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О СЛЕДАХ И ДИСТАНЦИЯХ В АНАЛИТИЧЕСКИХ ФУНКЦИОНАЛЬНЫХ ПРОСТРАНСТВАХ В C^n И ИНТЕГРАЛАХ МАРТИНЕЛЛИ – БОХНЕРА

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Аннотация

В этой работе мы приводим аналоги наших многочисленных результатов о следах и дистанциях в аналитических функциональных пространствах в C^n, полученных ранее, в терминах интегралов и ядер Мартинелли – Бохнера. Это первые результаты такого типа в терминах этих интегралов и ядер. Также нами будут обсуждаться некоторые новые утверждения для интегралов типа Мартинелли – Бохнера, связанные с классами типа Гельдера и точками Лебега.

В последние годы в большом цикле работ первого автора был получен ряд новых точных результатов, связанных со следами и расстояниями в различных функциональных пространствах. Во всех этих работах важную роль играют свойства ядер типа Бергмана и интегральные представления типа Бергмана. В этой статье мы получим некоторые аналоги этих результатов в терминах более общих интегральных представлений и более общих ядер в аналитических функциональных пространствах большей размерности. Это так называемое интегральное представление Мартинелли – Бохнера и ядра Мартинелли – Бохнера в C^n.

Наша работа состоит из трех частей. В первой части мы обобщаем полученные ранее результаты по следам. Во второй части мы получаем оценки функции расстояния в терминах ядер Мартинелли – Бохнера и интегралов Мартинелли – Бохнера. В третьей части представлены результаты для интегралов Мартинелли – Бохнера, связанные с классами Гельдера и точками Лебега. Эти вопросы естественно возникают из недавней серии работ первого автора о многофункциональных аналитических пространствах и связанными с ними вопросами.

Наши доказательтсва модифицируют методы и рассуждения известных ранее результатов и теорем для случая интегралов и ядер типа Мартинелли – Бохнера.

Ключевые слова: Интегралы и ядра Мартинелли – Бохнера, аналитическая функция, следы, дистанции, псевдовыпуклые области.

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TRACES AND DISTANCES IN ANALYTIC FUNCTION SPACES IN C^n AND MARTINELLY — BOCHNER INTEGRALS

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Abstract

In this note we provide some analogues of our numerous recent results on traces and distances in terms of Martinelly — Bochner integrals and kernels. These are first results of this type in terms of such kernels. Some assertions for Martinelly — Bochner integrals related with Holder classes and Lebegues points will be also discussed.

In recent years various new sharp results on traces and distances were provided in a big series of papers of the first author. In all these papers properties of Bergman-type kernels and Bergman-type integral representations are playing a critical role. The intension of this paper to find some analogues of these results in terms of or with the help of more general integral representations and more general kernels in analytic function spaces in higher dimension so-called Martinelly — Bochner integral representations and Martinelly — Bochner kernels in C^n.

Our work consists of three parts. In the first part we partially generalize our results on traces. In the second part we provide estimates of distance function in terms of Martinelly — Bochner kernels and Martinelly — Bochner integrals. In the third part we present results on Martinelly — Bochner integrals related with Holder classes and Lebegues points. This type of issues arise naturally in view of recent series of papers and new results of the first author on multifunctional analytic spaces and related issues.

In our proofs we modify the methods of earlier results and theorems for the case of Martinelli-Bochner integrals and kernels.

Keywords: Martinelly — Bochner integrals and kernels, analytic function, traces, distances, pseudoconvex domains.

Bibliography: 20 titles.

Introduction

In recent years various new sharp results on traces and distances were provided in a big series of papers of the first author (see [4], [9], [10], [11], [12], [13]). In all these papers properties of Bergman type kernels and Bergman-type integral representations are

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playing a critical role. The intension of this paper to find some analogues of these results in terms of or with the help of more general integral representations and more general kernels in analytic function spaces in higher dimension so-called Martinelly — Bochner integral representations and Martinelly — Bochner kernels in C^n (see [5]). Some new results on Martinelly — Bochner integrals related with the Lebegues points and Holder functional classes will be also presented.

1. Traces in pseudoconvex domains and Martinelly — Bochner integrals.

In this section we partially generalize our results on traces. The goal of this section is to provide a class of analytic functions on products of bounded strictly pseudoconvex domains $D \times \cdots \times D$ so that their traces allows Matrinelly-Bochner integral representation. To be more precise we first need standard definitions (see [5] and references there).

We define the unit ball in C^n by B. Let $\tilde{U}$ be an open set in $\mathbb{C}^n$, we say $f \in C^k(\tilde{U})$ if f is complex valued function and $f^{(k)} \in C^0(\tilde{U}) = C(U)$. We denote by $H(\tilde{U})$ the set of all analytic in $\tilde{U}$ functions. $\tilde{U}^m = \tilde{U} \times \cdots \times \tilde{U}$, $m \geq 1$ is a product domain and $H(\tilde{U}^m)$ is a space of all analytic function on $(\tilde{U}^m)$ (see [4], [9], [10], [11], [3]) or D^m. Using definitions of analytic function spaces in a domain, we define analytic function spaces in product domains in a natural way (see [10]).

We denote by D in $\mathbb{C}^n$ a region with ∂D boundary of C^k class in some neighborhood of closure of D and $d\rho \neq 0$ on ∂D. The ρ function we call as usual the defining function of D (see [10]). We consider an outer differential form (Bochner-Martinelly kernel) $U(\zeta, z)$ of $(n, n-1)$ type of the following form (see [5])

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} d\bar{\zeta}[K] \wedge d\zeta$$

where $d\bar{\zeta}[K] = d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{k-1} \wedge d\bar{\zeta}_{k+1} \wedge \cdots \wedge d\bar{\zeta}_m$, $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$. For $n = 1$ we have Causchy kernel $U(\zeta, z) = \frac{1}{2\pi i}\left(\frac{d\zeta}{\zeta - z}\right)$. Obviously coefficients of U are harmonic in $C^n \backslash \{z\}$ and $d_\zeta U(\zeta, z)$=0.

Moreover (see [5]) if $g(\zeta, z)$ is a fundamental solution of Laplace equation then we have

$$U(\zeta, z) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial g}{\partial \zeta_k} d\bar{\zeta}[K] \wedge d\zeta = (-1)^{n-1} \partial_\zeta g \wedge \sum_{k=1}^{n} d\bar{\zeta}[K] \wedge d\zeta[K]$$

where $\partial = \sum_{k=1}^{n} \partial\zeta_k \frac{\partial}{\partial \zeta_k}$.

