{43}\left(\frac{(\varepsilon+1)\left(\eta^2+1\right)}{\delta\eta(\varepsilon\eta+\eta+1)}\right). \end{aligned}$$

Now suppose $\operatorname{char} K \neq 2$, $\delta \neq -1/2$. For $\varepsilon \in K^*$, $\eta \in K^*/\left\{1/2, 2\right\}$, we have

$$\begin{aligned} z_{x_2}\left(\varepsilon, \frac{\delta+1}{\delta}, \eta\right) z_{x_2}\left(\varepsilon, \frac{\delta+1}{\delta}, \frac{\eta}{2\eta-1}\right) &= x_{34}\left(\frac{2(2\delta+1)(\varepsilon-1)\varepsilon(\eta-1)^2}{2\eta-1}\right),\\ z_{y_2}\left(\frac{\delta}{\delta+1}, \varepsilon, \eta\right) z_{y_2}\left(\frac{\delta}{\delta+1}, \varepsilon, \frac{-\varepsilon\eta+2\varepsilon+\eta-2}{\varepsilon-1}\right) &= x_{43}\left(\frac{2(2\delta+1)(\varepsilon-1)(\eta-1)^2}{\delta(\eta-2)\eta}\right). \end{aligned}$$

Let $\delta = -1/2$, for $\varepsilon \in K^*/\left\{-1\right\}$, we have

$$\begin{aligned} z_{x_2}\left(\frac{(\varepsilon-1)^2}{(\varepsilon+1)^2}, \varepsilon, 2\right) z_{x_2}\left(\frac{(\varepsilon-1)^2}{(\varepsilon+1)^2}, \varepsilon, \frac{1}{2}\right) z_{x_2}\left(\frac{1-\varepsilon}{\varepsilon+1}, -1, 2\right) &= x_{34}\left(\frac{\varepsilon^3-\varepsilon^2+\varepsilon-1}{4(\varepsilon+1)^3}\right),\\ z_{y_2}\left(\varepsilon, \frac{(\varepsilon-1)^2}{(\varepsilon+1)^2}, 2\right) z_{y_2}\left(\varepsilon, \frac{(\varepsilon-1)^2}{(\varepsilon+1)^2}, \frac{1}{2}\right) z_{y_2}\left(-1, \frac{\varepsilon(\varepsilon+3)}{(\varepsilon+1)^2}, 2\right) &= x_{43}\left(-\frac{2\left(\varepsilon^2+1\right)}{\varepsilon^2-1}\right), \end{aligned}$$

It follows that X_{34} and X_{43} are contained in H. After it we get $Q_{13}X_{12}$ and $Q_{13}X_{21}$ from the decomposition of $x(\varepsilon)$ and $y(\eta)$. Conjugating the elements $d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)$ and $d_1(\eta)d_3(\eta)x_{21}(\eta-1)$ by the element $x_{12}(-1)$, we get

$$x_2'(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon), \quad y_2'(\eta) = d_2(\eta)d_3(\eta)x_{21}\left(\frac{\eta-1}{\eta}\right),$$

For $\varepsilon \in K^*/\left\{1\right\}$, $\eta \in K^*$, $\theta \in K$, we calculate the product,

$$y_2'\left(\frac{\varepsilon+\theta-\eta}{\varepsilon-1}\right) x_2'(\varepsilon) y_2'\left(\frac{(\varepsilon-1)\eta}{\varepsilon+\theta-\eta}\right) x_2'\left(\frac{1}{\varepsilon}\right) y_2'\left(\frac{1}{\eta}\right) = x_{21}\left(1+\frac{\theta}{\varepsilon}-\eta\right).$$

Hence we extract the subgroups X_{21} and Q_{23}. They together with Q_{13}, X_{34}, and X_{43} generate H.

It is clear that H is embeded in $\mathrm{B}(2,K)\times\mathrm{GL}(2,K)$ via the map $\operatorname{diag}(x,y) \mapsto (x,y)$. The matrices in H can be viewed in a block form with respect to the decomposition of the vector space $K^4 = V_1 \oplus V_2$, where V_1 is spanned by $\{e_1, e_2\}$, V_2 is spanned by $\{e_3, e_4\}$.

The subgroup generated by matrices of the left block 2×2 is the group of lower triangular matrices $\mathrm{B}(2,K)$. At the same time matrices of the right block generate $\mathrm{GL}(2,K)$. At that for any $\operatorname{diag}(x,y) \in H$ we have $\det(x) = \det(y)$. Thus

$$H \leqslant \{\operatorname{diag}(x,y) \in \mathrm{B}(2,K)\times\mathrm{GL}(2,K) \mid \det(x) = \det(y)\} = H_2.$$

We claim that $H = H_2$. In fact, let $(x,y) \in H_2$. Write $y = d_3(\theta)g$, where $g \in \mathrm{SL}(2,K) = \langle X_{34}, X_{43}\rangle$. Then $\det(x) = \det(d_1(\varepsilon)d_2(\eta))$ and $\det(y) = \det(d_3(\theta))$. It follows that $\theta = \varepsilon\eta$ or $d_3(\theta) = d_1(\varepsilon)d_2(\eta)$ and $\operatorname{diag}(x,y) \in H$.

Define the determinant projection

$$\pi:\ H_2 \to K^*, \quad \pi(\mathrm{diag}(x,y)) = \det(x)(=\det(y)).$$

This is a group homomorphism and $\ker\pi = (\mathrm{B}(2,K)\cap \mathrm{SL}(2,K))\times \mathrm{SL}(2,K)$, so we obtain the short exact sequence,

$$1 \longrightarrow (\mathrm{B}(2,K)\cap \mathrm{SL}(2,K))\times \mathrm{SL}(2,K) \xrightarrow{i} H_2 \xrightarrow{\pi} K^* \longrightarrow 1.$$

Then the similar arguments as (1) give us that $H_2 \simeq (\mathrm{B}(2,K)\cap \mathrm{SL}(2,K))\times \mathrm{SL}(2,K)) \rtimes K^*$.

Since $\mathrm{B}(2,K)\cap \mathrm{SL}(2,K) \simeq \mathrm{U}(2,K)\rtimes_{\varphi_2} K^*$, the action of $\varphi_2(\lambda)$ on $\mathrm{U}(2,K)$ is determined by conjugation $\mathrm{diag}(\lambda,\lambda^{-1})x_{21}(\varepsilon)\,\mathrm{diag}(\lambda,\lambda^{-1})^{-1}$. Thus we have that $H_2 \simeq (\mathrm{U}(2,K)\rtimes K^*)\times$ $\times\, \mathrm{SL}(2,K))\rtimes K^*$.

Define the map

$$\begin{aligned}&\psi:\ (\mathrm{U}(2,K)\times \mathrm{SL}(2,K))\rtimes_\tau (K^*\times K^*) \to H_2,\\ &\psi\big((x_{21}(\varepsilon),g),(\eta,\lambda)\big) = \mathrm{diag}\left(x_{21}(\varepsilon)\,\mathrm{diag}(\eta\lambda,\eta^{-1}),\ g\,\mathrm{diag}(\eta\lambda,\eta^{-1})\right),\end{aligned}$$

where

$$\begin{aligned}&\tau:\ K^*\times K^* \to \mathrm{Aut}(\mathrm{U}(2,K)\times \mathrm{SL}(2,K)),\\ &\tau_{(\eta,\lambda)}(x_{21}(\varepsilon),g) = (M(\eta,\lambda)x_{21}(\varepsilon)M(\eta,\lambda)^{-1}, M(\eta,\lambda)gM(\eta,\lambda)^{-1}),\end{aligned}$$

here $\eta,\lambda\in K^*$, $M(\eta,\lambda)=\mathrm{diag}(\eta\lambda,\eta^{-1})$. It is easy to check that ψ is a isomorphism.

Finally we conclude that $H \simeq (\mathrm{U}(2,K)\times \mathrm{SL}(2,K))\rtimes (K^*\times K^*)$ when $\alpha\neq\delta$, $\alpha=-1$, $\delta\neq -1$.

For the case of $\alpha\neq -1$, $\delta=-1$, the span H can be obtained from above case by matrix transposition.

(3) Suppose that $\alpha=\delta$, $\alpha\neq -1$. When $\alpha=\delta=-1/2$, the generators are as follows.

$$\begin{aligned}x_3(\varepsilon) =&d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{2\varepsilon}\right)x_{34}\left(\frac{1-\varepsilon}{2\varepsilon}\right),\\ y_3(\eta) =&d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).\end{aligned}$$

We put

$$f_1(\varepsilon,\eta) = x_3(\varepsilon)y_3(\eta)x_3\left(\frac{\varepsilon\eta-\varepsilon+\eta+1}{\varepsilon\eta+\varepsilon+\eta-1}\right),\ f_2(\varepsilon,\eta) = y_3(\varepsilon)x_3(\eta)y_3\left(\frac{-\varepsilon\eta+\varepsilon+\eta+1}{\varepsilon\eta+\varepsilon-\eta+1}\right).$$

By calculating the commutator subgroups generated by $f_1(\varepsilon,\eta)$ and $f_2(\varepsilon,\eta)$, we get that $X_{21,43}, X_{12,34}\leqslant H$. In fact,

$$[f_1(\varepsilon_1,\eta_1), f_1(\varepsilon_2,\eta_2)] = X_{21,43}, \quad [f_2(\varepsilon_1,\eta_1), f_2(\varepsilon_2,\eta_2)] = X_{12,34}.$$

Thus we conclude that $H\simeq \mathrm{GL}(2,K)$ when $\alpha=\delta=-1/2$.

