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Явные законы взаимности в теории локальных полей

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Аннотация

В этой обзорной статье рассматриваются различные подходы к явному описанию символа Гильберта и их обобщения на p-адические представления в рамках p-адической теории Ходжа.

Ключевые слова: локальное поле, символ Гильберта, p-адическая теория Ходжа.

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Explicit reciprocity laws in the theory of local fields

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Abstract

This paper surveys different approaches to explicit formulas for the Hilbert symbol and their generalizations to p-adic representations in terms of p-adic Hodge theory.

Keywords: local field, Hilbert symbol, p-adic Hodge theory.

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Dedicated to the memory of Sergei Vladimirovich Vostokov

1. Introduction

An explicit computation of the Hilbert symbol is a classical problem in Number Theory, closely related to the search of global explicit reciprocity laws: vast generalizations of the famous quadratic reciprocity law to higher power residue symbols. Artin, Hasse, Shafarevich and Iwasawa proved important results in this direction, and a complete solution was independently obtained by Brückner and Vostokov in the late 1970s [18, 90].

About the same time, inspired by the papers of Iwasawa [54, 55], Coates and Wiles used an analog of the Hilbert symbol in the theory of elliptic curves to prove first results toward the Birch and Swinnerton-Dyer conjecture [21]. The universality of their approach became clear ten years later, thanks to a remarkable progress in two different branches of Number Theory: the work of Kolyvagin on Euler systems [63, 64] and the development of p-adic Hodge theory in a series of pioneering papers of Fontaine [35, 36, 37, 38]. Using p-adic Hodge theory, Bloch and Kato [15] extended the construction of the exponential map in the theory of p-adic Lie groups to a wide class of p-adic Galois representations. Its dual map $\exp^*$ is a morphism from the first continuous Galois cohomology $H^1(G_K, V)$ of a p-adic local field K with coefficients in a p-adic representation V to the de Rham module $\mathbf{D}_{\mathrm{dR}}(V)$ attached to V by the theory of Fontaine. It is expected that this map plays the role of a bridge between fundamental arithmetic objects (Selmer groups) of global Galois representations, and the underlying L-functions. This conception was stressed in Kato's paper [59]. Very roughly, it can be explained by the following diagram:

$$\left\{\begin{matrix}\text{localizations of cohomology classes}\\ \text{provided by Euler systems}\end{matrix}\right\} \xrightarrow{\exp^*} \left\{\begin{matrix}\text{special values}\\ \text{of } L\text{-functions}\end{matrix}\right\}.$$

In all cases where such a relationship is known, it is obtained as a result of two computations: an explicit formula for the special L-value, proved using analytical techniques, and an explicit description of the dual exponential map. In this picture, explicit formulas for the Hilbert symbol can be interpreted as an explicit description of the dual exponential map for the trivial Galois representation (or, equivalently, as an explicit description of the exponential map of the dual representation $\mathbf{Q}_p(1)$.) This observation and its relationship with general Bloch–Kato conjectures on special values of L-functions, gave a new impetus to the subject. The explicit reciprocity law proved by Bloch and Kato [15] describes the dual exponential map of Tate twists $\mathbf{Q}_p(m)$ and plays a key role in the arithmetic of the Riemann zeta function [15, 8, 19, 52].

The present paper can be seen as an introduction to local aspects of this programme. It delineates the hundred years history of explicit formulas in the theory of local fields. Our goal was not to provide an exhaustive guide to the current literature on the subject, but to give an overview of some ideas and methods. In that sense, this text is not complete. Some important results, as Kato's reciprocity law for modular curves and applications of (φ, Γ)-modules to local Iwasawa theory, deserve a more detailed separate survey. Also, explicit formulas for higher dimensional fields are only mentioned. The interested reader can refer to Venjakob's survey [89] for some additional information.

The author would like to thank I. B. Fesenko for his remarks on the first version of this survey.

This paper is dedicated to the memory of Sergei Vladimirovich Vostokov, whose papers on explicit formulas have greatly contributed to the development of the above theory.

2. Classical reciprocity laws

2.1. The story of explicit reciprocity laws starts with the famous quadratic reciprocity law conjectured by Euler and Legendre and proved in 1801 by Gauss:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}, \qquad \text{where } p \text{ and } q \text{ are distinct odd primes.}$$

Its generlization to higher power residue symbols was one of major themes in Number Theory in the 19th century. The most remarkable achievements during this period are the cubic and biquadratic reciprocity laws (Gauss, Eisenstein, Jacobi) and the reciprocity laws of Eisenstein and Kummer for p-cyclotomic fields $\mathbf{Q}(\zeta_p)$. Excellent expositions of these results can be found in the books of Ireland–Rosen [53], Lemmermayer [68], and in Weil's lecture at the Bourbaki Seminar [99].

Hilbert probably was the first who realized that explicit reciprocity laws reflect the local-global principle in class field theory. According to this conception, developed by him in the quadratic case for general number fields, the product of power residue symbols $\left(\frac{a}{b}\right)_n \left(\frac{b}{a}\right)_n^{-1}$ can be expressed as the product of local terms, the so-called norm residue symbols, now also known as Hilbert symbols. In his famous lecture at the ICM in Paris (1900), Hilbert formulated the problem of explicit reciprocity laws as Problem 9 in the following form:

For any field of numbers the law of reciprocity is to be proved for the residues of the ℓ-th power, when ℓ denotes an odd prime, and further when ℓ is a power of 2 or a power of an odd prime.