We denote various constants in estimates in this paper by C or by C with indexes, these constants are independent of functions in estimates.

Theorem 1 (**A.** (see [5]).). *Let D be a bounded region in $\mathbb{C}^n$ with partially-smooth boundary and let f be holomorphic in D of $C(\bar{D})$ class then*

$$\int_{\partial D} f(\zeta) U(\zeta, z) = \begin{cases} f(z), \ z \in D \\ 0, \ z \ z \notin \bar{D}. \end{cases} \tag{1}$$

This is a Martinelly — Bochner integral representation.

This theorem was obtained by Bochner then by Bochner then by Martinelly independently and by different methods. This formula is now classical and can be seen in various textbooks on complex analysis. Note for $n = 1$ we have classical Causchy formula, but for $n > 1$ we can easily see that the $U(\zeta, z)$ is not analytic by z or ζ. We will need the mentioned formula for Hardy spaces. We need some definitions. Below we omit the index of space if it is zero. Let D be a bounded region, $\partial D \in C^{1+\alpha}$, $\alpha > 0$. We say that analytic function f belongs to Hardy space $H^p(D)$, $p > 0$ if

$$\sup_{\varepsilon>0} \int_{\partial D} |f(\zeta - \varepsilon\nu(\zeta))|^p d\sigma(\zeta) < \infty,$$

where $d\sigma$ is a Lebegues measure on ∂D. And $\nu(\zeta)$ vector field of outer normals to ∂D (see [5]). For bounded pseudoconvex D domains we have another definition, for a defining function ρ, $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$; let $D_\varepsilon = \{z \in D : \rho(z) < -\varepsilon\}$ for $\varepsilon > 0$. Then

$$H^p(D) = \{f \in H(D) : \sup_{\varepsilon>0} \int_{\partial D_\varepsilon} |f(\zeta)|^p d\sigma_\varepsilon < \infty\};$$

$0 < p \leq \infty$.

We define the same space in product domains in a natural way as we define Hardy spaces in polydisk using definitions in the unit disk see [3], [4].

PROPOSITION 1 (**B.** (see [5]).). *Let D be bounded strongly pseudoconvex domain with smooth boundary. For all $p \geq 1$ and all f, $f \in H^p(D)$ Martinelly – Bochner representation is valid.*

Proof of proposition B.

Note first (see [5]) our proposition is valid for all bounded domains with $\partial D \in C^{1+\alpha}$, $\alpha > 0$. For each f, $f \in H^p$, $p \geq 1$, f has boundary values almost everywhere on ∂D so that these values are in $L^p(\partial D)$. This is a classical fact together with representation via boundary values

$$f(z) = \int_{\partial D} f(\zeta) P(\zeta, z) d\sigma,$$

where P is Poisson kernel of D domain.

Note the G Green function admits representation $G(\zeta, z) = g(\zeta, z) + h(\zeta, z)$, where g is a solution of Laplace equation and where h for all fixed $z \in D$ is a harmonic function in D of $C^{1+\alpha}(\bar{D})$ class hence

$$P(\zeta, z) d\sigma = U(\zeta, z)/\partial D + \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial h}{\partial \zeta_k d\bar{\zeta}} [K] \wedge d\zeta/\partial D = \Phi_1 + \Phi_2$$

but note

$$\int_{\partial D} (f(\zeta)) \Phi_2(\zeta) = \int_{\partial D} f(\zeta) \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial h}{\partial \zeta_k} d\bar{\zeta}[K] \wedge d\zeta =$$
$$= \int_D f(\zeta) d \left(\sum_{k=1}^{n} (-1)^{k-1} \frac{\partial h}{d\zeta_k} d\bar{\zeta}[K] \wedge d\zeta \right) = 0.$$

This follows directly from the fact that

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{\partial h}{d\zeta_k}d\bar{\zeta}[K]\wedge d\zeta$$

is a closed form (see [5]). And this gives directly what we need for each f, $f \in H^p(D)$, $p \geq 1$ Martially-Bochner represention is valid. □

The intention of this section to show such type results for other spaces in unit ball or general bounded strictly pseudoconvex domains D, based on our previous results and on embeddings from [1].

The natural question is the following:

Let X be a space of analytic functions, $X \subset H(D^m)$. Let also

$$Tr\ X = \{f(z,\dots,z), f \in X\}.$$

Can we say there is a function g_0, $g_0 \in Tr\ X$ so that it admits Martinelly — Bochner integral representation (1). Below we provide a general scheme of a solution for case of unit ball and bounded domains.

We define for a $\tilde{D}^\alpha$ — differential operator in pseudoconvex domain D (see [1]), $0 < p, q < \infty$ Hardy-Sobolev and Bergman–Besov spaces of analytic functions

$$H^p_{\alpha,\beta}(D) = \left\{ f \in H(D) : \sup_{\varepsilon>0}\left(\int_{\partial D_\varepsilon}|\tilde{D}^\alpha f(\zeta)|^p d\sigma_\zeta\right)^{\frac{1}{p}}\varepsilon^\beta < \infty \right\},\ \beta \geq 0,\ \alpha > 0.$$

$$A^{p,q}_{\delta,k}(D) = \left\{ f \in H(D) : \sum_{|\alpha|\leq k}\int_0^{r_0}\left(\int_{\partial D_r}|\tilde{D}^\alpha f(\zeta)|^p d\sigma_\zeta\right)^{\frac{q}{p}} r^{\delta\frac{q}{p}-1}dr < \infty \right\},\ \begin{matrix}\delta > 0,\\ \alpha > 0\end{matrix}.$$

We define same spaces in product domains in a natural way as we did in polydisk using definitions of spaces in the unit disk see [2], [3], [4].

We give a typical result in this direction on traces. Note for case of unit ball we have (see [2], [3], [4]).

Theorem 2 (**C.**).