Suppose $\alpha=\delta$, $\alpha\neq -1/2$. The generators are as follows.

$$\begin{aligned}x_4(\varepsilon) =&d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right),\\ y_4(\eta) =&d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).\end{aligned}$$

For $\varepsilon,\eta\in K^*$, $\varepsilon\neq\eta$, we have

$$\begin{aligned}\left[z\left(\varepsilon,\frac{\alpha+1}{\alpha}\right), z\left(\eta,\frac{\alpha+1}{\alpha}\right)\right] =&x_{12}\left((2\alpha+1)(\varepsilon-1)(\eta-1)(\varepsilon-\eta)\right)\times\\ &x_{34}\left((2\alpha+1)(\varepsilon-1)(\eta-1)(\varepsilon-\eta)\right),\\ \left[z\left(\frac{\alpha}{\alpha+1},\varepsilon\right), z\left(\frac{\alpha}{\alpha+1},\eta\right)\right] =&x_{21}\left(\frac{(2\alpha+1)(\varepsilon-1)(\eta-1)(\varepsilon-\eta)}{\alpha}\right)\times\\ &x_{43}\left(\frac{(2\alpha+1)(\varepsilon-1)(\eta-1)(\varepsilon-\eta)}{\alpha}\right).\end{aligned}$$

It follows that $X_{21,43}, X_{12,34} \leqslant H$. Thus we conclude that $H \simeq \mathrm{GL}(2, K)$ when $\alpha = \delta$, $\alpha \neq -1/2$.

(4) Let $\alpha = \delta = -1$. The generators are as follows.

$$\begin{aligned} x_5(\varepsilon) =& d_1(\varepsilon) d_3(\varepsilon) x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right) x_{34}\left(\frac{1-\varepsilon}{\varepsilon}\right), \\ y_5(\eta) =& d_1(\eta) d_3(\eta) x_{21}(\eta-1) x_{43}(\eta-1). \end{aligned}$$

Conjugating $x_5(\varepsilon)$ and $y_5(\eta)$ by the element $x_{12}(-1)x_{34}(-1)$, we have

$$\begin{aligned} &x_5'(\varepsilon) = x_{12}(-1)x_{34}(-1)x_5(\varepsilon)x_{34}(1)x_{12}(1) = d_1(\varepsilon)d_3(\varepsilon), \\ &y_5'(\eta) = x_{12}(-1)x_{34}(-1)y_5(\eta)x_{34}(1)x_{12}(1) = d_2(\eta)d_4(\eta)x_{21}\left(\frac{\eta-1}{\eta}\right) x_{43}\left(\frac{\eta-1}{\eta}\right). \end{aligned}$$

For $\varepsilon \in K^*/\{1\}$, $\eta \in K^*$, $\theta \in K$, we calculate the product,

$$y_5'\left(\frac{\varepsilon+\theta-\eta}{\varepsilon-1}\right) x_5'(\varepsilon) y_5'\left(\frac{(\varepsilon-1)\eta}{\varepsilon+\theta-\eta}\right) = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ \theta & \eta & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & \theta & \eta \end{bmatrix} \simeq \mathrm{B}(2, K).$$

□

4. Main results

In this section we give a description of the undegenerate cases. For this we use some ideas of projective geometry (see, for example, [14]). The idea is to use similar arguments was motivated by work [12].

Let V be a 4-dimensional vector space over a field K. The *projective space* $\overline{V} = \mathbb{P}(V)$ is defined as the set of all 1-dimensional linear subspaces of V. The correspondence between geometric objects in $\overline{V}$ and linear subspaces of V is given by:

• A point $p = [p_1 : p_2 : p_3 : p_4]$ corresponds to a 1-dimensional subspace $\mathrm{span}\{(p_1, p_2, p_3, p_4)^t\}$.

• A line ℓ corresponds to a 2-dimensional subspace W, represented by a nonzero skew-symmetric matrix $P = (p_{ij})$ satisfying: (a) $p_{ij} = -p_{ji}, \forall i, j$. (b) Plücker relation $\det(P) = 0 \iff$ $\iff p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$.

• A plane Π corresponds to a 3-dimensional subspace U, represented dually by a covector (q^1, q^2, q^3, q^4) via the hyperplane equation $\sum_{i=1}^4 q^i x_i = 0$.

The linear transformations $x(\varepsilon)$ and $y(\eta)$ induce projective transformations $\overline{x}(\varepsilon)$ and $\overline{y}(\eta)$ on $\overline{V}$ with the following actions:

- On points: $p \mapsto \overline{x}(\varepsilon)p$ and $p \mapsto \overline{y}(\eta)p$.
- On planes: $q \mapsto q\,\overline{x}(\varepsilon)^{-1}$ and $q \mapsto q\,\overline{y}(\eta)^{-1}$.
- On lines: $P \mapsto \overline{x}(\varepsilon)^t P\,\overline{x}(\varepsilon)$ and $P \mapsto \overline{y}(\eta)^t P\,\overline{y}(\eta)$.

Theorem 2. *Let X, Y be a pair of 2-tori in $\mathrm{GL}(4, K)$, $K \neq \mathbb{F}_2, \mathbb{F}_3$. Suppose that $r = s = 4$, $p = 0$, $\mathrm{char}\, K \neq 2$, then up to simultaneous conjugation X and Y generate one of the following subgroups H, listed in Table 2.*

Таблица 2: For $r=4$, $s=4$.

	base	**H**
1	(r4s4a), $\alpha=-1$, $\gamma\neq 0,-3/2,\ -1/4$, $\delta=0$, $4\gamma+1\in K^{*2}$ (r4s4a), $\alpha=0$, $\gamma\neq 0$, $\delta\neq 0,-1$, $\gamma\neq\delta+1$, $\delta^2+4\gamma\in K^{*2}$ (r4s4a), $\alpha=0$, $\gamma\neq 0,1$, $\delta=0$, $\gamma\in K^{*2}$ (r4s4a), $\alpha=-1$, $\gamma\neq 0,1$, $\delta=-1$, $\gamma\in K^{*2}$ (r4s4a), $\alpha\neq 0,\pm1,-3$, $\gamma\neq 0,-1$, $\delta=-1$, $(\alpha+1)^2+4\gamma\in K^{*2}$ (r4s4a), $\alpha\neq 0,-1$, $\gamma\neq 0$, $\delta\neq 0,-1$, $\gamma\neq\alpha\delta$, $\gamma\neq(\alpha+1)(\delta+1)$, $\alpha^2+4\gamma-2\alpha\delta+\delta^2\in K^{*2}$ (r4s4b), $\alpha\neq 0,-1$, $\delta\neq 0,-1$, $\alpha\neq\delta$	$(\mathrm{SL}(2,K)\times\mathrm{SL}(2,K))\rtimes K^*$
2	(r4s4a), $\alpha=-1$, $\gamma=-3/2$, $\delta=0$, $1+4\gamma\in K^{*2}$ (r4s4b), $\alpha\neq 0,-1$, $\delta\neq 0,-1$, $\alpha=\delta$	$\mathrm{GL}(2,K)$
3	(r4s4a), $\alpha=-1$, $\gamma\neq 0,-3/2,-1/4$, $\delta=0$, $4\gamma+1\notin K^{*2}$, $K'=K(\sqrt{4\gamma+1})$ (r4s4a), $\alpha=0$, $\gamma\neq 0$, $\delta\neq 0,-1$, $\gamma\neq\delta+1$, $\delta^2+4\gamma\notin K^{*2}$, $K'=K(\sqrt{\delta^2+4\gamma})$ (r4s4a), $\alpha=0$, $\gamma\neq 0,1$, $\delta=0$, $\gamma\notin K^{*2}$, $K'=K(\sqrt{\gamma})$ (r4s4a), $\alpha=-1$, $\gamma\neq 0,1$, $\delta=-1$, $\gamma\notin K^{*2}$, $K'=K(\sqrt{\gamma})$ (r4s4a), $\alpha\neq 0,\pm1,-3$, $\gamma\neq 0,-1$, $\delta=-1$, $(\alpha+1)^2+4\gamma\notin K^{*2}$, $K'=K(\sqrt{(1+\alpha)^2+4\gamma})$ (r4s4a), $\alpha\neq 0,-1$, $\gamma\neq 0$, $\delta\neq 0,-1$, $\gamma\neq\alpha\delta$, $\gamma\neq(\alpha+1)(\delta+1)$, $\alpha^2+4\gamma-2\alpha\delta+\delta^2\notin K^{*2}$, $K'=K(\sqrt{\alpha^2+4\gamma-2\alpha\delta+\delta^2})$	$\mathrm{SL}(2,K')\rtimes K^*$
4	(r4s4a), $\alpha=-1$, $\gamma=-1/4$, $\delta=0$ (r4s4a), $\alpha=0$, $\gamma=-1$, $\delta=2$ (r4s4a), $\alpha=-3$, $\gamma=-1$, $\delta=-1$ (r4s4a), $\alpha\neq 0,-1$, $\gamma=0$, $\delta\neq 0,-1$, $\alpha=\delta$ (r4s4a), $\alpha\neq 0,-1$, $\gamma=-1$, $\delta\neq 0,-1$, $\delta-\alpha=\pm2$, $\gamma\neq\alpha\delta$, $\gamma\neq(\alpha+1)(\delta+1)$	$\mathrm{GL}(2,K).K^3$
5	(r4s4a), $\alpha=1$, $\gamma=-1$, $\delta=-1$	$Q_{24}X_{12,34}X_{14}X_{13,24}^{-1}X_{23}$
6	(r4s4a), $\alpha\neq 0,-1$, $\gamma=0$, $\delta\neq 0,-1$, $\alpha\neq\delta$	$\mathrm{GL}(2,K).K$
7	(r4s4a), $\alpha=-1$, $\gamma=0$, $\delta=-1$.	$\mathrm{B}(2,K).K^3$
8	(r4s4a), $\alpha=-1$, $\gamma=0$, $\delta\neq 0,-1$	$\mathrm{B}(2,K).K$
9	(r4s4a), $\alpha=0$, $\gamma=1$, $\delta=0$ (r4s4a), $\alpha=0$, $\gamma\neq 0,\pm1$, $\delta\neq 0,-1$, $\gamma=\delta+1$ (r4s4a), $\alpha\neq 0,-1$, $\gamma\neq 0$, $\delta\neq 0,-1$, $\gamma\neq\alpha\delta$, $\gamma=(\alpha+1)(\delta+1)$, $\alpha+\delta\neq-2$ (r4s4b), $\alpha=-1$, $\delta\neq 0,-1$	$(\mathrm{U}(2,K)\times\mathrm{SL}(2,K))\rtimes$ $\rtimes(K^*\times K^*)$
10	(r4s4a), $\alpha=0$, $\gamma=-1$, $\delta=-2$ (r4s4a), $\alpha\neq 0,-1$, $\gamma\neq 0$, $\delta\neq 0,-1$, $\gamma\neq\alpha\delta$, $\gamma=(\alpha+1)(\delta+1)$, $\alpha+\delta=-2$ (r4s4b), $\alpha=-1$, $\delta=-1$	$\mathrm{B}(2,K)$