The development of the class field theory by Takagi, Artin and Hasse in the 1910s and 1920s included this problem in the general theory of abelian extensions of number fields. The central result of the class field theory is the construction of the so-called reciprocity map

$$\theta_F \,:\, C_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

from the idele class group C_F of a number field F to the Galois group of its maximal abelian extension. According to the local–global principle, which was taking root in class field theory after the works of Hasse, Schmidt and Chevalley in 1930s, this morphism can be constructed as the product of local reciprocity maps

$$\theta_{F_v} \,:\, F_v^* \to \mathrm{Gal}(F_v^{\mathrm{ab}}/F_v),$$

where v runs over all places (Archimedean and non-Archimedean) of the field F. In particular, we have the product formula

$$\prod_v \theta_{F_v}(a) = 1, \qquad a \in F^*, \tag{1}$$

which is a distant descendent of the product formula for normalized absolute values:

$$\prod_v \|a\|_v = 1, \qquad a \in F^*. \tag{2}$$

Assume now that F contains the group μ_n of nth roots of the unity for some fixed $n \geqslant 2$. The Hilbert symbol on F_v is the bilinear pairing

$$(\,\cdot\,,\,\cdot\,)_{v,n} \,:\, F_v^* \times F_v^* \to \mu_n$$

defined by the formula

$$(a,b)_{v,n} = \frac{\left(\sqrt[n]{b}\right)^{\theta_{F_v}(a)}}{\sqrt[n]{b}}.$$

Then (1) implies the general reciprocity law for the product of Hilbert symbols:

$$\prod_v (a,b)_{v,n} = 1, \qquad a,b \in F^*, \tag{3}$$

which also can be written as a general reciprocity law for nth power residue symbols:

$$\left(\frac{a}{b}\right)_n \left(\frac{b}{a}\right)_n^{-1} = \prod_{v\in S} (a,b)_{v,n}, \qquad \text{for all coprime } a,b \in F^*, \tag{4}$$

where S includes all Archimedean places and all places dividing n.

The reciprocity law (4) reduces the computation of the product of power residue symbols to the purely local problem of explicit computation of Hilbert symbols. For example, in the case of $F = \mathbf{Q}$ and odd integers $a,b > 0$, we have $S = \{2,\infty\}$, $(a,b)_\infty = 1$ and

$$(a,b)_2 = (-1)^{\frac{a-1}{2}\frac{b-1}{2}},$$

which leads to the quadratic reciprocity law. In general, the computation of $(a,b)_{v,n}$ is a highly nontrivial problem, because from the construction of the reciprocity map it is not a priori clear how the automorphism $\theta_{F_v}(a)$ acts on $\sqrt[n]{b}$. In some sense, the Hilbert symbol establishes a connection between class field theory and Kummer theory of local fields.

2.2. Hilbert observed that formula (2) has a direct analog in the theory of analytic functions. Namely, let f be a nonzero meromorphic function on a compact Riemann surface X, and let $v_x(f)$ denote the order of f at a point $x \in X$. Then the classical formula

$$\sum_{x\in X} v_x(f) = 0$$

can be written in the equivalent form $\prod\limits_{x\in X} \|f\|_x = 1$, where $\|\cdot\|_x$ denotes the normalized absolute value attached to the point x. This analogy between number fields and Riemann surfaces, emphasised by Hilbert, suggests that the reciprocity law (3) should be seen as an analog of Cauchy's integral formula or, rather, of the residue formula for abelian differentials:

$$\sum_{x\in X} \operatorname{res}_x (f dg) = 0. \tag{5}$$

From this point of view, the Hilbert symbol $(a,b)_{v,n}$ is a direct analog of the residue $\operatorname{res}_x (f dg)$.

This analogy can be pushed forward if we consider the function field F of an algebraic curve over a finite field k of characteristic p. The analog of Kummer theory is Artin–Schreier–Witt theory, which we recall below. Let $W_n(F)$ denote the ring of truncated p-typical Witt vectors with coefficients in F; it is equipped with the natural Frobenius endomorphism $\varphi(x_0, x_1, \dots) = (x_0^p, x_1^p, \dots)$, and we set $\wp(x) = \varphi(x) - x$ for each $x \in W_n(F)$. Artin–Schreier–Witt theory describes Galois extensions of F generated by the roots of equations

$$\wp(X) = b, \qquad a \in W_n(F),$$

which are analogs of Kummer equations $X^{p^n} = b$ in characteristic p. The analog of the p^nth Hilbert symbol is the Artin–Schreier–Witt symbol:

$$(\ ,\ ]_n \,:\, F_v^* \times W_n(F_v) \to \mathbf{Z}/p^n\mathbf{Z}, \qquad (a,b]_n = \wp^{-1}(b)^{\theta_{F_v}(a)} - \wp^{-1}(b).$$

This pairing was studied by Witt, who proved the formula

$$(a,b]_n = \operatorname{Tr}_{W(k)/\mathbf{Z}_p}\bigl(\operatorname{res}_{\pi_v}(d\log(\widehat{a})\, b)\bigr), \tag{6}$$

where $\widehat{a}$ is some lifting of a in $W(F_v)$, which can be defined in terms of the Artin–Hasse–Shafarevich exponential (see below), π_v is a fixed uniformizer of F_v, and $\mathrm{Tr}_{W(k)/\mathbf{Z}_p}$ denotes the trace map [102]. The case $n = 1$ was treated before by Schmid; in that case, Witt's formula reduces to:

$$(a, b]_1 = \mathrm{Tr}_{k/\mathbf{F}_p}\big(\mathrm{res}(d\log(a)\, b)\big). \tag{7}$$

The global reciprocity law in characteristic p reads:

$$\sum_v (a, b]_n = 0, \qquad a \in F^*, b \in W_n(F).$$

Witt's formula allows to see it as a direct analog of the residue formula (5).

2.3. From now on, we turn our attention to the case of characteristic 0. In the literature, various explicit formulas for the Hilbert symbol are often quoted as *explicit reciprocity laws*, but we prefer to call them *explicit formulas* and keep the name of reciprocity laws for global statements.