1. *Let $p \leq 1$, $\beta \geq \alpha \geq 0$, then $A^{p,p}_{nm+(\beta-\alpha)pm-n}(B) \subset Trace\ H^p_{\alpha,\beta}(B^m)$;*
2. *Let $p \leq 1$; $\alpha_j \geq 0$, $t \geq \frac{\sum_{j=1}^m \alpha_j}{m}$ then $A^{p,p}_{mt+mn-n-\sum_{j=1}^m \alpha_j}(B) \subset Trace\ (A^{p,p}_{t,\vec{\beta}}(B^m))$; $\beta_j = \alpha_j + 1$, $j = 1,\dots,m$.*

Obviously we have also (see [1], [3], [4]) the following embeddings ($0 < p \leq \infty$, $p \geq q$ and s_1, $s_2 \geq 0$).

$$A^p_{p(s_2-s_1),s_2}(B) \subset H^p_{s_1}(B) \subset H^p(B) \subset H^q(B)$$

then we have ($A^{p,p} = A^p$, $H^p_0 = H^p$)

$$A^p_{\alpha,\beta}(B) \subset A^p_{\alpha,0}(B) = A^p_\alpha(B),\ \beta \geq 0.$$

Now according to proposition B for any analytic f function, $f \in H^p(B)$, $1 \le p < \infty$ the Martinelly — Bochner integral representation is valid. This in combination with embeddings above gives

THEOREM 3.

1. *Let $p \le 1$, $\alpha \ge \beta$ then $Trace H^p_{\alpha,\beta}(B^m)$ contains a f function for which*

$$f(z) = \int_{\partial D} f(\zeta) U(\zeta, z); \; z \in D \tag{M}$$

2. *Let $p \le 1$, $\alpha_j \ge 0$, $\beta_j = \alpha_j + 1$, $t \ge \frac{\sum \alpha_j}{m}$, $j = 1, \dots, m$ then $Trace\ A^p_{t,\vec{\beta}}(B^m)$ contain a f function for which (M) integral representation is valid.*

REMARK 1. *Note previously this assertion was known for Bergman representation and Bergman kernels.*

The case of general D domains (pseudoconvex) can be considered similarly. It is based on our recent results from [11] and [12] and similar embeddings for pseudoconvex domains from [1]. Analytic Herz type spaces can be considered similarly (see [13]).

We need some definitions to formulate a theorem we need. Let $K(w)$ be entire function of one variable $w = u + iv$, $K : R \to R$, $K(u) \ne 0$ for all $u \in C$. Let

$$c_n K(Im\ z_n)\Phi(\zeta, z) = \frac{\partial^{n-2}}{\partial s^{n-2}} Im \frac{K(i\alpha + Im\ z_n)}{\alpha(i\alpha + Im\ (\zeta_n - z_n))},$$

$s = \sum_{k=1}^{n-1} |\zeta_k - z_k|^2$, $\alpha^2 = s + Re(\zeta_n - z_n)$, $c_n = (-1)^{n-1}(2\pi i)^n$.

THEOREM 4 (**D1.** (see [5]).). *Let $n > 1$ then $\Phi(\zeta, z) = g(\zeta, z) + h(\zeta, z)$ where g is fundamental solution of Laplace equation, $h(\zeta, z)$ is harmonic by ζ in C^n for all fixed z.*

We need an extension of Martinelly — Bochner integral to unbounded domains D in C^n then for tubular domains the same problem can be posed and solved based on our recent results on traced in Bergman type spaces in tubular domains over symmetric cones with smooth boundary.

If $f \in H(D) \cap C(\bar{D})$ and if $h \in C'(\bar{D})$ and h is harmonic by ζ for all $z \in D$. Then $\bar{\mu}_{\bar{h}}$ form is closed in $\bar{D}$

$$\bar{\mu}_{\bar{h}} = \sum_{k=1}^{m} (-1)^{n+k-1} \frac{\partial h}{\partial \zeta_k} d\bar{\zeta}[K] \wedge d\zeta$$

and

$$\int_{\partial D} f(\zeta) \bar{\mu}_{\bar{h}}(\zeta) = 0, \; f \in H(D) \cap C(\bar{D}).$$

Note in bounded domain then we have (see [5])

$$f(z) = \int_{\partial D} f(\zeta)[U(\zeta, z) + \bar{\mu}_{\bar{h}}(\zeta)], \; z \in D \tag{G_1}$$

with coefficients $(K \times K)$ matrices $x, t \in R^m$.

Remark 2. *Note if* $K \equiv 1$ *then* $\Phi = g$.

Theorem 5 (**D2.** (see [5]).). *If* D *is unbounded domain in* C^n, $n > 1$ *with smooth boundary,* $f \in H(D) \cap C(\bar{D})$ *where* $B(0, R)$ *is a ball of radius* R *then if*

$$\lim_{R\to\infty} \left(\int_{\partial D \setminus B(0,R)} f(\zeta)\Omega(\zeta, z) \right) = 0 \qquad (K_0)$$

for all fixed $z \in D$ *then*

$$f(z) = \int_{\partial D} f(\zeta)\Omega(\zeta, z) \qquad (K)$$

where

$$\Omega(\zeta, z) = \sum_{k=1}^{n} (-1)^{k-1} \left(\frac{\partial \Phi}{\partial \zeta_k} \right) d\zeta[\bar{k}] \wedge d\zeta.$$

Proof.

Just use G_1 for domain $\tilde{D} \cap B(0, R)$ then pass $R \to \infty$.

To formulate our theorem 6 we will need basic facts of theory of analytic function spaces in tubular domains over symmetric cones (see [20]).

A subset Ω of $\mathbb{R}^n$ or V, so that $dim\ V = n$ to be a cone if $\lambda x \in \Omega$, for all $x \in \Omega$, $\lambda > 0$, if $\lambda x + \mu y \in \Omega$ for all $x, y \in \Omega$, $\lambda, \mu > 0$ then it is convex. Let in addition $\Omega^* = \{y \in \mathbb{R}^n : (y/x) > 0, \forall x \in \bar{\Omega}\backslash\{0\}\}$ and $\Omega^* = \Omega$. This type open cone is selfdual (Ω^* is dual cone).

Let $G(\Omega) = \{g \in Gl(\mathbb{R}^n) : g\Omega = \Omega\}$, where $Gl(\mathbb{R}^n)$ denotes the group of all linear invertible transformation of $\mathbb{R}^n$. If for all $x, y \in \Omega$, $y = gx$, for some $g \in G(\Omega)$ then our open convex cone Ω is homogeneous, if also $\Omega^* = \Omega$ then it is symmetric cone.

If the equation $\Omega = \Omega_1 + \Omega_2$ is not possible for each $V_1 \subset \mathbb{R}^n$, $V_2 \subset \mathbb{R}^n$, then our cone is irreducible, here $V_i \neq \varnothing$, $i = 1, 2$ (Ω_1, Ω_2 are symmetric cones), where also $\Omega_i \subset V_i$, $i = 1, 2$.