Proof. (1) For the base (r4s4a), $\alpha=-1$, $\gamma\neq 0$, $\delta=0$. The generators have the following form.

$$\begin{aligned} x(\varepsilon) =& d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{\gamma(\varepsilon-1)}{\varepsilon}\right),\\ y(\eta) =& d_1(\eta)d_2(\eta)x_{31}(\eta-1)x_{42}(\eta-1). \end{aligned}$$

Next we find the subspaces of $\overline{V}$ invariant under the action of $\overline{X}$. It is easy to check that there are no points and planes in $\overline{V}$ invariant under the action of $\overline{X}$. Moreover, a line P in $\overline{V}$ is invariant under $\overline{X}$ if and only if

$$\overline{x}(\varepsilon)^t P = aP\overline{x}(\varepsilon)^{-1},\quad a\in K^*,\quad \overline{y}(\eta)P = bP\overline{y}(\eta)^{-1},\quad b\in K^*. \tag{c}$$

And P satisfying (c) has the entries $p_{12}=p_{34}=0$, $p_{23}=p_{14}$, $p_{24}=(p_{13}+p_{14})/\gamma$ and $p_{13}^2+p_{13}p_{14}-\gamma p_{14}^2=0$. The solvability of this equation is equivalent to that of $t^2+t-\gamma=0$. This leads to 3 different cases.

• The equation $t^2+t-\gamma=0$ has a unique root when $\gamma=-1/4$. Then a simultaneous conjugation by the element $\omega_{23}x_{21}(-1/2)x_{43}(-1/2)$ leads to the following generators.

$$\begin{aligned} x_1(\varepsilon) =& d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{2\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{1-\varepsilon}{2\varepsilon}\right),\\ y_1(\eta) =& d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1). \end{aligned}$$

For $\varepsilon,\eta\in K^*$, we put

$$r_1(\varepsilon,\eta)=z_{x_1}(\varepsilon,-1,\eta),\quad r_2(\varepsilon,\eta)=z_{y_1}(-1,\varepsilon,\eta).$$

Then

$$\begin{aligned} r_1\left(\varepsilon,-\frac{1}{\varepsilon^3}\right) &= x_{13}(-\varepsilon')x_{23}(-2\varepsilon')x_{24}(\varepsilon'),\\ r_2(\varepsilon,-\varepsilon) &= x_{13}\left(-\frac{2}{\varepsilon}+2\varepsilon\right)x_{14}\left(\frac{2}{\varepsilon}-2\varepsilon\right)x_{24}\left(\frac{2}{\varepsilon}-2\varepsilon\right). \end{aligned}$$

where $\varepsilon'=\frac{2(\varepsilon-1)(\varepsilon^3+1)}{\varepsilon^2}$. It follows that the subgroups $X_{13,23,24}^{-1,-2}$ and $X_{13,14,24}^{-1}$ are contained in H. We take the elements from these two subgroups,

$$m_1(\varepsilon)=x_{13}(-\varepsilon)x_{23}(-2\varepsilon)x_{24}(\varepsilon),\quad m_2(\varepsilon)=x_{13}(-\varepsilon)x_{14}(\varepsilon)x_{24}(\varepsilon),$$

and calculate the following products,

$$\begin{aligned} m_1\left(-\frac{2(\varepsilon-1)(\eta-1)}{\varepsilon^2\eta}\right)r_1(\varepsilon,\eta) &= x_{14}\left(\frac{(\varepsilon-1)(\eta-1)\left(\varepsilon^3\eta+1\right)}{\varepsilon^2\eta}\right),\\ m_2\left(-\frac{2(\varepsilon-1)(\eta-1)}{\varepsilon}\right)r_2(\varepsilon,\eta) &= x_{23}\left(-\frac{2(\varepsilon-1)(\eta-1)(\varepsilon+\eta)}{\varepsilon\eta}\right). \end{aligned}$$

Then we get that the subgroups X_{14}, X_{23} and $X_{13,24}^{-1}$ are contained in H. From the decomposition of $x_1(\varepsilon)$ we have that $Q_{13}X_{12}X_{34}\leqslant H$, and take its element,

$$u(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{2\varepsilon}\right)x_{34}\left(\frac{1-\varepsilon}{2\varepsilon}\right).$$

Due to the lemma 2(3), we get $\langle u(\varepsilon),y_1(\eta)\rangle=\mathrm{GL}(2,K)$, thus we conclude that
$H=\mathrm{GL}(2,K)X_{14}X_{23}X_{13,24}^{-1}=\mathrm{GL}(2,K).K^3$.

• The equation $t^2+t-\gamma=0$ has two distinct roots t_1 and t_2, i.e. $4\gamma+1\in K^{*2}$. Then the subgroup H has two 2-dimensional invariant subspaces in V, namely, the planes spanned by $(-1,t_i,0,0)$ and $(0,0,-1,t_i)$, $i=1,2$. Thus conjugating this base by the element z_1^{-1}, the generators have the following form.

$$\begin{aligned}x_2(\varepsilon)=&d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{c(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{(-1-c)(\varepsilon-1)}{\varepsilon}\right),\\ y_2(\eta)=&d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).\end{aligned}$$

where $c=\frac{1+\gamma-t_2}{2t_2+1}$.

Suppose that $c\neq-1/2$, i.e. $\gamma\neq-3/2$. Applying the lemma 2(1), we have $H=\langle x_2(\varepsilon),y_2(\eta)\rangle=(\mathrm{SL}(2,K)\times\mathrm{SL}(2,K))\rtimes K^*$.

Suppose that $c=-1/2$, i.e. $\gamma=-3/2$. New generators are as follows.

$$x_3(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{2\varepsilon}\right)x_{34}\left(\frac{1-\varepsilon}{2\varepsilon}\right),\quad y_3(\eta)=d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).$$

Due to lemma 2(3), $H=\langle x_3(\varepsilon),y_3(\eta)\rangle=\mathrm{GL}(2,K)$.

• The equation $t^2+t-\gamma=0$ has no roots in K, i.e. $4\gamma+1$ does not belong to K^{*2}. We consider the extension $K'=K(\sqrt{4\gamma+1})$ and apply the same argument as in above case.

(2) For the base (r4s4a), $\alpha=0$, $\gamma\neq0$, $\delta\neq0,-1$, $\gamma\neq\delta+1$. The generators have the following form.

$$\begin{aligned}x(\varepsilon)=&d_1(\varepsilon)d_2(\varepsilon)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{\gamma(\varepsilon-1)}{\varepsilon}\right)x_{24}\left(\frac{\delta(\varepsilon-1)}{\varepsilon}\right),\\ y(\eta)=&d_1(\eta)d_2(\eta)x_{31}(\eta-1)x_{42}(\eta-1).\end{aligned}$$

P satisfying (c) has the entries $p_{12}=p_{34}=0$, $p_{23}=p_{14}$, $p_{24}=(p_{13}+p_{14}\delta)/\gamma$ and $p_{13}^2+p_{13}p_{14}\delta-\gamma p_{14}^2=0$. The solvability of this equation is equivalent to that of $t^2+\delta t-\gamma=0$. This leads to 3 different cases.

• The equation $t^2+\delta t-\gamma=0$ has a unique root when $\gamma=-1$ and $\delta=2$. Then a simultaneous conjugation by the element $\omega_{23}x_{21}(-1)x_{43}(-1)$ leads to the following generators.

$$\begin{aligned}x_1(\varepsilon)=&d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right),\\ y_1(\eta)=&d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).\end{aligned}$$

Suppose that $\operatorname{char}K\neq3$. For $\varepsilon\neq\eta$, $\eta\neq\varepsilon/(\varepsilon-1)$ and $\varepsilon\neq1,2$. We put

$$q_1(\varepsilon)=z_{x_1}\left(\varepsilon,2,-\frac{1}{3(\varepsilon-1)\varepsilon}\right),\quad q_2(\eta)=z_{y_1}\left(\frac{1}{2},\frac{2}{3},\eta\right),$$

and

$$[q_1(\varepsilon),q_1(\eta)]=k_1(\varepsilon,\eta),\quad [q_2(\varepsilon),q_2(\eta)]=k_2(\varepsilon,\eta),\quad \left[q_2(\varepsilon),q_2\left(\frac{\varepsilon}{\varepsilon-1}\right)\right]=k_3(\varepsilon).$$

Multiplying $k_1(\varepsilon,\eta)$, $k_2(\varepsilon,\eta)$ by suitable $k_3(\varepsilon)$, we can get the subgroup X_{23} and X_{14}. Moreover it follows from $k_3(\varepsilon)$ that $X_{13,24}^{-1}\leqslant H$.