Fix a finite extension L of the field of p-adic numbers $\mathbf{Q}_p$ and denote by v_L the canonical valuation on L. The residue field k_L is finite with $q = p^f$ elements. Assume that L contains the group μ_n of all nth roots of the unity. The Hilbert symbol $(\ ,\ )_{L,n} : L^* \times L^* \to \mu_n$ is a bilinear, antisymmetric map, which satisfies the symbol property:

$$(a, 1-a)_n = 1, \quad a \neq 0, 1;$$

therefore it factors through the Milnor K-group $\mathrm{K}_2(L)$. Moreover, from the norm property of the reciprocity map $\theta_K : K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$, it follows that $(a, b)_n = 1$ if a belongs to the norm subgroup of the extension $L\left[\sqrt[n]{b}\right]/L$.

If n is coprime with p, the computation of the nth Hilbert symbol on L reduces to the case $n = q-1$. The group μ_{q-1} is canonically isomorphic to the multiplicative group k_L^*, and from the above properties of the Hilbert symbol it follows that an explicit formula for $(a, b)_{q-1}$ is provided by the *tame symbol*:

$$(a, b)_{q-1} = \overline{(-1)^{v(a)v(b)}\left(\frac{b^{v(a)}}{a^{v(b)}}\right)} \in k_L^*,$$

where the overline denotes the reduction map.

The power p case is much more difficult and had been the subject of intensive study. At a very early stage, it was realized that the Hilbert symbol can be alternatively described either in terms of residues of differential forms arising from expansion of the elements a and b in power series of a fixed uniformizer of L or in terms of the trace map.

A classical example of a formula of the first type is a formula proved by Kummer [65] for the cyclotomic extension $\mathbf{Q}_p[\mu_p]$. Here we assume that $p \neq 2$. Let ζ_p be a fixed primitive pth root of the unity and $\pi = \zeta_p - 1$. For principal units a and b in $\mathbf{Z}_p[\mu_p]$, we choose invertible power series $a(X)$ and $b(X)$ with coefficients in $\mathbf{Z}_p$ such that $a = a(\pi)$ and $b = b(\pi)$. In Hilbert's formulation [50], the result proved by Kummer is equivalent to the formula:

$$(a, b)_p = \zeta_p^{\Phi_p(a,b)}, \qquad \text{where } \Phi_p(a, b) = \mathrm{res}\left(\frac{d\log a(X)\log b(X)}{X^p}\right).$$

(Of cause, Kummer formulated his theorem differently, as a global statement.)

Now we recall the definition of the exponential map introduced by Artin and Hasse [5], which appears in many questions related to the theory of local fields. Let L_0 denote the maximal absolutely unramified subfield of L and O_{L_0} be its ring of integers. Extend the absolute Frobenius automorphism σ of $L_0/\mathbf{Q}_p$ to an operator Δ on formal power series with coefficients in L_0 setting

$\Delta(X) = X^p$. For any power series $f(X) \in XO_{L_0}[[X]]$, its Artin–Hasse–Shafarevich exponential is defined as

$$E(f) = \exp\left(\left(1 - \frac{\Delta}{p}\right) f(X)\right).$$

It is not difficult to prove that $E(f) \in 1 + XO_{L_0}[[X]]$ and satisfies the main property of the exponential function: $E(f+g) = E(f)E(g)$. The Artin–Hasse–Shafarevich logarithm [1] is defined as the inverse map:

$$\ell(f) = \frac{1}{p}\log\left(\frac{f^p}{f^\Delta}\right).$$

Fix an uniformizer π_L of L. The evaluation morphism $O_{L_0}[[X]] \to O_L$, $X \mapsto \pi_L$ is surjective; it can be interpreted as an embedding of $\mathrm{Spec}(O_K)$ into the scheme $\mathrm{Spec}(O_{L_0}[[X]])$ equipped with the Frobenius operator Δ.

For any $a \in O_{L_0}$, let A denote a solution of the equation $(\sigma - 1)A = a$ in the p-adic completion of the unramified closure of $\mathbf{Q}_p$. Fix a primitive p^nth root of unity $\zeta_{p^n} \in L$ and a power series $\zeta_{p^n}(X) \in O_{L_0}[[X]]$ such that $\zeta_{p^n} = \zeta_{p^n}(\pi_L)$. In [46], Hasse introduced the element

$$E(a) = E(p^n A\ell(\zeta_{p^n}(X)))|_{X=\pi_L}.$$

From formal properties of the Artin–Hasse–Shafarevich exponential it follows that $E(a)$ is a unit in L; moreover one notices that $E(a)^{1/p^n} = E(A\ell(\zeta_{p^n}(X)))|_{X=\pi_L}$ belongs to the unramified extension L^{ur} of L. Since the reciprocity automorphism $\theta_L(\pi_L)$ acts on L^{ur} as the relative Frobenius, this gives the formula

$$(\pi_L, E(a))_{p^n} = \zeta_{p^n}^{\mathrm{Tr}_{L_0/\mathbf{Q}_p}(a)},$$

where $\mathrm{Tr}_{L_0/\mathbf{Q}_p}$ denotes the trace map.

A completely general description of the Hilbert symbol was first obtained by Shafarevich [2] [84]. For a local field L containing p^nth roots of the unity ($p \neq 2$), he computed the Hilbert symbol on the basis of L^*/L^{*p^n} formed by the elements

$$\pi_L, \quad E(a), \qquad E(\eta_i, \pi_L^j) = E(\eta_i X^j)\big|_{X=\pi_L} \quad \text{for } 1 \leqslant j < \frac{pe}{p-1}, \tag{8}$$

where $\{\eta_i\}_{i=1}^f$ is a fixed basis of O_{L_0} over $\mathbf{Z}_p$, and e denotes the ramification index of $L/\mathbf{Q}_p$.