We need also determinant $\Delta^t(Im\ z)$, $z \in \mathbb{C}^n$, $t \in (0, \infty)$. We fix V — a simple Euclidean Jordan algebra with rank r.

If $x \in V$, $m(x) = \min\{k > 0 : (l, x, x^2, \dots, x^k)$ are linearly dependent$\}$, then $1 \leq m(x) \leq dim\ V$ and $r = \max\{m(x) : x \in V\}$, we say rank of V is r.

According to spectral theorem if V has rank r, then $x = \sum_{i=1}^{r} \lambda_i c_i$; $\lambda_i \in R$; c_i — are elements of so called Jourdan frame, and $\{\lambda_i\}$ are determined uniquely by x (with their multiplicities). We fix now a Peirce decomposition of $V = \oplus_{1\leq i\leq j\leq r} V_{ij}$; (we formally look at V as a space of symmetric matrices (V_{ij}), $V_{ii} = Rc_i$, where R is a special mapping, $V_{ij} = V(c_i, 1/2) \cap V(c_i, 1/2) = \{x \in V : c_i x = c_j x = \frac{x}{2}\}$, $i < j$, $dim\ V_{ij} = d = 2\frac{n/r-1}{r-1}$). We denote by P_{ij} the orthogonal projection of V onto V_{ij} for $i \leq j$. Finally we denote by $\Delta_j(x)$, $j = 1, \dots, r$, the principal minors of $x \in V$ with respect to the fixed Jordan frame $\{c_1, \dots, c_r\}$. That is $\Delta_k(x)$ is the determinant of the projection $P_k x$ of x in the Jordan subalgebra $V^{(k)} = \oplus_{1\leq i\leq j\leq r} V_{ij}$.

It is well-known that $\Omega = \{x \in V : \Delta_k(x) > 0 : k = 1, \dots, r\}$. We have also $\Delta_k(mx) = \Delta_k(x)$, $x \in V$, $m \in Z_+$, $m > 0$. See other properties of Δ_k in [20].

We define $\Delta_s(x) = \prod_{j=1}^{r} \Delta_j^{s_j - s_j + 1}(x) = \Delta_1^{s_1 - s_2}(x) \dots \Delta_r^{S_r}(x)$, $x \in \Omega$, $s \in C^r$. We have that $|\Delta_s| = \Delta(Im\ z)$ and $\Delta_s \sum_{i=1}^{r} a_i c_i = \prod_{i=1}^{r} a_i^{s_i}$; $a_i > 0$, $i = 1, \dots, r$.

Let Ω be an irreducible symmetric cone in the Euclidean space V, and $T_\Omega = V + i\Omega$ the corresponding tube domain in the complexified space $V^{\mathbb{C}}$, $T_\Omega^m = T_\Omega \times \dots T_\Omega$.

As combination of this theorem D2 and theorems on traces [16] in tube we have

THEOREM 6. *Let T_Ω be tubular domain over symmetric cone, then let*

$$A_\alpha^p(T_\Omega^m) = \left\{ f \in H(T_\Omega^m) : \int_{T_\Omega} \dots \int_{T_\Omega} |f(z_1, \dots, z_m)|^p \prod_{j=1}^m \Delta^\alpha(Im\ z_j) dv(z_j) < \infty \right\},$$

$0 < p < \infty$, $\alpha > -1$. *Then* $Trace A_\alpha^p = \{f(z, \dots, z),\ f \in A_\alpha^p(T_\Omega^m)\}$ *contain a function, so that representation (K) is valid if (K_0) holds.*

Similar results can be obtained based on integral representation based on second Green formula (see [5]) and our recent results on traces of Bergman harmonic function spaces in the unit ball (see [17]) $B = \{|x| < 1\} \subset R^n$ and R_+^{n+1}.

We give more detail on this integral representation.

Let in R^m, $m \geq 1$ us consider a differential operator in R^m $A = \sum_{j=1}^m A_j \frac{\partial}{\partial x_j}$, where A_j are matrices of order $l \times K$ on C, $S_m = \{|x| = 1, x \in R^m\}$ if

$$A_i^* A_j + A_j^* A_i = \begin{cases} 0 & \text{if } i \neq j \\ 2I_k & \text{if } i = j. \end{cases}$$

where $i, j \in [1, n]$, $A_i^* = \bar{A}_i^T$ complex conjugate of A_i, I_k is unit matrix. Then A is Dirac operator. For solutions of Dirac operator we have analogue of Martinelly — Bochner theorem

For definition of vector function space in theorem E we refer the reader to [5].

THEOREM 7 (**E.** (see [5]).). *Let A be Dirac operator, $D \subset R^n$ is a bounded domain with smooth boundary then if $f \in [C(\bar{D}) \cap C^1(D)]^k$, $Af = 0$ then*

$$\int_{\partial D} U_A(t, x) f(t) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{if } x \notin \bar{D}. \end{cases}$$

where U_A is a special Martinelly — Bochner type kernel in R^m see [5] a differential form type of $(n-1)$ type

$$U_A(t, x) = \frac{1}{Vol(S_m)} \sum_{j=1}^m \sum_{i=1}^m (-1)^{i-1} A_j^* A_i \frac{t_j - x_j}{|t - x|^m} dt[i].$$

Details of proofs of these assertions analogues of previous theorems 3, 6 for harmonic function spaces based on theorem E we leave to readers (see [12], [13]).

2. Distance functions and Martinelly — Bochner integrals

The goal of this section to provide estimates of distance function in terms of Martinelly — Bochner kernels and Martinelly — Bochner integrals based on methods and results of [9], [10] and [11] and our earlier work cited there.

Note previously we obtained such type theorems in various spaces of analytic and harmonic functions of one and several variables in various domains in terms of Bergman kernel and Bergman-type projections (see [9], [10], [11] and various references there). The line of our proof is rather similar however to those simpler cases but it is based on some results on more general kernels as Martinelly — Bochner kernels and Martinelly — Bochner integrals (see [5] and references there).

Let D be bounded domain in C^n. X, Y be quazinormed subspaces of $H(D)$, where $H(D)$ is the space of all analytic functions in D. Let $X \subset Y$, let $f \in Y$. We search for those estimates of a function $dist_Y(f, X)$ which generalize previously known such type estimates obtained via similar Bergman type kernels.