Suppose that $\operatorname{char}K=3$. We calculate

$$\begin{aligned}z_{x_1}\left(\varepsilon,2,\frac{2}{\varepsilon^3}\right)&=x_{13}\left(\frac{(2\varepsilon+1)\varepsilon'}{\varepsilon^2}\right)x_{24}\left(\frac{(\varepsilon+2)\varepsilon'}{\varepsilon^2}\right)x_{23}\left(\frac{(\varepsilon+2)\varepsilon'}{\varepsilon^2}\right),\\ z_{y_1}(2,\varepsilon,2\varepsilon)&=x_{13}\left(\frac{2(\varepsilon+1)(\varepsilon+2)}{\varepsilon}\right)x_{24}\left(\frac{(\varepsilon+1)(\varepsilon+2)}{\varepsilon}\right)x_{14}\left(\varepsilon+\frac{2}{\varepsilon}\right),\end{aligned}$$

where $\varepsilon' = 2\varepsilon^3 + 2$. In particular, for $2\varepsilon^4\eta^2 + \varepsilon^3\eta^2 + 2\varepsilon^2\eta + \varepsilon^2 + 2\varepsilon\eta^2 + \eta^2 \neq 0$, $2\eta + 1 \neq 0$ and $\varepsilon \in K^*$, we have

$$z_{x_1}\left(\varepsilon, 2, \frac{2}{\varepsilon^3}\right) z_{x_1}\left(\eta, 2, \frac{2\varepsilon^2\eta + \varepsilon^2}{2\varepsilon^4\eta^2 + \varepsilon^3\eta^2 + 2\varepsilon^2\eta + \varepsilon^2 + 2\varepsilon\eta^2 + \eta^2}\right) =$$
$$x_{14}\left(\frac{2\left(2\varepsilon^4 + \varepsilon^3 + 2\varepsilon + 1\right)(\eta + 2)\left(2\varepsilon^4\eta^2 + \varepsilon^3\eta^2 + \varepsilon^2\left(2\eta^4 + \eta^3 + 2\eta + 1\right) + 2\varepsilon\eta^2 + \eta^2\right)}{\varepsilon^2(2\eta + 1)\left(2\varepsilon^4\eta^2 + \varepsilon^3\eta^2 + 2\varepsilon^2\eta + \varepsilon^2 + 2\varepsilon\eta^2 + \eta^2\right)}\right).$$

Thus we extract the subgroup X_{14}. From the decomposition of $z_{x_1}\left(\varepsilon, 2, \frac{2}{\varepsilon^3}\right)$ and $z_{y_1}(2, \varepsilon, 2\varepsilon)$ we get the subgroup $X^2_{13,24}$ and X_{23}.

Now, from the decomposition of $x_1(\varepsilon)$ we have that $Q_{13}X_{12}X_{34}$ are contained in H, and we take its element $m(\varepsilon)$,

$$m(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\varepsilon - 1}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon - 1}{\varepsilon}\right).$$

This is the case of lemma 2(3), so $\langle m(\varepsilon), y_1(\eta)\rangle = \mathrm{GL}(2, K)$. Finally we conclude that $H = \mathrm{GL}(2, K).X_{14}X_{23}X^{-1}_{13,24} = \mathrm{GL}(2, K).K^3$.

• The equation $t^2 + \delta t - \gamma = 0$ has two roots t_1 and t_2, i.e. $\delta^2 + 4\gamma \in K^{*2}$. Then the subgroup H has two 2-dimensional invariant subspaces in V. Simultaneous conjugation by z_1^{-1} yields the following generators.

$$\begin{aligned} x_2(\varepsilon) =& d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{c_1(\varepsilon - 1)}{\varepsilon}\right)x_{34}\left(\frac{(\delta - c_1)(\varepsilon - 1)}{\varepsilon}\right), \\ y_2(\eta) =& d_1(\eta)d_3(\eta)x_{21}(\eta - 1)x_{43}(\eta - 1), \end{aligned}$$

where $c_1 = \frac{-2\gamma + t_1\delta}{2t_1 + \delta}$. From the lemma 2(1), we conclude that $H = \langle x_2(\varepsilon), y_2(\eta)\rangle = (\mathrm{SL}(2, K) \times \times \mathrm{SL}(2, K)) \rtimes K^*$.

• The equation $t^2 + \delta t - \gamma = 0$ has no roots in K, i.e. $\delta^2 + 4\gamma$ does not belong to K^{*2}. We consider the extension $K' = K(\sqrt{\delta^2 + 4\gamma})$ and apply the same argument as in above case.

(3) For the base (r4s4a), $\alpha = 0$, $\gamma \neq 0, 1$, $\delta = 0$. The generators have the following form.

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{14}\left(\frac{\varepsilon - 1}{\varepsilon}\right)x_{23}\left(\frac{\gamma(\varepsilon - 1)}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta - 1)x_{42}(\eta - 1).$$

P satisfying (c) has the entries $p_{12} = p_{34} = 0$, $p_{23} = p_{14}$, $p_{24} = p_{13}/\gamma$ and $p_{13}^2 - p_{14}^2\gamma = 0$. The solvability of this equation is equivalent to that of $t^2 - \gamma = 0$.

• The equation $t^2 - \gamma = 0$ has two roots t_1 and t_2, i.e. $\gamma \in K^{*2}$. Then the subgroup H has two 2-dimensional invariant subspaces in V. Simultaneous conjugation by z_1^{-1} yields the following generators.

$$\begin{aligned} x_1(\varepsilon) =& d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{-c(\varepsilon - 1)}{\varepsilon}\right)x_{34}\left(\frac{c(\varepsilon - 1)}{\varepsilon}\right), \\ y_1(\eta) =& d_1(\eta)d_3(\eta)x_{21}(\eta - 1)x_{43}(\eta - 1), \end{aligned}$$

where $c = \frac{\gamma}{t_1}$. From the lemma 2(1), we get $H = \langle x_1(\varepsilon), y_1(\eta)\rangle = (\mathrm{SL}(2, K) \times \mathrm{SL}(2, K)) \rtimes K^*$.

• The equation $t^2 - \gamma = 0$ has no roots in K, i.e. γ does not belong to K^{*2}. We consider the extension $K' = K(\sqrt{\gamma})$ and apply the same argument as in above case.

(4) For the base (r4s4a), $\alpha = -1$, $\gamma \neq 0, 1$, $\delta = -1$. The generators have the following form.

$$\begin{aligned} x(\varepsilon) =& d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{1 - \varepsilon}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon - 1}{\varepsilon}\right)x_{23}\left(\frac{\gamma(\varepsilon - 1)}{\varepsilon}\right)x_{24}\left(\frac{1 - \varepsilon}{\varepsilon}\right), \\ y(\eta) =& d_1(\eta)d_2(\eta)x_{31}(\eta - 1)x_{42}(\eta - 1). \end{aligned}$$

P satisfying (c) has the entries $p_{12}=p_{34}=0$, $p_{23}=p_{14}$, $p_{24}=p_{13}/\gamma$ and $p_{13}^2-p_{14}^2\gamma=0$. The solvability of this equation is equivalent to that of $t^2-\gamma=0$.

• The equation $t^2-\gamma=0$ has two roots t_1 and t_2, i.e. $\gamma\in K^{*2}$. Then the subgroup H has two 2-dimensional invariant subspaces in V. Simultaneous conjugation by z_1^{-1} yields the following generators.

$$\begin{aligned} x_1(\varepsilon)=&d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{c(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{(-2-c)(\varepsilon-1)}{\varepsilon}\right),\\ y_1(\eta)=&d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1),\end{aligned}$$

where $c=\frac{-\gamma-t_1}{t_1}$. From the lemma 2(1), $H=\langle x_1(\varepsilon),y_1(\eta)\rangle=(\mathrm{SL}(2,K)\times\mathrm{SL}(2,K))\rtimes K^*$.

• The equation $t^2-\gamma=0$ has no roots in K, i.e. γ does not belong to K^{*2}. We consider the extension $K'=K(\sqrt{\gamma})$ and apply the same argument as in above case.

(5) For the base (r4s4a), $\alpha\neq0,-1$, $\gamma\neq0$, $\delta=-1$. The generators have the following form.

$$\begin{aligned} x(\varepsilon)=&d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{\gamma(\varepsilon-1)}{\varepsilon}\right)x_{24}\left(\frac{1-\varepsilon}{\varepsilon}\right),\\ y(\eta)=&d_1(\eta)d_2(\eta)x_{31}(\eta-1)x_{42}(\eta-1).\end{aligned}$$

P satisfying (c) has the entries $p_{12}=p_{34}=0$, $p_{23}=p_{14}$, $p_{24}=(p_{13}-p_{14}(\alpha+1))/\gamma$ and $-p_{13}^2+p_{13}p_{14}(\alpha+1)+\gamma p_{14}^2=0$. The solvability of this equation is equivalent to that of $t^2-(\alpha+1)t-\gamma=0$. This leads to 3 different cases.