In Shafarevich's formulas, the necessity to decompose given elements a and b in the basis (8) to compute the symbol $(a,b)_{p^n}$ has a disadvantageous fault of beauty. This was obviated independently by Brückner and Vostokov in the papers [17, 18, 90], where a completely explicit formula for the Hilbert symbol was proved in full generality. The main idea of Vostokov's approach, which proved to be very fruitful, was to construct, firstly, an explicit pairing $\langle\ ,\ \rangle_{L,n} : L^* \times L^* \to \mu_{p^n}$ in the form

$$\langle a, b\rangle_{L,p^n} = \zeta_{p^n}^{\mathrm{Tr}_{L_0/\mathbf{Q}_p}(\mathrm{res}\,\Phi_{p^n}(a,b))},$$

where the differential form $\Phi_{p^n}(a,b)$ is given in terms of expansions of a and b in power series of a fixed uniformizer, and secondly prove that it coincides with the Hilbert symbol on the Shafarevich basis. The main result can be stated as follows:

Theorem 1 (Brückner, Vostokov). *Assume that $p \neq 2$. The p^nth Hilbert symbol is given by the formula*

$$(a,b)_{p^n} = \zeta_{p^n}^{\mathrm{Tr}_{L_0/\mathbf{Q}_p}(\mathrm{res}\,\Phi_{p^n}(a,b))}$$

[1] This map first appears explicitly in [90].

[2] Strickly speaking, Shafarevich gives detailed proofs only for $n = 1$. The general case was studied in detail in the paper of Lapin [66].

with

$$\Phi_{p^n}(a,b)=\left(\frac{1}{\zeta_{p^n}(X)^{p^n}-1}+\frac{1}{2}\right)\left(d\log(a(X))\ell(b(X))-\frac{1}{p}\ell(a(X))d\log b(X)^{\Delta}\right),$$

where $a(\pi)=a$ *and* $b(\pi)=b$, *and* $1/(\zeta_{p^n}(X)^{p^n}-1)$ *is taken in the* p*-adic completion* $O_{L_0}\{\{X\}\}$ *of* $O_{L_0}[[X]][1/X]$.

We refer the reader to the book [34] for a compact exposition of Vostokov's proof. The papers of Brückner and Henniart [18, 47] treat also the case $p=2$.

The main advantage of the Brückner–Vostokov formula is that it gives a closed-form expression of the Hilbert symbol without any restriction on the arguments and therefore provides a complete solution [3] of Hilbert's 9th problem.

Now we turn to explicit formulas of the second type; they were first discovered, in some particular cases, by Artin and Hasse [5], and we will call them *Artin-Hasse's type formulas.* In the most general form, they were proved by Sen [85]. The module of differentials $\Omega_{O_L/\mathbf{Z}_p}$ of the ring of integers of L is the cyclic O_L-module generated by $d\pi_L$ and annihilated by the different ideal $\mathscr{D}_{L/\mathbf{Z}_p}$. This gives sense to the derivative $\frac{d\alpha}{d\pi_L}$ of $\alpha\in O_L$; it is an element of O_L well defined modulo $\mathscr{D}_{L/\mathbf{Z}_p}$.

Theorem 2 (Sen)**.** *For all* $a,b\in L^*$ *such that* $v_L(b)>\frac{2v_L(p)}{p-1}$ *one has:*

$$(a,b)_{p^n}=\zeta_{p^n}^{\frac{1}{p^n}\mathrm{Tr}_{L/\mathbf{Q}_p}\left(\frac{d\log(a)}{d\log(\zeta_{p^n})}\log(b)\right)}.$$

Sen's formula is simpler that Brückner–Vostokov's one, but is valid only under an additional assumption on the second argument. However it seems very tedious to deduce it directly from Brückner–Vostokov formula except in the case of cyclotomic extensions of $\mathbf{Q}_p$. On the other hand, the formulas of Artin–Hasse type are more directly connected to Iwasawa theory of number fields.

Until the work of Sen, the proofs of explicit formulas were based on local class field theory. Sen was the first to apply to this problem techniques of p-adic Hodge theory, namely Tate's computation of continuous cohomology $H^1(G_L,\mathbf{C}_p)$ of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$ with coefficients in the p-adic completion $\mathbf{C}_p$ of $\overline{\mathbf{Q}}_p$.

3. Preliminaries from p-adic Hodge theory

3.1. In this section, we give an overview of basic constructions and notions of p-adic Hodge theory, which will be used in the rest of this survey. Readers familiar with p-adic period rings and (φ,Γ)-modules can skip it.

We start with the definition of the field of de Rham periods $\mathbf{B}_{\mathrm{dR}}$ and its subring of crystalline periods $\mathbf{B}_{\mathrm{cris}}$ introduced by Fontaine [37, 40]. Let $O_{\mathbf{C}_p}$ denote the ring of integers of the p-adic completion of $\overline{\mathbf{Q}}_p$. Fontaine constructs first the perfectization[4] of the quotient $O_{\mathbf{C}_p}/pO_{\mathbf{C}_p}$; it is defined as the projective limit

$$R=\varprojlim\left(O_{\mathbf{C}_p}/pO_{\mathbf{C}_p}\xleftarrow{\varphi}O_{\mathbf{C}_p}/pO_{\mathbf{C}_p}\xleftarrow{\varphi}\cdots\right),$$

where φ denotes the Frobenius endomorphism $\varphi(x)=x^p$. Note that R is a valuation ring of characteristic p; its field of fractions is algebraically closed and equipped with a natural action of the absolute Galois group of $\mathbf{Q}_p$. Alternatively, R can be constructed as the projective limit

[3] One can, of cause, dream of a non-abelian version, in the framework of non-abelian class field theory.
[4] Also called "the tilting"in the modern terminology.