We have the following result in this direction. First if $X \subset H^p$, $p \geq 1$ then the problem of estimates of $dist_{H^p}(f, X)$, $f \in H^p$ appears naturally. In our previous papers to role of Bergman type kernels for this problem was critical.

Let $H^p \subset X$, $X \subset H(D^m)$, then let $f \in X$. We also can consider a problem of finding estimates of function $dist_X(f, H^p)$, $1 \leq p < \infty$, $H^p = H^p(D^m)$ on product domains in terms of Martinelly — Bochner kernels or integrals.

We have following results following arguments of proofs of our previous papers [10], [11], [14], [15].

Let $X \subset H^p(D^m)$, $1 \leq p < \infty$, $m \in N$, we assume X is a quazinormed subspace of $H(D^m)$. To estimate $dist_{H^p}(f, X)$, $f \in H^p$, $1 \leq p < \infty$ we follow our arguments from [14] in the unit disk $\{|z| < 1\}$ and unit ball.

Since $f \in H^p(D^m)$, then applying Martinelly — Bochner representation by each variable

$$f(z_1, \ldots, z_m) = \int_{(\partial D)^m} f(\vec{\zeta}) U(\vec{\zeta}, \vec{z}) = \int_{(\partial D)} \cdots \int_{\partial D} f(\zeta_1, \ldots, \zeta_m) \prod_{j=1}^{m} U(\zeta_j, z_j);$$

$z_j \in D$, $j = 1, \ldots, m$ where

$$H^p(D^m) = \{f \in H(D^m) : \sup_{R>0} \int_{\partial D} \cdots \int_{\partial D} |f(\vec{\zeta}_R)|^p d\sigma < \infty\},\ 1 \leq p < \infty$$

is a Hardy space on products of D domains, $m \in N$, D is bounded in C^n, $\partial D \in C^{1+\alpha}$, $\alpha > 0$ (note these are Banach spaces) and where $\vec{\zeta}_R = (\zeta_1 - R\nu(\zeta_1), \ldots, \zeta_m - R\nu(\zeta_m))$, ν is a vector field of outer unit normals to ∂D.

Let $X^{\Psi}_{\varepsilon,f} = \{\zeta \in \partial(D^m) : \Psi(\zeta)|f(\zeta)| \geq \varepsilon\}$; $\Psi \in C^1(\partial(D^m))$ fixed. Then $f = f_1 + f_2$

$$f_1(z) = \int_{X^{\Psi}_{\varepsilon,f}} f(\zeta) \prod_{j=1}^{m} U(\zeta_j, z); \qquad f_2(z) = \int_{\partial(D^m) \setminus X^{\Psi}_{\varepsilon,f}} f(\zeta) \prod_{j=1}^{m} U(\zeta_j, z).$$

Assuming $f_2 \in H^p$ and it is norm is less than ε, we get

$$dist_{H^p}(f, X) \le c \inf \left\{ \varepsilon > 0 : \| \int_{X^{\Psi}_{\varepsilon,f}} f(\zeta) \prod_{j=1}^{m} U(\zeta_j, z) \|_{X(D^m)} < \infty \right\}.$$

Since $dist_{H^p}(f, X) \le c\|f - f_1\|_{H^p} = \|f_2\|_{H^p}$.

Theorem 8. *Let $p \ge 1$. Let $\Psi \in C^1(\partial(D^m))$, $\Psi \ge 0$, be fixed function, let $f \in H^p(D^m)$, $\int_{\partial(D^m)\setminus X^{\Psi}_{\varepsilon,f}} f(\zeta) \prod_{j=1}^{m} U(\zeta_j, z) \in H^p$ and it is norm is less than ε. Then*

$$dist_{H^p}(f, Y) \le c \inf \left\{ \varepsilon > 0 : \left\| \int_{X^{\Psi}_{\varepsilon,f}} f(\vec{\zeta}) \prod_{j=1}^{m} U(\zeta_j, z) \right\|_{Y(D^m)} < \infty \right\}$$

where

$$X^{\Psi}_{\varepsilon,f} = \{\zeta \in \partial(D^m) : \Psi(\zeta)|f(\zeta)| \ge \varepsilon\},$$

where $Y \subset H^p$ is a quazinormed subspace of $H^p(D^m)$ for any bounded D domain with $\partial D \in C^{\alpha+1}$, $\alpha > 0$ boundary.

Remark 3. *Similar results are valid for Martinelly — Bochner type integrals and harmonic function spaces in R^{n+1}_{+} and unit ball in R^m. See [17] where similar problems where solved via Bergman kernel.*

Theorem 8 is a typical assertion, other assertions of such type in terms of $U(z, \zeta)$ kernels can be also formulated similarly.

3. Some remarks on Martinelly – Bochner integrals related with Holder classes and Lebegues points.

The goal of this section is among other things from [5] related with expressions like $|\Phi_f(z_1) - \Phi_f(z_2)|$ to the case of two functional similar type expressions like $|\Phi_f(z_1) - \Phi_g(z_2)|$, where Φ is a certain operator, f, g are certain functional class members, z_1, $z_2 \in D$, where D is a fixed domain in C^n.

This type of issues arise naturally in view of recent series of papers and new results of the first author on multifunctional analytic spaces and related issues (see [8], [17] and various references there).

To obtain this natural tranfsormation of known results from one functional case to two functional case we follow the proof of [5] and at the same time modify proofs from there to our case. First we provide certain equalities (A), (B), (C), (D), which can be checked directly, then use them in our proofs. We alert the reader that in those places where arguments are close or the same as in [5], we omit details making the exposition of this note rather concise we note also that our problems we considered below is an attempt to provide natural extensions of known one functional results to two functional case.