• The equation $t^2-(\alpha+1)t-\gamma=0$ has a unique root when $\gamma=-1$ and $\alpha+1=\pm2$. If $\alpha=1$, a simultaneous conjugation by $w_{23}w_{24}w_{13}x_{21}(1)x_{43}(1)$ leads to the following generators.

$$x_1(\varepsilon)=d_2(\varepsilon)d_4(\varepsilon)x_{23}\left(\frac{\varepsilon-1}{\varepsilon}\right),\quad y_1(\eta)=d_2(\eta)d_4(\eta)x_{12}(\eta-1)x_{34}(\eta-1).$$

Performing the straightforward calculation, we obtain

$$\begin{aligned} z(\varepsilon,\eta)=&x_{12}\left(\frac{-\theta_0}{\varepsilon\eta}\right)x_{13}\left(\frac{-\theta_0}{\varepsilon}\right)x_{14}\left(\frac{(\eta-1)\theta_0}{\varepsilon\eta}\right)x_{23}(-\theta_0)\times\\ &x_{24}\left(\frac{\theta_0(1+\varepsilon(\eta-1))}{\varepsilon\eta}\right)x_{34}\left(\frac{-\theta_0}{\varepsilon\eta}\right),\end{aligned}$$

where $\theta_0=(\varepsilon-1)(\eta-1)$. Then for $\varepsilon\eta-\varepsilon-3\eta+1\neq0$, we have

$$\left[z(\varepsilon,\eta),z\left(\varepsilon,\frac{(\varepsilon-1)\eta}{\varepsilon\eta-\varepsilon-3\eta+1}\right)\right]=x_{13}(\theta_1)x_{24}(\theta_1),\quad [z(\varepsilon,\eta),z(\eta,\varepsilon)]=x_{14}(\theta_2),$$

where $\theta_1=\frac{(\varepsilon-1)(\eta-1)(\varepsilon(\eta-2)-3\eta+2)(\varepsilon+2\eta-1)}{\varepsilon\eta(\varepsilon(\eta-1)-3\eta+1)}$, $\theta_2=-\frac{2(\varepsilon-1)^2(\eta-1)^2(\varepsilon-\eta)}{\varepsilon^2\eta^2}$. Take $\varepsilon\neq\eta$, $\varepsilon\neq1,1-2\eta$, $(3\eta-2)/(\eta-2)$, $\eta\neq1,2,1/2$. Hence for some $\theta\in K^*$, $x_{13}(-\theta)x_{24}(\theta)$ and $x_{14}(\theta)$ lie in H.

On the other hand, for $\eta\neq1,2,1/2$,

$$\begin{aligned} z(\varepsilon,\eta)z\left(\varepsilon,2-\eta\right)=&x_{12}\left(-\frac{2(\varepsilon-1)(\eta-1)^2}{\varepsilon(\eta-2)\eta}\right)x_{34}\left(-\frac{2(\varepsilon-1)(\eta-1)^2}{\varepsilon(\eta-2)\eta}\right)\times\\ &x_{13}(-q_1(\varepsilon,\eta))x_{24}(q_1(\varepsilon,\eta))x_{14}(q_2(\varepsilon,\eta)),\\ z(\varepsilon,\eta)z\left(\varepsilon,\frac{\eta}{2\eta-1}\right)=&x_{23}\left(-\frac{2(\varepsilon-1)(\eta-1)^2}{2\eta-1}\right)x_{13}(-q_3(\varepsilon,\eta))x_{24}(q_3(\varepsilon,\eta))x_{14}(q_4(\varepsilon,\eta)),\end{aligned}$$

where $q_i(\varepsilon,\eta)$, $i=1,2,3,4$ are rational functions, which have poles at point $\eta=-1/2,2$. But we know that $X_{13,24}^{-1}$ and X_{14} are contained in H. Hence we can extract the elements of the groups $X_{12,34}$ and X_{23}, after it we get the whole groups. Finally, we conclude that $H=Q_{24}X_{12,34}X_{14}X_{13,24}^{-1}X_{23}$.

If $\alpha = -3$, at this time, $\operatorname{char} K \neq 3$, then a simultaneous conjugation by the element $\omega_{23}x_{21}(-1)x_{43}(-1)$ leads to the following generators.

$$\begin{aligned} x_2(\varepsilon) =& d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{2-2\varepsilon}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{2-2\varepsilon}{\varepsilon}\right), \\ y_2(\eta) =& d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1). \end{aligned}$$

For $\varepsilon \neq \eta$, $\eta \neq \varepsilon/(\varepsilon-1)$ and $\varepsilon \neq 1, 2$. We put

$$q_1(\varepsilon) = z_{x_2}\left(\varepsilon, \frac{1}{2}, -\frac{2}{3(\varepsilon-1)\varepsilon}\right), \quad q_2(\eta) = z_{y_2}\left(2, \frac{1}{3}, \eta\right),$$

and

$$[q_1(\varepsilon), q_1(\eta)] = k_1(\varepsilon,\eta), \quad [q_2(\varepsilon), q_2(\eta)] = k_2(\varepsilon,\eta), \quad \left[q_2(\varepsilon), q_2\left(\frac{\varepsilon}{\varepsilon-1}\right)\right] = k_3(\varepsilon).$$

Multiplying $k_1(\varepsilon,\eta)$, $k_2(\varepsilon,\eta)$ by suitable $k_3(\varepsilon)$, we can get the subgroup X_{23} and X_{14}. Moreover it follows from $k_3(\varepsilon)$ that $X_{13,24}^{-1} \leqslant H$.

Now, from the decomposition of $x_2(\varepsilon)$ we have that $Q_{13}X_{12}X_{34}$ are contained in H, and take its element $m(\varepsilon)$,

$$m(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{2-2\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{2-2\varepsilon}{\varepsilon}\right).$$

From the lemma 2(3), we get $\langle m(\varepsilon), y_2(\eta)\rangle = \mathrm{GL}(2,K)$. Thus we conclude that $H = \mathrm{GL}(2,K).X_{14}X_{23}X_{13,24}^{-1} = \mathrm{GL}(2,K).K^3$.

• The equation $t^2-(\alpha+1)t-\gamma = 0$ has two roots t_1 and t_2, i.e. $(\alpha+1)^2+\gamma \in K^{*2}$. Then the subgroup H has two 2-dimensional invariant subspaces in V. Simultaneous conjugation by z_1^{-1} yields the following generators.

$$\begin{aligned} x_3(\varepsilon) =& d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{c(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{(\alpha-1-c)(\varepsilon-1)}{\varepsilon}\right), \\ y_3(\eta) =& d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1), \end{aligned}$$

where $c = \frac{-2\gamma-\alpha(\alpha-t_1+1)-t_1}{-\alpha+2t_1-1}$. From the lemma 2(1), we conclude that $H = \langle x_3(\varepsilon), y_3(\eta)\rangle = (\mathrm{SL}(2,K)\times\mathrm{SL}(2,K)) \rtimes K^*$.

• The equation $t^2-(\alpha+1)t-\gamma = 0$ has no roots in K, i.e. $(\alpha+1)^2+4\gamma$ does not belong to K^{*2}. We consider the extension $K' = K(\sqrt{(\alpha+1)^2+4\gamma})$ and apply the same argument as in above case.

(6) For the base (r4s4a), $\alpha \neq 0,-1$, $\gamma = 0$, $\delta \neq 0,-1$. The generators have the following form.

$$\begin{aligned} x(\varepsilon) =& d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{24}\left(\frac{\delta(\varepsilon-1)}{\varepsilon}\right), \\ y(\eta) =& d_1(\eta)d_2(\eta)x_{31}(\eta-1)x_{42}(\eta-1). \end{aligned}$$

P satisfying (c) has the entries $p_{12} = p_{34} = 0$, $p_{23} = p_{14}$, $p_{13} = -p_{14}(\delta-\alpha)$ and $p_{14}+p_{24}(\delta-\alpha) = 0$. The solvability of this equation is equivalent to that of $(\delta-\alpha)t+1 = 0$. It has only one root $t = 1/(\alpha-\delta)$.

• Let $\delta \neq \alpha$. Conjugating original base by the element $w_{23}x_{21}(\alpha-\delta)x_{43}(\alpha-\delta)$. Then the generators of the group X and Y have the following form.

$$\begin{aligned} x_1(\varepsilon) =& d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\delta(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right), \\ y_1(\eta) =& d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1). \end{aligned}$$

If $\alpha+\delta+1\neq 0$. For $\eta\in K^*$, we put

$$q_1(\eta)=z_{x_1}\left(a,\frac{1+\delta}{\delta},\eta\right),\quad q_2(\eta)=z_{y_1}\left(\frac{\delta}{\delta+1},a,\eta\right),$$
$$q_3(\eta)=z_{x_1}\left(b,\frac{1+\alpha}{\alpha},\eta\right),\quad q_4(\eta)=z_{y_1}\left(\frac{\alpha}{\alpha+1},b,\eta\right),$$

where $a=\frac{\alpha^2+\alpha}{\alpha^2+\alpha-\delta^2-\delta}$, $b=-\frac{\delta^2+\delta}{\alpha^2+\alpha-\delta^2-\delta}$. The straightforward calculations show that

$$[q_1(\varepsilon),q_1(\eta)]=x_{14}(\theta_3)x_{34}\left((\alpha-\delta)\theta_3\right),\quad [q_2(\varepsilon),q_2(\eta)]=x_{23}(\theta_4)x_{43}\left((\alpha-\delta)\theta_4\right),$$
$$[q_3(\varepsilon),q_3(\eta)]=x_{12}\left((\delta-\alpha)\theta_5\right)x_{14}(\theta_5),\quad [q_4(\varepsilon),q_4(\eta)]=x_{21}\left((\delta-\alpha)\theta_6\right)x_{23}(\theta_6),$$

where

$$\theta_3=\frac{\alpha(\alpha+1)(\varepsilon-1)(\eta-1)\left(2\alpha^2+2\alpha-\delta(\delta+1)\right)(\varepsilon-\eta)}{\varepsilon^2\eta^2(\alpha-\delta)^3(\alpha+\delta+1)},$$
$$\theta_4=\frac{(\varepsilon-1)(\eta-1)\left(2\alpha^2+2\alpha-\delta(\delta+1)\right)(\varepsilon-\eta)}{\alpha(\alpha-\delta)^2},$$
$$\theta_5=\frac{\delta(\delta+1)(\varepsilon-1)(\eta-1)\left(\alpha^2+\alpha-2\delta(\delta+1)\right)(\varepsilon-\eta)}{\varepsilon^2\eta^2(\alpha-\delta)^3(\alpha+\delta+1)},$$
$$\theta_6=\frac{(\varepsilon-1)(\eta-1)\left(\alpha^2+\alpha-2\delta(\delta+1)\right)(\varepsilon-\eta)}{\delta(\alpha-\delta)^2}.$$