$\varprojlim_{\varphi} O_{\mathbf{C}_p}$ of a countable number of copies of the set $O_{\mathbf{C}_p}$ with the coordinatewise multiplication and the addition defined by the formula:

$$(x+y)^{(n)} = \lim_{m\to+\infty} \left(x^{(n+m)} + y^{(n+m)}\right)^{p^m}, \quad \text{if } x = \left(x^{(n)}\right)_{n\geqslant 0},\ y = \left(y^{(n)}\right)_{n\geqslant 0}.$$

The ring $\mathbf{A}_{\inf} = W(R)$ of Witt vectors with coefficients in R is equipped with a natural surjective morphism $\theta : \mathbf{A}_{\inf} \to O_{\mathbf{C}_p}$ given by $\theta(x_1, x_2, \ldots) = \sum\limits_{m=0} p^m x_m^{(m)}$; its kernel J is a principal ideal in $\mathbf{A}_{\inf}$. Extending scalars, one obtains a map $\theta_{\mathbf{Q}_p} : \mathbf{A}_{\inf}[1/p] \to \mathbf{C}_p$. The $J[1/p]$-adic completion $\mathbf{B}^+_{\mathrm{dR}} = \varprojlim_{n} \mathbf{A}_{\inf}[1/p]/J[1/p]^n$ of $\mathbf{A}_{\inf}[1/p]$ is a discrete valuation ring with the maximal ideal $\mathfrak{m}_{\mathrm{dR}} = J\mathbf{B}^+_{\mathrm{dR}}$ and residue field $\mathbf{C}_p$. To each compartible system $\varepsilon = (\zeta_{p^n})_{n\geqslant 0}$, of p^nth roots of unity one can associate a canonical uniformizer $t = \log[\varepsilon]$ of $\mathbf{B}^+_{\mathrm{dR}}$; here $[\varepsilon]$ is the Teichmüller lifting of ε in the ring of Witt vectors. This element plays the role of the period $\int_C dz/z = 2\pi i$ in p-adic Hodge theory. The field $\mathbf{B}_{\mathrm{dR}}$ is defined as the field of fractions of $\mathbf{B}^+_{\mathrm{dR}}$. It is equipped with a natural action of the absolute Galois group of $\mathbf{Q}_p$ and the filtration $\left(\mathbf{B}^i_{\mathrm{dR}}\right)_{i\in\mathbf{Z}}$ induced by the discrete valuation. The Galois group acts on t as multiplication by the cyclotomic character and $\mathbf{B}^i_{\mathrm{dR}}/\mathbf{B}^{i+1}_{\mathrm{dR}} \simeq \mathbf{C}_p(i)$ as Galois modules [5].

To construct the ring $\mathbf{B}_{\mathrm{cris}}$, we consider the divided power envelope $\mathbf{A}^{\mathrm{PD}}_{\inf}$ of $\mathbf{A}_{\inf}$. More explicitly,

$$\mathbf{A}^{\mathrm{PD}}_{\inf} = \mathbf{A}_{\inf}\left[\frac{\gamma^2}{2!}, \frac{\gamma^3}{3!}, \ldots\right] \subset \mathbf{B}^+_{\mathrm{dR}}, \qquad \gamma \text{ is any fixed generator of } J.$$

Let $\mathbf{A}^+_{\mathrm{cris}}$ denote the p-adic completion of $\mathbf{A}^{\mathrm{PD}}_{\inf}$. Then $t \in \mathbf{A}^+_{\mathrm{cris}}$, and we set $\mathbf{B}^+_{\mathrm{cris}} = \mathbf{A}^+_{\mathrm{cris}}[1/p]$ and $\mathbf{B}_{\mathrm{cris}} = \mathbf{B}^+_{\mathrm{cris}}[1/t]$. The ring $\mathbf{B}_{\mathrm{cris}}$ is stable under the action of the Galois group and is equipped with an operator φ induced from the canonical Frobenius endomorphism on $\mathbf{A}_{\inf}$.

The above constructions were motivated by the theory of p-adic representations of local fields. Let K be a finite extension of $\mathbf{Q}_p$ with absolute Galois group $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. To each p-adic representation V of G_K, Fontaine [41] attached the modules of Galois invariants $\mathbf{D}_{\mathrm{dR}}(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_K}$ and $\mathbf{D}_{\mathrm{cris}}(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{cris}})^{G_K}$ having the following properties:

- $\mathbf{D}_{\mathrm{dR}}(V)$ is a K-vector space of dimension $\leqslant \dim_{\mathbf{Q}_p}(V)$ equipped with the decreasing filtration $\mathbf{D}^i_{\mathrm{dR}}(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B}^i_{\mathrm{dR}})^{G_K}$. One says that V is *de Rham* if $\dim_K \mathbf{D}_{\mathrm{dR}}(V) = \dim_{\mathbf{Q}_p}(V)$.

- $\mathbf{D}_{\mathrm{cris}}(V)$ is a vector space of dimension $\leqslant \dim_{\mathbf{Q}_p}(V)$ over the maximal unramified subextension K_0 of K, equipped with a semilinear Frobenius operator φ. There exists a canonical embedding $\mathbf{D}_{\mathrm{cris}}(V) \otimes_{K_0} K \hookrightarrow \mathbf{D}_{\mathrm{dR}}(V)$. One says that V is *crystalline* if $\dim_{K_0} \mathbf{D}_{\mathrm{cris}}(V) = \dim_{\mathbf{Q}_p}(V)$.

For the resulting categories one has:

$$\mathbf{Rep}_{\mathrm{cris}}(G_K) \subset \mathbf{Rep}_{\mathrm{dR}}(G_K) \subset \mathbf{Rep}_{\mathbf{Q}_p}(G_K).$$

Comparision theorems in p-adic Hodge theory relate this classification to p-adic representations arising in p-adic étale cohomology.