Let

$$(Mf)(z^+) = \int_{\partial D} f(\zeta)U(\zeta, z^+);$$
$$(Mf)(z^-) = \int_{\partial D} f(\zeta)U(\zeta, z^-);$$

then we have following equalities

$$(Mf)(z^+) - (Mg)(z^-) - f(z) - g(z) = \int_{\partial D}(f(\zeta) - f(z))(U(\zeta, z^+) - U(\zeta, z^-))-$$

$$- \int_{\partial D} g(\zeta)U(\zeta, z^-) - \int_{\partial D} g(z)(U(\zeta, z^+) - U(\zeta, z^-)) + \int_{\partial D} f(\zeta)U(\zeta, z^-) = \qquad (A)$$
$$= \int_{\partial D}(f(\zeta) - f(z))(U(\zeta, z^+) - U(\zeta, z^-)) + M(f-g)(z^-) - g(z);$$

$$(Mf)(\tilde{z}^+) - (Mg)(\tilde{z}^-) = \int_{\partial D}(f(\zeta) - f(z^0))U(\zeta, \tilde{z}^+)-$$
$$- \int_{\partial D}(f(\zeta)-f(z^0))U(\zeta, \tilde{z}^-) + f(z^0)\int_{\partial D}(U(\zeta, \tilde{z}^+) - U(\zeta, \tilde{z}^-)) - \int_{\partial D} g(\zeta)U(\zeta, \tilde{z}^-)+$$
$$(B)$$
$$+ \int_{\partial D} f(\zeta)U(\zeta, \tilde{z}^-) = \int_{\partial D}(f(\zeta) - f(z^0))U(\zeta, \tilde{z}^+) - \int_{\partial D}(f(\zeta) - f(z^0))U(\zeta, \tilde{z}^-)+$$
$$+f(z^0)\int_{\partial D}(U(\zeta, \tilde{z}^+) - U(\zeta, \tilde{z}^-)) + M(f-g)(\tilde{z}^-);$$

we omit technical calculations leaving them to readers. Note all equalities are based on simple properties of Martinelly — Bochner integrals (see [5]).

It can be checked directly that

$$\int_{\partial D}(f(\zeta) - f(0))U(\zeta, z) - \int_{\partial D\setminus B(0,\varepsilon)}(g(\zeta) - g(0))U(\zeta, 0) =$$
$$= \int_{\partial D\setminus B(0,\varepsilon)}(f(\zeta) - f(0))(U(\zeta, z) - U(\zeta, 0)) + \int_{\partial D\cap B(0,\varepsilon)}(f(\zeta) - f(0))U(\zeta, z)-$$
$$- \int_{\partial D\setminus B(0,\varepsilon)}(g(\zeta) - g(0))U(\zeta, 0) + \int_{\partial D\setminus B(0,\varepsilon)}(f(\zeta) - f(0))U(\zeta, 0) =$$

$$= \int_{\partial D\setminus B(0,\varepsilon)}(f(\zeta) - f(0))(U(\zeta, z) - U(\zeta, 0))+ \qquad (C)$$

$$+ \int_{\partial D\cap B(0,\varepsilon)}(f(\zeta) - f(0))U(\zeta, z) + \int_{\partial D\setminus B(0,\varepsilon)}((f-g)(\zeta) - (f-g)(0))U(\zeta, 0);$$

$$- \int_{\partial D\setminus\sigma_\delta}(f(\zeta) - f(z^1))U(\zeta, z^1) + \int_{\partial D\setminus\sigma_\delta}(f(\zeta) - g(z^2))U(\zeta, z^2) =$$

$$= \int_{\partial D \setminus \sigma_\delta} (g(\zeta) - g(z^2))(U(\zeta, z^2) - U(\zeta, z^1)) + (f(z^1) - f(z^2)) \int_{\partial D \setminus \sigma_\delta} U(\zeta, z^1) + \qquad (D)$$

$$+ \int_{\partial D \setminus \sigma_\delta} (g(\zeta) - g(z^2))U(\zeta, z^1) - \int_{\partial D \setminus \sigma_\delta} f(\zeta)U(\zeta, z^1) + f(z^2) \int_{\partial D \setminus \sigma_\delta} U(\zeta, z^1) =$$

$$= \int_{\partial D \setminus \sigma_\delta} (g(\zeta) - g(z^2))(U(\zeta, z^2) - U(\zeta, z^1)) +$$

$$+ (f(z^1) - f(z^2)) \int_{\partial D \setminus \sigma_\delta} U(\zeta, z^1) + A_{f,g} + B_{f,g} + C_{f,g} =$$

$$= \int_{\partial D \setminus \sigma_\delta} (g(\zeta) - g(z^2))(U(\zeta, z^2) - U(\zeta, z^1)) + (f(z^1) - f(z^2)) \int_{\partial D \setminus \sigma_\delta} U(\zeta, z^1) +$$

$$+ \int_{\partial D \setminus \sigma_\delta} ((g - f)(\zeta) - (g - f)(z^2))U(\zeta, z^1);$$

where σ_δ is arc on ∂D and $B(0, \varepsilon)$ is a standard ball in a domain.

REMARK 4. *For those cases when $g = f$ all equalities (A), (B), (C), (D) can be seen in [5] (see lemma 5.3, theorem 6.1, 6.2).*

Let Ω be a domain, we say f is Holder class α if $|f(\zeta) - f(z)| \le c|\zeta - z|^\alpha$, $\zeta, z \in \Omega$, $\alpha > 0$. We consider bounded domains with smooth boundary, $f \in L^1(\partial D)$, $D = \{z \in C^n, \rho(z) < 0\}$, $\rho \in C^1(C^n)$, $d\rho \neq 0$ on ∂D. Let $V(\partial D)$ be a neighborhood of ∂D. We assume f can be extended to $V(\partial D)$. Consider integrals

$$\Phi_f(z) = \int_{\partial D} (f(\zeta) - f(z))U(\zeta, z).$$

Note if $z \notin \partial D$ then the integral has no singularity, if $z \in \partial D$, then

$$|f(\zeta) - f(z)| \times |U(\zeta, z)| \le c|\zeta - z|^{\alpha + 1 - 2n} d\sigma(\zeta) \qquad (2)$$

if f is α Holder class function. The natural question is let f is of Holder class α in $V(\partial D)$ then can we say $\Phi_f(z)$ is of same class α. The answer is positive (see [5]).

The next more general question is if f, g are of Holder class α then what can we say on $|\Phi_f(z_1) - \Phi_g(z_2)|$, $z_1, z_2 \in V(\partial D)$. We will use formula (D) for our proof. But first note that the following simple observation is valid.