It follows that the subgroups $X^{\alpha-\delta}_{34,14}$, $X^{\alpha-\delta}_{43,23}$, $X^{\delta-\alpha}_{12,14}$ and $X^{\delta-\alpha}_{21,23}$ are contained in H. We take the elements from these subgroups,

$$f_1(\varepsilon)=x_{14}(\varepsilon)x_{34}\left((\alpha-\delta)\varepsilon\right),\quad f_2(\varepsilon)=x_{23}(\varepsilon)x_{43}\left((\alpha-\delta)\varepsilon\right),$$
$$f_3(\varepsilon)=x_{12}\left((\delta-\alpha)\varepsilon\right)x_{14}(\varepsilon),\quad f_4(\varepsilon)=x_{21}\left((\delta-\alpha)\varepsilon\right)x_{23}(\varepsilon),$$

then we have,

$$f_1(\varepsilon)f_3(-\varepsilon)=x_{12}((\alpha-\delta)\varepsilon)x_{34}((\alpha-\delta)\varepsilon),\quad f_2(\varepsilon)f_4(-\varepsilon)=x_{21}((\alpha-\delta)\varepsilon)x_{43}((\alpha-\delta)\varepsilon),$$

So we get that $X_{12,34}$ and $X_{21,43}$ are the subgroups of H. Also we obtain Q_{13} and $X^{\alpha-\delta}_{12,14}$ from the decomposition of $x(\varepsilon)$ and $y(\eta)$. Finally, we conclude that $H=\mathrm{GL}(2,K).X^{\alpha-\delta}_{12,14}=\mathrm{GL}(2,K).K$, when $\alpha\neq\delta$ and $\alpha+\delta+1\neq 0$.

If $\alpha+\delta+1=0$. At this time $\alpha\neq -1/2$. Conjugating this base by the element $w_{23}x_{21}(2\alpha+1)x_{43}(2\alpha+1)$, the generators have the following form.

$$x_2(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{(\alpha+1)(1-\varepsilon)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right),$$
$$y_2(\eta)=d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).$$

For $\varepsilon\neq\pm 1$, we put

$$q_1(\varepsilon)=z_{y_2}\left(\frac{\alpha+1}{\alpha},\varepsilon,-\varepsilon\right),\quad q_2(\varepsilon)=z_{y_2}\left(\frac{\alpha}{\alpha+1},\varepsilon,-\varepsilon\right),$$

and calculate the products,

$$p_1(\varepsilon)=q_1(\varepsilon)z\left(\frac{\alpha+1}{\alpha},\frac{\varepsilon}{\varepsilon^2+\varepsilon-1}\right),\quad p_2(\varepsilon)=q_2(\varepsilon)z\left(\frac{\alpha}{\alpha+1},\frac{\varepsilon}{\varepsilon^2+\varepsilon-1}\right).$$

Then we have

$$
\begin{aligned}
[p_1(\varepsilon),p_1(\eta)] =& x_{21}(-(2\alpha+1)q_5(\varepsilon,\eta))x_{23}(q_5(\varepsilon,\eta)),\\
[p_2(\varepsilon),p_2(\eta)] =& x_{23}(q_6(\varepsilon,\eta))x_{43}((2\alpha+1)q_6(\varepsilon,\eta)).
\end{aligned}
$$

where $q_5(\varepsilon,\eta)$ and $q_6(\varepsilon,\eta)$ are the rational functions of ε and η, which have the poles at points $\varepsilon^2+\varepsilon-1=0$ and $\eta^2+\eta-1=0$. Thus we get the subgroups $X_{21,23}^{-2\alpha-1}$ and $X_{43,23}^{2\alpha+1}$. And then,

$$
x_{21}((2\alpha+1)\varepsilon)x_{23}(-\varepsilon)x_{43}((2\alpha+1)\varepsilon)x_{23}(\varepsilon)=x_{21}((2\alpha+1)\varepsilon)x_{43}((2\alpha+1)\varepsilon),
$$

It follows that $X_{21,43}$ is contained in H. From the decomposition of generators we obtain the subgroups Q_{13} and $X_{12,34,14}^{\alpha,-\alpha-1}$. Now put

$$
t(\varepsilon)=x_{12}\left(-\frac{\alpha\varepsilon}{\alpha+1}\right)x_{34}(\varepsilon)x_{14}\left(\frac{-\varepsilon}{\alpha+1}\right),\quad s(\eta)=x_{21}(\eta)x_{43}(\eta).
$$

We set $w(\xi)=t(\xi)s(-\xi^{-1})t(\xi)$, and calculate the product,

$$
u(\xi,\varepsilon)=w(\xi)t(\varepsilon)w(-\xi)s(\xi^{-2}\varepsilon),
$$

Put $u_0(\varepsilon)=u\left(-\frac{\alpha\varepsilon}{\alpha+1},\varepsilon\right)$, for $\varepsilon\neq\eta$, we have

$$
[u_0(\varepsilon),u_0(\eta)]=x_{12}\left(\frac{(2\alpha+1)^3(\eta-\varepsilon)}{\alpha^2(\alpha+1)^2}\right)x_{14}\left(\frac{(2\alpha+1)^2(\eta-\varepsilon)}{\alpha^2(\alpha+1)^2}\right).
$$

Thus we get the whole subgroup $X_{12,14}^{2\alpha+1}$. Then

$$
t(\varepsilon)x_{12}\left(\frac{(2\alpha+1)\varepsilon}{\alpha+1}\right)x_{14}\left(\frac{\varepsilon}{\alpha+1}\right)=x_{12}(\varepsilon)x_{34}(\varepsilon).
$$

It follows that we extract the subgroup $X_{12,34}$. Finally we conclude that $H=\mathrm{GL}(2,K)X_{12,14}^{2\alpha+1}=$ $=\mathrm{GL}(2,K).K$, when $\alpha\neq\delta$, $\alpha+\delta+1=0$.

• Let $\alpha=\delta$ and $\alpha\neq-1/2$. Conjugating original base by the element w_{23} of the Weyl group. The new generators have the following form.

$$
\begin{aligned}
x_3(\varepsilon)&=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right),\\
y_3(\eta)&=d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).
\end{aligned}
$$

For $\varepsilon,\eta\in K^*$, we put

$$
q_1(\varepsilon)=z_{x_3}\left(\varepsilon,\frac{\alpha+1}{\alpha},-\frac{\alpha}{(2\alpha+1)(\varepsilon-1)\varepsilon}\right),\quad q_2(\eta)=z_{y_3}\left(\frac{\alpha}{\alpha+1},\frac{\alpha+1}{2\alpha+1},\eta\right),
$$

and

$$
[q_1(\varepsilon),q_1(\eta)]=k_1(\varepsilon,\eta),\quad [q_2(\varepsilon),q_2(\eta)]=k_2(\varepsilon,\eta),\quad \left[q_2(\varepsilon),q_2\left(\frac{\varepsilon}{\varepsilon-1}\right)\right]=k_3(\varepsilon).
$$

Calculate a commutator subgroup $k_3(\varepsilon)$ generated by $q_2(\varepsilon)$, we get that $X_{13,24}^{-1}\leqslant H$. Multiplying $k_1(\varepsilon,\eta)$, $k_2(\varepsilon,\eta)$ by suitable $k_3(\varepsilon)$, we can get the subgroup X_{23} and X_{14}.

Now, from the decomposition of $x_3(\varepsilon)$ we have that $Q_{13}X_{12}X_{34}$ are contained in H, and we take its element $m(\varepsilon)$,

$$
m(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right).
$$

From the lemma 2(3), we get $\langle m(\varepsilon), y_3(\eta)\rangle = \mathrm{GL}(2,K)$. Thus we conclude that $H = \mathrm{GL}(2,K).X_{14}X_{23}X_{13,24}^{-1} = \mathrm{GL}(2,K).K^3$, when $\alpha = \delta$, $\alpha \neq -1/2$.

If $\alpha = \delta = -1/2$. The new generators have the following form.

$$x_4(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{2\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{1-\varepsilon}{2\varepsilon}\right),$$
$$y_4(\eta) = d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).$$

For $\varepsilon, \eta \in K^*$,

$$z_{y_4}(-1,\varepsilon,\eta)z_{y_4}(-1,\varepsilon,2-\eta) = x_{23}\left(-\frac{4(\varepsilon-1)(\eta-1)^2}{(\eta-2)\eta}\right)$$

we get that $X_{23} \leqslant H$. Multiplying $z_{x_4}\left(\varepsilon,-1,-\frac{1}{\varepsilon^3}\right)$ by suitable element from X_{23}, we have the subgroup $X_{13,24}^{-1}$. In fact,

$$z_{x_4}\left(\varepsilon,-1,-\frac{1}{\varepsilon^3}\right)x_{23}(2\varepsilon') = x_{13}(-\varepsilon')x_{24}(\varepsilon'),$$

where $\varepsilon' = \frac{2(\varepsilon-1)(\varepsilon^3+1)}{\varepsilon^2}$. Since

$$z_{y_4}(-1,\varepsilon,-\varepsilon)\,x_{13}\left(-\frac{2\left(\varepsilon^2-1\right)}{\varepsilon}\right)x_{24}\left(\frac{2\left(\varepsilon^2-1\right)}{\varepsilon}\right) = x_{14}\left(\frac{2}{\varepsilon}-2\varepsilon\right),$$

It follows that X_{14} are contained in H. Thus we get the element,

$$m(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{2\varepsilon}\right)x_{34}\left(\frac{1-\varepsilon}{2\varepsilon}\right).$$

From the lemma 2(3), we get $\langle m(\varepsilon), y_4(\eta)\rangle = \mathrm{GL}(2,K)$. Thus we conclude that $H = \mathrm{GL}(2,K).X_{14}X_{23}X_{13,24}^{-1} = \mathrm{GL}(2,K).K^3$, when $\alpha = \delta$, $\alpha = -1/2$.