3.2. We review basic results about the field of norms functor of Fontaine–Wintenberger [101]. This construction associates to each infinite arithmetically profinite extension L/K of local fields of charactersitic zero a local field $\mathscr{X}(L/K)$ of characteristic p. Arithmetically profinite extensions are defined in terms of ramification theory. A theorem of Sen [83] says that infinite totally ramified Lie extensions are arithmetically profinite. The multiplicative group of $\mathscr{X}(L/K)$ is defined as the projective limit

$$\mathscr{X}(L/K)^* = \varprojlim_{E} E^*,$$

[5] As usual, $\mathbf{C}_p(i)$ denotes the twist of $\mathbf{C}_p$ by the ith power of the cyclotomic character.

where L/K runs over the finite subextensions $K \subset E \subset L$, and the transition morphisms are the norm maps $N_{E''/E'} : E'' \to E'$. The addition on $\mathscr{X}(L/K) = \mathscr{X}(L/K)^* \cup \{0\}$ is defined as follows: if $x, y \in \mathscr{X}(L/K)$ are written as compatible systems of elements $x = (x_E)_E, y = (y_E)_E$, then

$$(x+y)_E = \varinjlim_{E'/E} N_{E'/E}(x_{E'} + y_{E'}).$$

The key property of this construction is the existence of a functorial one-to-one correspondence between finite extensions of L and separable extensions of $\mathscr{X}(L/K)$; in particular, there exists a natural isomorphism of Galois groups:

$$\mathrm{Gal}(\overline{K}/L) \simeq \mathrm{Gal}(\mathscr{X}(L/K)^{\mathrm{sep}}/\mathscr{X}(L/K)).$$

The field $\mathscr{X}(L/K)$ has a canonical embedding in the field of fractions $\mathrm{Fr}(R)$ of R.

The field of norms $\mathscr{X}_{\mathrm{cyc}}$ of the p-cyclotomic extension $K_\infty = K(\zeta_{p^\infty})$ of K plays a distinguished role in the theory of p-adic representations. The embedding $\mathscr{X}_{\mathrm{cyc}} \hookrightarrow \mathrm{Fr}(R)$ can be canonically lifted in characteristic 0; its lifting is a two-dimensional p-adically complete subring O_{cyc} of $W(\mathrm{Fr}(R))$ with residue ring $\mathscr{X}_{\mathrm{cyc}}$, stable under natural actions of the cyclotomic Galois group $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ and the Frobenius endomorphism induced from $W(\mathrm{Fr}(R))$.

Definition 1. *A (φ, Γ_K)-module is a free O_{cyc}-module of finite rank D equipped with semilinear actions of φ and Γ_K, commuting to each other. One says that D is étale if $\varphi(D)$ generates D over O_{cyc}.*

We denote by $\mathbf{Mod}_{\text{ét}}^{\varphi,\Gamma_K}$ the category of étale (φ, Γ_K)-modules. Let $\mathbf{Rep}_{\mathbf{Z}_p}(G_K)$ denote the category of finitely generated free $\mathbf{Z}_p$-modules equipped with a continuous linear action of G_K. Fontaine constructed an equivalence of categories [6]

$$\mathbf{D} : \mathbf{Rep}_{\mathbf{Z}_p}(G_K) \to \mathbf{Mod}_{\text{ét}}^{\varphi,\Gamma_K}.$$

Analogous equivalences exist for p-adic representations and in p-torsion case. This result plays a fundamental role in p-adic Hodge theory. We don't discuss this theory here and refer the reader to [12, 27] for a survey of main results and further references. In particular, for a p-adic representation V, one can recover $\mathbf{D}_{\mathrm{dR}}(V)$ and $\mathbf{D}_{\mathrm{cris}}(V)$ from $\mathbf{D}(V)$.

One of stricking outcomes of this equivalence is the computation of Galois cohomology in terms of (φ, Γ_K)-modules [48, 49]. Namely, the cohomology $H^*(G_K, T)$ is functorially isomorphic to the cohomology of the complex

$$C^\bullet_{\varphi,\gamma_0}(\mathbf{D}(T)) : 0 \to \mathbf{D}(T) \xrightarrow{\partial_0} \mathbf{D}(T) \oplus \mathbf{D}(T) \xrightarrow{\partial_1} \mathbf{D}(T) \to 0,$$

where $\partial_0(x) = ((\varphi-1)x, (\gamma_0-1)x)$, $\partial_1(y,z) = (\gamma_0-1)y - (\varphi-1)z$, and γ_0 denote a fixed generator of Γ_K. The cup product has a simple description in these terms.

4. Other approaches to explicit formulas

4.1. In this section, we discuss some other approaches to explicit formulas for the Hilbert symbol. Most of them rely on p-adic Hodge theory.

Abrashkin [1] deduced Brückner–Vostokov's formula from Witt's formula (6) for the Artin–Schreier–Witt symbol in characteristic p using the field of norms $\mathscr{X}_{\mathrm{Kum}}$ of the Kummer extension $L(\sqrt[p^\infty]{\pi_L})/L$. As in the cyclotomic case, it has a natural lifting O_{Kum} in characteristic 0. Namely,

[6] $\mathbf{D}(T) = (T \otimes_{\mathbf{Z}_p} \widehat{O}_{\mathrm{cyc}}^{\mathrm{ur}})^{H_K}$, where $\widehat{O}_{\mathrm{cyc}}^{\mathrm{ur}}$ is the p-adic completion of the maximal unramified extension of O_{cyc} and $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$.

let $\widetilde{\pi}_L \in \mathscr{X}_{\mathrm{Kum}}$ denote a compatible system $\big(\pi_L^{1/p^m}\big)_{m\geqslant 0}$ of p^nth roots of a uniformizer $\pi_L \in L$, and $\omega_{\mathrm{Kum}} = [\widetilde{\pi}_L] \in W(\mathrm{Fr}(R))$ the Teichmüller lifting of $\widetilde{\pi}_L$. Then we define O_{Kum} as the p-adic completion of $O_{L_0}[[\omega_{\mathrm{Kum}}]][1/\omega_{\mathrm{Kum}}]$. Since $\varphi(\omega_{\mathrm{Kum}}) = \omega_{\mathrm{Kum}}^p$, the isomorphism

$$O_{\mathrm{Kum}} \simeq O_{L_0}\{\{X\}\}, \qquad \omega_{\mathrm{Kum}} \mapsto X$$

is compatible with the action of φ on the left hand side and the operator Δ on the right hand side. The restriction of Fontaine's theta map $\theta : \mathbf{A}_{\mathrm{inf}} \to O_{\mathbf{C}_p}$ to $O_{L_0}[[\omega_{\mathrm{Kum}}]]$ corresponds, under the above isomorphism, to the evaluation map $X \mapsto \pi_L$. These are first signs that the field $\mathscr{X}_{\mathrm{Kum}}$ provides the right framework for the interpretation of Brückner–Vostokov formulas. The relation between the Kummer equation $x^{p^n} = b$ in characteristic 0 and the corresponding Artin–Schreier–Witt equation involves the crystalline ring $\mathbf{A}_{\mathrm{cris}}^+$.