Let $z^1, z^2 \in V(\partial D)$, $|z^1 - z^2| = \delta$, δ is small, if $B(z^1, 2\delta) \subset V(\partial D)$, $\sigma_\delta = \partial D \cap B(z^1, 2\delta)$. Then (see [5])

$$\left| \int_{\sigma_\delta} (f(\zeta) - f(z^j)U(\zeta, z^j) \right| \le c_1 \int_{\sigma_\delta} |\zeta - z^j|^{1 + \alpha - 2n} d\sigma \le c_2 \delta^\alpha, j = 1, 2.$$

This follows from (2) directly and the fact that σ_δ is smooth. So for $f = g$ case it is remains to estimate integrals by $D \setminus \sigma_\delta$ to get estimate for $|\Phi_f(z_1) - \Phi_f(z_2)|$. But for $|\Phi_f(z_1) - \Phi_g(z_2)|$ we have formula (D) and estimates

$$A_f = \left| \int_{\sigma_\delta} (f(\zeta) - f(z^1))U(\zeta, z^1) \right| \le \tilde{c}_1 \delta^\alpha, \qquad 0 < \alpha < 1$$

$$B_g = |\int_{\sigma_\delta} (g(\zeta) - g(z^2)U(\zeta, z^2)| \leq \tilde{\tilde{c}}_1 \delta^\alpha, \qquad 0 < \alpha < 1.$$

Since obviously we have

$$|\Phi_f(z_1) - \Phi_g(z_2)| =$$

$$= \left| \int_{\partial D} (f(\zeta) - f(z_1))U(\zeta, z_1) - \int_{\partial D} (g(\zeta) - g(z_2))U(\zeta, z_2) \right| \leq A_f + B_g + C_{f,g};$$

$$|C_{f,g} = \left| \int_{\partial D \setminus \sigma_\delta} (f(\zeta) - f(z_1))U(\zeta, z_1) - \int_{\partial D \setminus \sigma_\delta} (g(\zeta) - g(z_2))U(\zeta, z_2) \right|;$$

Using (D) we have now that $c_{f,g} \leq c_g^1 + c_f^2 + c_g^3 + c_f^4 + c_f^5$. Note $c_g^3 + c_f^4 + c_f^5 = \int_{\partial D \setminus \sigma_\delta} ((g - f)(\zeta) - (g - f)(z^2);\ U(\zeta, z^1) = \Phi_{f,g}(z_1, z_2)$.

Remark 5. *Note we also have*

$$|\Phi_{f,g}(z_1, z_2)| \leq |\Phi_{g-f}(z_2)| + \tilde{c}^1_{g-f},$$

$$\tilde{c}^1_{g-f} = \int_{\partial D \setminus \sigma_\delta} ((g - f)(\zeta) - (g - f)(z^2))(U(\zeta, z^1) - U(\zeta, z^2)).$$

So we have that

$$|\Phi_f(z_1) - \Phi_g(z_2)| \leq c_g^1 + \tilde{c}^1_{g-f} + c_f^2 + c\delta^\alpha + \tilde{c}|\Phi_{g-f}(z_2)|.$$

We will stop on estimate with $\Phi_{g,f}$. We have $c_f^2 \leq c\delta^\alpha$ since $|\int_{\partial D \setminus \sigma_\delta} (U(\zeta, z^1))| \leq 1$, see [5]. We estimate c_g^1 and $\tilde{c}^1_{g-f}$. This is based on simple estimates of $U(\zeta, z^2) - U(\zeta, z^1)$, $z_1, z_2 \in V(\partial D)$.

We have that (see [5])

$$\left| \int_{\partial D \setminus \sigma_\delta} (g(\zeta) - g(z^2))(U(\zeta, z^2) - U(\zeta, z_1)) \right| \leq \tilde{c}\delta \int_{\partial D \setminus \sigma_\delta} |\zeta - z^1|^{\alpha - 2n} d\sigma \leq \tilde{c}(\delta^\alpha),$$

$$\left| \int_{\partial D \setminus \sigma_\delta} ((f - g)(\zeta) - (g - f)(z^2))(U(\zeta, z^1) - U(\zeta, z^2)) \right| \leq$$

$$\leq \tilde{\tilde{c}}\delta \int_{\partial D \setminus \sigma_\delta} |\zeta - z^1|^{\alpha - 2n} d\sigma \leq \tilde{\tilde{c}}(\delta^\alpha).$$

Since $\int_{\partial D \setminus \sigma_\delta} |\zeta - z^1|^{\alpha - 2n} d\sigma \leq \bar{c}\delta^{\alpha - 1}$ (see [5]).

So finally we have an estimate

$$|\Phi_f(z_1) - \Phi_g(z_2)| \leq c\delta^\alpha + |\Phi_{g,f}(z_1, z_2)|,\ 0 < \alpha < 1$$

and

$$|\Phi_f(z_1) - \Phi_g(z_2)| - |\Phi_{g,f}(z_1, z_2)| \leq c\delta^\alpha.$$

Note if $f \equiv g$ this obtained in [5] and [6].

We now formulate the final result.

Theorem 9. *Let f, g be of Holder class α, $0<\alpha<1$ in $V(\partial D)$. Then in $V(\partial D)$ we have the following estimate*

$$|\Phi_f(z_1)-\Phi_g(z_2)|\le c\delta^\alpha+|\Phi_{g,f}(z_1,z_2)|;\ z_1,z_2\in V(\partial D),$$

$|z^1-z^2|=\delta$, $\delta<\delta_0$, *for some fixed* δ_0, $\delta_0>0$.

We assume $Mf(z)=\int_{\partial D} f(\zeta)U(\zeta,z)$, $z\notin\partial D$, $f\in C^1(\bar{D})$ below.

Now we use (A), (B), (C) to obtain such type results for other mentioned assertions from [5]. Note to get the proof we must follow one functional proof from [5] modify it to use mentioned equalities (A), (B), (C).

Let D be bounded domain with C^1 boundary and $f\in L^1(\partial D)$ and $z^0\in\partial D$ and V_{z^0} be a cone with z^0 as peak (vertex), with conic angle equal or less to $\pi/2$. Let $z\in D\cap V_{z^0}$. We say z^0 is a Lebegues point for f if (see [5])

$$\lim_{\varepsilon\to+0}\varepsilon^{1-2n}\int_{\partial D\cap B(z^0,\varepsilon)}|f(\zeta)-f(z^0)|d\sigma=0.$$

We will be using (C) and arguments from proof of theorem 5.2 [5]. We note first (see [5])

$$\left|\int_{\partial D\cap B(0,\varepsilon)}|f(\zeta)-f(0)|U(\zeta,z)\right|\le K\int_{\partial D\cap B(0,\varepsilon)}|f(\zeta)-f(0)|d\sigma\to 0;$$
$$K=\frac{c_2}{\varepsilon^{2n-1}}\ as\ \varepsilon\to 0.$$

This and (C) lead to theorem.

So based on (C) we have the following.