(7) For the base (r4s4a), $\alpha \neq 0,-1$, $\gamma \neq 0$, $\delta \neq 0,-1$, $\gamma \neq \alpha\delta$, $\gamma \neq (\alpha+1)(\delta+1)$. The generators have the following form.

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{\gamma(\varepsilon-1)}{\varepsilon}\right)x_{24}\left(\frac{\delta(\varepsilon-1)}{\varepsilon}\right),$$
$$y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta-1)x_{42}(\eta-1).$$

P satisfying (c) has the entries $p_{12} = p_{34} = 0$, $p_{23} = p_{14}$, $p_{24} = (p_{13} - p_{14}(\alpha-\delta))/\gamma$ and $p_{13}^2 - p_{14}^2\gamma + p_{13}p_{14}(\delta-\alpha) = 0$. The solvability of this equation is equivalent to that of $t^2 + (\delta-\alpha)t - \gamma = 0$. This leads to 3 different cases.

• The equation $t^2 + (\delta-\alpha)t - \gamma = 0$ has a unique root when $\gamma = -1$ and $\delta - \alpha = \pm 2$. That is to say, (I) If $\delta = \alpha + 2$, then $t = -1$. (II) If $\alpha = \delta + 2$, then $t = 1$. These two situations lead to same result, here we consider case (I) only.

Suppose that $\gamma = -1$, $\delta = \alpha + 2$. A simultaneous conjugation by the element $\omega_{23}x_{21}(-1)x_{43}(-1)$ leads to the following generators.

$$x_1(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{(\alpha+1)(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{(\alpha+1)(\varepsilon-1)}{\varepsilon}\right),$$
$$y_1(\eta) = d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).$$

For $\varepsilon \neq 1$, we put

$$q_1(\varepsilon) = z_{x_1}\left(\varepsilon, \frac{\alpha+2}{\alpha+1}, \frac{\alpha+1}{(\varepsilon-1)\varepsilon}\right), \quad q_2(\eta) = z_{y_1}\left(\frac{\alpha+1}{\alpha+2}, \alpha+2, \eta\right),$$

and for $\varepsilon \neq \eta$, we have

$$q_1(\varepsilon,\eta) = [q_1(\varepsilon), q_1(\eta)] = x_{13}\left(g_1(\varepsilon,\eta)\right) x_{24}\left(-g_1(\varepsilon,\eta)\right) x_{14}\left(g_2(\varepsilon,\eta)\right),$$
$$q_2(\varepsilon,\eta) = [q_2(\varepsilon), q_2(\eta)] = x_{13}\left(g_3(\varepsilon,\eta)\right) x_{24}\left(-g_3(\varepsilon,\eta)\right) x_{23}\left(g_4(\varepsilon,\eta)\right),$$

where $g_i(\varepsilon,\eta), i = 1,2,3,4$ are rational functions of ε and η.

In particular, if $\alpha \neq -11/6$, we have

$$\left[q_2(\varepsilon), q_2\left(\frac{(2\alpha+3)\varepsilon}{6\alpha\varepsilon - 2\alpha + 11\varepsilon - 3}\right)\right] = x_{13}\left(g_5(\varepsilon)\right) x_{24}\left(-g_5(\varepsilon)\right),$$

where $g_5(\varepsilon)$ is rational function, which has pole at $\varepsilon = \frac{3+2\alpha}{11+6\alpha}$.

Let $\alpha = -11/6$. For $\varepsilon \neq -7/6, 7/3, -7, 1$, we have

$$\left[q_2(\varepsilon), q_2\left(\frac{7\varepsilon}{6\varepsilon - 7}\right)\right] = x_{13}(-g_6(\varepsilon)) x_{24}(g_6(\varepsilon)),$$

where $g_6(\varepsilon)$ is rational function, which has pole at $\varepsilon = -7/6$. So we extract the subgroup $X_{13,24}^{-1}$. And then multiplying $q_1(\varepsilon,\eta)$, $q_2(\varepsilon,\eta)$ by suitable elements from $X_{13,24}^{-1}$, we can get the subgroup X_{23} and X_{14}.

Now, from the decomposition of $x_1(\varepsilon)$ we have element $m(\varepsilon)$,

$$m(\varepsilon) = d_1(\varepsilon) d_3(\varepsilon) x_{12}\left(\frac{(1+\alpha)(\varepsilon-1)}{\varepsilon}\right) x_{34}\left(\frac{(1+\alpha)(\varepsilon-1)}{\varepsilon}\right).$$

From the lemma 2(3), we get $\langle m(\varepsilon), y_1(\eta)\rangle = \mathrm{GL}(2,K)$. Thus we conclude that $H = \mathrm{GL}(2,K).X_{14}X_{23}X_{13,24}^{-1} = \mathrm{GL}(2,K).K^3$.

• The equation $t^2 + (\delta - \alpha)t - \gamma = 0$ has two roots t_1 and t_2, i.e. $\alpha^2 + 4\gamma - 2\alpha\delta + \delta^2 \in K^{*2}$. Then the subgroup H has two 2-dimensional invarian subspaces in V. Simultaneous conjugation by z_1^{-1} yields the following generators.

$$x_2(\varepsilon) = d_1(\varepsilon) d_3(\varepsilon) x_{12}\left(\frac{c(\varepsilon-1)}{\varepsilon}\right) x_{34}\left(\frac{(\alpha+\delta-c)(\varepsilon-1)}{\varepsilon}\right),$$
$$y_2(\eta) = d_1(\eta) d_3(\eta) x_{21}(\eta-1) x_{43}(\eta-1),$$

where $c = \frac{-\gamma + t_1(\delta+t_2) - \alpha t_2}{t_1 - t_2}$. From the lemma 2(1), we get $H = \langle x_2(\varepsilon), y_2(\eta)\rangle = (\mathrm{SL}(2,K) \times \mathrm{SL}(2,K)) \rtimes K^*$.

• The equation $t^2 + (\delta - \alpha)t - \gamma = 0$ has no roots in K, i.e. $\alpha^2 + 4\gamma - 2\alpha\delta + \delta^2$ does not belong to K. We consider the extension $K' = K(\sqrt{\alpha^2 + 4\gamma - 2\alpha\delta + \delta^2})$ and apply the same argument as in above case.

(8) For the base (r4s4a), $\alpha = -1$, $\gamma = 0$, $\delta = -1$. Conjugating this base by the element w_{23} of the Weyl group, the generators have the following form.

$$x_1(\varepsilon) = d_1(\varepsilon) d_3(\varepsilon) x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right) x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right) x_{34}\left(\frac{1-\varepsilon}{\varepsilon}\right),$$
$$y_1(\eta) = d_1(\eta) d_3(\eta) x_{21}(\eta-1) x_{43}(\eta-1).$$

Then for $\varepsilon \neq 2$, we have

$$[z(2,\varepsilon), z(2,1-\varepsilon)] = x_{14}\left(\frac{1}{2} - \varepsilon\right) x_{23}\left(\frac{1}{2} - \varepsilon\right),$$

$$z(2,\varepsilon) z(2,2-\varepsilon) = x_{14}\left(\frac{(\varepsilon-1)^3}{4\varepsilon}\right) x_{23}\left(-\frac{(\varepsilon-1)^2\left(\varepsilon^2+\varepsilon+2\right)}{4(\varepsilon-2)\varepsilon}\right) x_{13}(-\varepsilon') x_{24}(\varepsilon'),$$

where $\varepsilon' = \frac{(\varepsilon-1)^2(\varepsilon+1)}{4\varepsilon}$. For $\eta \neq -1, 3$, we calculate the following commutators,

$$[z(2,\eta), z(2,3-\eta)] = x_{13}(\theta_7)x_{23}(2\theta_7)x_{24}(-\theta_7),$$
$$[z(2,\eta), z(2,-1-\eta)] = x_{13}(\theta_8)x_{14}(-2\theta_8)x_{24}(-\theta_8),$$

where $\theta_7 = \frac{-2\eta^3+9\eta^2-13\eta+6}{2(\eta-3)\eta}$, $\theta_8 = \frac{2\eta^3+3\eta^2-3\eta-2}{2\eta(\eta+1)}$. Multiplying $z(2,\varepsilon)z(2,2-\varepsilon)$ by suitable elements of $X_{14,23}$, $X^{2,-1}_{23,24,13}$ and $X^{-2,-1}_{14,24,13}$, we have that the subgroups X_{14} and X_{23} are contained in H, then we consecutively extract the subgroup $X^{-1}_{13,24}$.

From the decomposition of $x(\varepsilon)$ we take the element $m(\varepsilon)$.

$$m(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{1-\varepsilon}{\varepsilon}\right),$$

From the lemma 2(4) we get $\langle m(\varepsilon), y_1(\eta)\rangle \simeq \mathrm{B}(2,K)$. Finally, we conclude that $H = \mathrm{B}(2,K).X_{14}X_{23}X^{-1}_{13,24} = \mathrm{B}(2,K).K^3$.