This approach was inspired by Fontaine's proof of the reciprocity law of Bloch–Kato [42].

4.2. Coleman [23] proved an explicit formula for the Hilbert symbol for cyclotomic extensions. The structure of his formula is very similar to Brückner–Vostokov's one with the Frobenius operator φ acting as $\varphi(X) = (X+1)^p - 1$. To formulate his result, we introduce the cyclotomic analog of the Artin–Hasse–Shafarevich logarithm:

$$\ell_{\mathrm{cyc}}(f) = \frac{1}{p}\log\left(\frac{f^p}{f^{\varphi}}\right).$$

Let $L_n = L_0(\zeta_{p^n})$ be a finite p-cyclotomic p-extension of an absolutely unramified local field L_0. We fix a primitive p^nth root of unity ζ_{p^n} and set $\pi_n = \zeta_{p^n} - 1$. Although Coleman's formula computes the Hilbert symbol $(a,b)_{p^n}$ for all $a, b \in L_n^*$, we reproduce it below only for units to avoid additional notation.

Theorem 3 (Coleman). *Let $a, b \in U_{L_n}$, and let $a(X), b(X) \in O_{L_0}[[X]]$ be such that $a = a(\pi_n)$ and $b = b(\pi_n)$. Then*

$$(a,b)_{p^n} = \zeta_{p^n}^{\mathrm{Tr}_{L_0/\mathbf{Q}_p}\operatorname{res}\Phi_{p^n}^{\mathrm{cyc}}(a,b)},$$

where

$$\Phi_{p^n}^{\mathrm{cyc}}(a,b) = \frac{1}{(X+1)^{p^n}-1}\left(d\log(a(X))\ell_{\mathrm{cyc}}(b(X)) - \frac{1}{p}\ell_{\mathrm{cyc}}(a(X))\,d\log b(X)^{\varphi}\right).$$

The proof of Coleman is close in spirit to Brückner's approach, but also involves techniques of local Iwasawa theory, which we recall below. The local analog of the Iwasawa module is the projective limit of principal units in the p-cyclotomic tower:

$$\mathscr{U}_\infty = \varprojlim_n U_{L_n}^{(1)}.$$

To study this projective limit, define a continuous action of the cyclotomic Galois group $\Gamma_0 = \mathrm{Gal}(L_\infty/L_0)$ on the ring of formal power series $O_{L_0}[[X]]$ by the formula

$$f(X)^{\gamma} = f((X+1)^{\chi_0(\gamma)} - 1), \quad \gamma \in \Gamma_0,$$

where $\chi_0 : \Gamma_0 \to \mathbf{Z}_p^*$ denotes the cyclotomic character. Note that this action commutes with the Frobenius operator φ. The set

$$\mathfrak{X}_\infty = \{f \in 1 + XO_{L_0}[[X]] \mid f^{\varphi} = \prod_{\zeta^p=1} f((X+1)\zeta - 1)\}.$$

is a multiplicative $\mathbf{Z}_p[[\Gamma_0]]$-module.

Theorem 4 (Coleman [22]). *For each compatible system of units $u=(u_n)_{n\geqslant 0}\in \mathscr{U}_\infty$, there exists a unique $f_u\in \mathfrak{X}_\infty$ such that*
$$u_n = f^{\sigma^{-n}}(\pi_n), \qquad \forall n\geqslant 1.$$
Here σ acts as the absolute Frobenius on the coefficients of power series and $\sigma(X)=X$. The correspondence $u\mapsto f_u$ is an isomorphism of $\mathbf{Z}_p[[\Gamma_0]]$-modules:
$$\mathscr{U}_\infty \simeq \mathfrak{X}_\infty.$$
The above statements generalize to projective limits of units in Lubin–Tate towers (see Section 5).

The proof of this theorem involves the norm operator $\mathscr{N}$ characterized by the property
$$\mathscr{N}(f)(\varphi(X)) = \prod_{\zeta^p=1} f((X+1)\zeta-1).$$
This operator plays an important role also in the proof of Theorem 3. We remark that $\mathfrak{X}_\infty = (1+XO_{L_0}[[X]])^{\mathscr{N}=\sigma}$.

If a and b are universal norms in the cyclotomic extension, we can take $a(X), b(X)\in\mathfrak{X}_\infty$, and a formal computation shows that the residue in Coleman's formula can be computed as
$$\operatorname{res}\Phi^{\mathrm{cyc}}_{p^n}(a,b) = \operatorname{res}\left(\frac{\ell_{\mathrm{cyc}}(b(X))d\ell_{\mathrm{cyc}}(a(X))}{(X+1)^{p^n}-1}\right).$$
In this form, Coleman's formula can be seen as a particular case of Perrin-Riou's explicit reciprocity law (see Section 7 below).