Theorem 10. *Let $z\in D\cap V_{z^0}$ be Lebegues point for f and g, $f,g\in L^1(\partial D)$. Then*

$$\lim_{\substack{z\to z^0\\ z\in V_{z^0}}}\int_{\partial D}(f(\zeta)-f(z^0))U(\zeta,z)-\int_{\partial D\setminus B(z^0,|z-z^0|)}(g(\zeta)-g(z^0)U(\zeta,z^0))=$$

$$=\lim_{\substack{z\to z^0\\ z\in V_{z^0}}}\int_{\partial D\setminus B(z^0,|z-z^0|)}((f-g)(\zeta)-(f-g)(z^0))U(\zeta,z^0).$$

Using (A) and (B) and following the proof of theorem 6.1, 6.2 (see [5]). We can have assertions concerning

$$\lim_{z\pm\to z^0}Mf(z^+)-Mg(z^-)-f(z^0)-g(z^0),$$

where, $f,g\in L^1(\partial D)$, z^0 is a Lebegues point of f and g simultaneously and

$$\int_{\partial D}|Mf(\tilde{z}^+)-Mg(\tilde{z}^-)-f(z)-g(z)|^p d\sigma$$

if D is bounded, $\partial D\in C^1$, $f\in L^p(\partial D)$, $p\ge 1$, $\nu(\zeta)$ is a unit vector of outer normal to ∂D on ζ, $\tilde{z}^+=z-\varepsilon\nu(z)$, $\tilde{z}^-=z+\varepsilon\nu(z)$. We omit details here.

So based on (B) we have the following

Theorem 11. *Let z^0 be Lebegues point of f and g, $f \in L^1(\partial D)$, $g \in L^1(\partial D)$, $z^+ \in V_{z_0} \cap D$, $z^- = V_{z_0} \cap (C^n \backslash \bar{D})$, $a|z^+ - z^0| \leq |z^- - z^0| \leq b|z^+ - z^0|$, $a, b \in R$, $a \leq b$, $z_0 \in \partial D$, D is bounded, $\partial D \in C^1$, z^0 is a peak of cone V_{z_0}, cone angle is $\beta < \pi/2$. Then*

$$\lim_{z\pm \to z^0} (Mf(z^+) - Mg(z^-)) = f(z_0) + \lim_{z\pm \to z^0} (M(f-g)(z^-)).$$

Remark 6. *For $f = g$ this is a theorem from [5].*

Based on (A) we have the following theorem 12.

Theorem 12. *Let D be a bounded domain with smooth boundary. Let $f, g \in L^p(\partial D)$, $p \geq 1$. Let $\tilde{z}^+$, $\tilde{z}^-$ belong to D and $C^n \backslash \bar{D}$ for ε small enough. Then*

$$A_{f,g} = \int_{\partial D} |Mf(z - \varepsilon\nu(z)) - Mg(z + \varepsilon\nu(z))|^p d\sigma \leq$$
$$\leq c \int_{\partial D} |f|^p d\sigma + \int_{\partial D} |M(f-g)(z^-)|^p d\sigma;$$
$$\lim_{\varepsilon \to +0} \int_{\partial D} |Mf(z - \varepsilon\nu(z)) - Mg(z + \varepsilon\nu(z)) - f(z) - g(z)|^p d\sigma \leq$$
$$\leq \lim_{\varepsilon \to 0} \|M(f-g)(z^-)\|_{L^p} + \|g\|_{L^p}.$$

Remark 7. *At the end of this paper we find interesting to suggest another idea related from one hand to Martinelly—Bochner integrals from the other hand to our recent work on traces in analytic functions Bergman spaces. In [7] the role of expanded Bergman projection based on expaned Bergman kernel was crucial for solution of certain problems related with traces of anlytic functions in polyballs. It will be nice to study their analogues expanded Martinelly—Bochner kernels and based on them expanded Martinelly—Bochner integrals based in particular on the following simple idea.*

Let as usual $g(\zeta, z) = c_n \frac{1}{|\zeta - z|^{2n-2}}$, $n > 1$, ρ be defining function of $D \subset C^n$, $D = \{\rho < 0\}$, $d\rho \neq 0$ on ∂D, $\rho \in C^1$.

$P_{k,s}$ be a complete orthonormal system of homogeneous harmonic polynomials in $L^2(S)$, * is well-known Hodze operator for differential forms see [5] $\sigma(\zeta) = \sum\limits_{k=1}^{n} (-1)^{k-1} \bar{\zeta}_k d\bar{\zeta}[k] \wedge d\zeta$.

We suggest to study kernels of type on product domains (expanded Martinelly — Bochner kernels)

$$U(\zeta, z_1, \ldots, z_m) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial}{\partial \zeta_k} \prod_{j>1}^{m} g(\zeta, z_j) d\bar{\zeta}[K] \wedge d\zeta$$

$$\tilde{U}(\zeta, z_1, \ldots, z_m) = \frac{(n-1)!}{\pi^n} \sum_{k=1}^{n} \frac{\partial \rho}{\partial \zeta}_k \prod_{j=1}^{m} \frac{(\bar{\zeta}_k - \bar{z}_j^k)}{|\zeta - z_j|} d\sigma$$

$$\bar{U}(\zeta, z_1, \ldots, z_m) = \left(- \sum_{k,s} \frac{P_{k,s}(z_1)}{n+k-1} \left[*\partial \frac{\overline{P_{k,s}(\zeta)}}{|\zeta|^{2n+2k-2}} \right] \right)$$

$$\cdots\left(-\sum_{k,s}\frac{P_{k,s}(z_m)}{n+k-1}\left[*\partial\frac{\overline{P_{k,s}(\zeta)}}{|\zeta|^{2n+2k-2}}\right]\right)$$

Note for $m=1$ these kernels coincide with classical Martinenelly—Bochner integrals (see [5]). Same idea used for Bergman kernel in series of papers of first author (see [7] and references there).

$$B(z_1,\ldots,z_m,w)=\prod_{j=1}^{m}\frac{(1-|z_j|)^{\alpha_j}}{(1-\bar{z}_jw)^{\alpha_j+2}},\ \alpha_j>-1,\ z_j,w\in D,\ j=1,\ldots,m.$$

These types of extensions of Bergman kernel was considered by author in [7].

4. Conclusion.

In our paper we got some analogues of our numerous recent results on traces and distances in terms of Martinelly — Bochner integrals and kernels. These are first results of this type in terms of such kernels. We also discussed some assertions for Martinelly — Bochner integrals related with Holder classes and Lebegues points.

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