(9) For the base (r4s4a), $\alpha = -1$, $\gamma = 0$, $\delta \neq 0, -1$. Conjugating this base by the element w_{23} of the Weyl group, the generators have the following form.

$$\begin{aligned} x_1(\varepsilon) =& d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{\delta(\varepsilon-1)}{\varepsilon}\right),\\ y_1(\eta) =& d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1). \end{aligned}$$

If $\delta \neq -1/2$, for $\varepsilon, \eta \in K^*$, $\varepsilon \neq \eta$, we have

$$\begin{aligned} \left[z\left(\varepsilon, \frac{\delta+1}{\delta}\right)z\left(\eta, \frac{\delta+1}{\delta}\right)\right] =& x_{14}\left(\frac{(2\delta+1)(\varepsilon-1)(\eta-1)(\varepsilon-\eta)}{\delta+1}\right)\\ & x_{34}\left((2\delta+1)(\varepsilon-1)(\eta-1)(\varepsilon-\eta)\right). \end{aligned}$$

If $\delta = -1/2$, for $\varepsilon \in K^*/\{-1\}$, we have

$$\begin{aligned} & z_{x_1}\left(\frac{(\varepsilon-1)^2}{(\varepsilon+1)^2}, \varepsilon, 2\right)z_{x_1}\left(\frac{(\varepsilon-1)^2}{(\varepsilon+1)^2}, \varepsilon, \frac{1}{2}\right)z_{x_1}\left(\frac{1-\varepsilon}{\varepsilon+1}, -1, 2\right)\\ &= x_{14}\left(\frac{(\varepsilon-1)\left(\varepsilon^2+1\right)}{2(\varepsilon+1)^3}\right)x_{34}\left(\frac{(\varepsilon-1)\left(\varepsilon^2+1\right)}{4(\varepsilon+1)^3}\right). \end{aligned}$$

In all cases it follows that $X^{\delta+1}_{34,14}$is contained in H. Then we have

$$m(\varepsilon) = x_1(\varepsilon)x_{14}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{(\delta+1)(1-\varepsilon)}{\varepsilon}\right) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{1-\varepsilon}{\varepsilon}\right).$$

From the decomposition of $x_1(\varepsilon)$, we also get X_{14}. So applying lemma 2(4), we conclude that $H = \mathrm{B}(2,K).X_{14} = \mathrm{B}(2,K).K$.

(10) For the base (r4s4a), $\alpha = 0$, $\gamma = 1$, $\delta = 0$. The generators have the following forms.

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{\varepsilon-1}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta-1)x_{42}(\eta-1).$$

P satisfying (c) has the entries $p_{12} = p_{34} = 0$, $p_{23} = p_{14}$, $p_{24} = p_{13}$ and $p_{13}^2 - p_{14}^2 = 0$. The solvability of this equation is equivalent to that of $t^2 - 1 = 0$. This equation has two roots $t_1 = 1$ and $t_2 = -1$. Then the subgroup H has two 2-dimensional invariant subspaces in V. Simultaneous conjugation by z_1^{-1} yields the following generators.

$$x_1(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right), \quad y_1(\eta) = d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1),$$

From the lemma 2(2), we conclude that $H = \langle x_1(\varepsilon), y_1(\eta)\rangle = (\mathrm{U}(2,K)\times \mathrm{SL}(2,K)) \rtimes (K^*\times K^*)$.

(11) For the base (r4s4a), $\alpha = 0$, $\gamma \neq 0, 1$, $\delta \neq 0, -1$, $\gamma = \delta + 1$. The generators have the following form.

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{14}\left(\frac{\varepsilon - 1}{\varepsilon}\right)x_{23}\left(\frac{\gamma(\varepsilon-1)}{\varepsilon}\right)x_{24}\left(\frac{(\gamma-1)(\varepsilon-1)}{\varepsilon}\right),$$
$$y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta-1)x_{42}(\eta-1).$$

P satisfying (c) has the entries $p_{12} = p_{34} = 0$, $p_{23} = p_{14}$, $p_{24} = (p_{13} + p_{14}(\gamma - 1))/\gamma$ and $(p_{13} - p_{14})^2(p_{13} + p_{14}\gamma)^2 = 0$. The solvability of this equation is equivalent to that of $(t-1)^2(t+\gamma)^2 = 0$. The equation $(t-1)^2(t+\gamma)^2 = 0$ has two roots $t_1 = 1$ and $t_2 = -\gamma$. Then the subgroup H has two 2-dimensional invariant subspaces in V. Simultaneous conjugation by z_1^{-1} yields the following generators.

$$x_1(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{\gamma(\varepsilon-1)}{\varepsilon}\right),$$
$$y_1(\eta) = d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1),$$

From the lemma 2, if $\gamma = -1$, then $H = \langle x_1(\varepsilon), y_1(\eta)\rangle = \mathrm{B}(2,K)$. If $\gamma \neq -1$, then $H = \langle x_1(\varepsilon), y_1(\eta)\rangle = (\mathrm{U}(2,K)\times \mathrm{SL}(2,K)) \rtimes (K^*\times K^*)$.

(12) For the base (r4s4a), $\alpha \neq 0, -1$, $\gamma \neq 0$, $\delta \neq 0, -1$, $\gamma \neq \alpha\delta$, $\gamma = (\alpha+1)(\delta+1)$. The generators have the following form.

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{\gamma(\varepsilon-1)}{\varepsilon}\right)x_{24}\left(\frac{\delta(\varepsilon-1)}{\varepsilon}\right),$$
$$y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta-1)x_{42}(\eta-1).$$

P satisfying (c) has the entries $p_{12} = p_{34} = 0$, $p_{23} = p_{14}$, $p_{13} = -p_{14}(\delta-\alpha) + p_{24}(\alpha+1)(\delta+1)$ and $((p_{14} - p_{24}(\alpha+1))^2(p_{14} + p_{24} + p_{24}\delta)^2 = 0$. The solvability of this equation is equivalent to that of $(t-(\alpha+1))^2(t+\delta+1)^2 = 0$. This equation has two roots $t_1 = \alpha+1$ and $t_2 = -\delta-1$. Then the subgroup H has two 2-dimensional invariant subspaces in V. Simultaneous conjugation by z_1^{-1} yields the following generators.

$$x_1(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{(\alpha+\delta+1)(\varepsilon-1)}{\varepsilon}\right),$$
$$y_1(\eta) = d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1),$$

From the lemma 2, if $\alpha+\delta+1 = -1$, then $H = \langle x_1(\varepsilon), y_1(\eta)\rangle = \mathrm{B}(2,K)$. If $\alpha+\delta+1 \neq -1$, then $H = \langle x_1(\varepsilon), y_1(\eta)\rangle = (\mathrm{U}(2,K)\times \mathrm{SL}(2,K)) \rtimes (K^*\times K^*)$.

(13) For the base (r4s4b), $\alpha \neq 0, -1$, $\delta \neq 0, -1$. Conjugating this base by the element w_{23} of the Weyl group, the generators have the following form.

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{\delta(\varepsilon-1)}{\varepsilon}\right),$$
$$y(\eta) = d_1(\eta)d_3(\eta)x_{21}(\eta-1)x_{43}(\eta-1).$$

From the lemma 2, if $\alpha \neq \delta$, then $H = (\mathrm{SL}(2,K)\times \mathrm{SL}(2,K)) \rtimes K^*$. If $\alpha = \delta$, then $H = \mathrm{GL}(2,K)$.

(14) For the base (r4s4b), $\alpha = -1$, $\delta \neq 0, -1$. Due to the lemma 2(2), we conclude that $H = (\mathrm{U}(2,K)\times \mathrm{SL}(2,K)) \rtimes (K^*\times K^*)$.

(15) For the base (r4s4b), $\alpha = -1$, $\delta = -1$. Due to the lemma 2(4), we conclude that $H = \mathrm{B}(2,K)$.

□

Theorem 3. *Let X, Y be a pair of 2-tori in* $\mathrm{GL}(4,K)$*. Assume that $r = s = 4$, one of p and q is 2 or $p = q = 1$, then X and Y generate one of the following subgroup H, listed in Table* 3.

Таблица 3

	base	**H**
1	(p2q0a), $\alpha = \delta = 0$, $\gamma \neq 0$	$Q_{12}Q_{34}X^{\gamma}_{23,14}$
2	(p2q0a) $\alpha, \delta \neq 0$, $\gamma = 0$	$Q_{12}Q_{34}X^{\alpha,\delta}_{13,24,14}$
3	(p2q0a) $\alpha, \gamma \neq 0$, $\delta = 0$	$Q_{12}Q_{34}X^{\alpha,\gamma}_{13,23,14}$
4	(p2q0a) $\delta, \gamma \neq 0$, $\alpha = 0$	$Q_{12}Q_{34}X^{\gamma,\delta}_{23,24,14}$
5	(p2q0a) $\alpha, \delta, \gamma \neq 0$, $\alpha\delta \neq \gamma$	$Q_{12}Q_{34}X^{\alpha,\gamma,\delta}_{13,23,24,14}$
6	(p2q1a) $\alpha = 0$, $\delta = 0$ (p1q1c)	$Q_{12}Q_{34}X_{14}$
7	(p2q1a) $\alpha = 0$, $\delta \neq 0$	$Q_{12}Q_{34}X^{\delta}_{24,14}$
8	(p2q1a) $\alpha \neq 0$, $\delta = 0$	$Q_{12}Q_{34}X^{\alpha}_{13,14}$
9	(p2q1a) $\alpha \neq 0$, $\delta \neq 0$	$Q_{12}Q_{34}X^{\alpha,\alpha\delta,\delta}_{13,23,24,14}$
10	(p2q2a)	$Q_{12}Q_{34}$

Proof. We observe the bases (p2q0a), (p2q1a) and (p2q2a) in lemma 1, the group $\langle X, Y\rangle$ is embedded in a group of upper triangular matrices. We consider commutator subgroup $z(\varepsilon,\eta)$ generated by $x(\varepsilon)$ and $y(\eta)$. Moreover, from the decomposition of $x(\varepsilon)$and $y(\eta)$, we obtain their spans of X and Y.

For the base (p1q1c), a simultaneous conjugation by $w_{34}x_{31}(-1)w_{13}w_{34}w_{23}$ leads to the following generators.

$$x(\varepsilon) = d_3(\varepsilon)d_4(\varepsilon), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{14}(\eta - 1).$$

Performing the straightforward calculation, we obtain

$$z(\varepsilon,\eta) = x_{14}\left(\frac{-(\varepsilon-1)(\eta-1)}{\varepsilon}\right).$$

It follows that the subgroups Q_{12}, Q_{34} and X_{14} are contained in H, thus we conclude that $H = Q_{12}Q_{34}X_{14}$.

□

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