4.3. In [7, §2], we gave a new proof of Coleman's explicit formula using the theory of (φ,Γ)-modules. We use the cohomological interpretation of the Hilbert pairing in terms of the cup product
$$H^1(G_L,\mu_{p^n})\times H^1(G_L,\mu_{p^n}) \xrightarrow{\cup} H^2(G_L,\mu_{p^n}^{\otimes 2}).$$
The short exact sequence
$$0\to\mu_{p^n}\to\overline{L}^*\to\overline{L}^*\to 0$$
gives rise to the coboundary map $\delta_{p^n}\,:\,L^*\to H^1(G_L,\mu_{p^n})$. Assume that L contains the group of p^nth roots of the unity. From the cohomological approach to local class field theory, it follows that the diagram below commutes:
$$\begin{tikzcd}
L^*\times L^* \arrow[r, "(\ ,\ )_{p^n}"] \arrow[d, "(\delta_{p^n},\delta_{p^n})"] & \mu_{p^n} \\
H^1(G_L,\mu_{p^n})\times H^1(G_L,\mu_{p^n}) \arrow[r, "\cup"] & H^2(G_L,\mu_{p^n}^{\otimes 2}). \arrow[u]
\end{tikzcd}$$
Here the right vertical map is induced by the canonical isomorphism (which follows from the computation of the Brauer group):
$$\mathrm{inv}_{p^n}\,:\,H^2(G_L,\mu_{p^n})\xrightarrow{\sim}\mathbf{Z}/p^n\mathbf{Z}.$$
This isomorphism has an explicit description in terms of the residue map on the associated (φ,Γ_L)-module $\mathbf{D}(\mu_{p^n})$ [7, 49].

If $L_n=L_0(\mu_{p^n})$, the lifting O_{cyc} of the field of norms of the cyclotomic extension L_∞/L_n can be described explicitly. Namely, let $\varepsilon\in\mathscr{X}_{\mathrm{cyc}}$ denote a compatible system $(\zeta_{p^n})_{n\geqslant 0}$ of primitive p^nth roots of unity and $[\varepsilon]\in W(\mathrm{Fr}(R))$ its Teichmüller lifting. Set $\omega_{\mathrm{cyc}}=[\varepsilon]-1$. Then O_{cyc} is the p-adic completion of $O_{L_0}[[\omega_{\mathrm{cyc}}]][1/\omega_{\mathrm{cyc}}]$. The isomorphism

$$O_{\mathrm{cyc}}\simeq O_{L_0}\{\{X\}\},\qquad \omega_{\mathrm{cyc}}\mapsto X,$$

is compatible with the action of Γ and φ on the both sides. This explains the nature of Coleman's operators in terms of the cyclotomic field of norms just as the Kummerian field of norms explains the nature of the operator Δ in Brückner–Vostokov formulas.

The map $\delta_{p^n}\,:\,L^*\to H^1(G_L,\mu_{p^n})$ has an explicit description in terms of (φ,Γ_{L_n})-modules, which involves the logarithm ℓ_{cyc} and logarithmic derivatives. Computing the cup product

$$H^1(G_{L_n},\mu_{p^n})\times H^1(G_{L_n},\mu_{p^n})\to H^2(G_{L_n},\mu_{p^n}^{\otimes 2})$$

in terms of (φ,Γ_{Ln})-modules we obtain Coleman's formula.

Note that for non cyclotomic extensions, the action of φ and the cyclotomic Galois group on O_{Kum} is very complicated, and the above approach fails. See also [20] for a partial generalization of Coleman power series to an arbitrary ground field and [13].

4.4. In [58], Kato gave a new proof of Brückner–Vostokov's formula using the cohomological interpretation of Hilbert's pairing and computing cup products in Galois cohomology in terms of syntomic complexes. The same method allows to recover the formula of Coleman and establish a link between two types of explicit formulas: each of them corresponds to a specific choice of the lift of the Frobenius operator in the syntomic complex [45].

5. Explicit formulas for formal groups

5.1. Let K be a fixed finite extension of $\mathbf{Q}_p$ with residue field k_K, and O_K its ring of integers. A one-dimensional commutative formal group over O_K is a formal power series $\mathcal{F}(X,Y)=X+Y+\ldots\in O_K[[X,Y]]$ satisfying the following axioms:

i) *Associativity:* $\mathcal{F}(\mathcal{F}(X,Y),Z)=\mathcal{F}(X,\mathcal{F}(Y,Z))$;

ii) *Commutativity:* $\mathcal{F}(X,Y)=\mathcal{F}(Y,X)$;

iii) *Identity element:* $\mathcal{F}(0,X)=\mathcal{F}(X,0)=X$;

iv) *Inverse element:* there exists $i(X)=X+\ldots\in O_K[[X]]$ such that $\mathcal{F}(X,i(X))=0$.

More generally, a formal group of dimension d is defined as a collection $\mathcal{F}(X,Y)=\{\mathcal{F}_i(X,Y)\}_{i=1}^d$ of d power series in $2d$ variables $X=(X_1,\ldots,X_d), Y=(Y_1,\ldots,Y_d)$ satisfying the above properties. Formal groups appeared first in the theory of Lie groups in 1940's. Their role in the arithmetic of local fields became clear after the work of Lubin–Tate [70]. Let $\overline{\mathfrak{m}}$ denote the maximal ideal of the ring of integers of the algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. For all $\alpha,\beta\in\overline{\mathfrak{m}}^d$, power series $\mathcal{F}(\alpha,\beta)$ converge to an element of $\overline{\mathfrak{m}}^d$, and the rule $(\alpha,\beta)\mapsto\mathcal{F}(\alpha,\beta)$ defines a structure of $\mathbf{Z}_p$-module on $\overline{\mathfrak{m}}^d$, which we will denote by $\mathcal{F}(\overline{\mathfrak{m}})$.

For any uniformizer π of K, Lubin and Tate constructed a one-dimensional formal group $\mathcal{F}_\pi$ with the isogeny $[\pi](X)=\pi X+X^q$; here $q=|k_K|$. For each $a\in O_K$ there exists a unique endomorphism $[a](X)\in\mathrm{End}(\mathcal{F})$ such that $[a](X)\equiv aX \pmod{X^2}$; this establishes an isomorphism $[\ \ ]:O_K\simeq\mathrm{End}(\mathcal{F})$. Let K_π denote the extension generated over K by all $[\pi]^n$-torsion points of $\mathcal{F}_\pi$. The compositum of the maximal unramified extension of K with K_π coincides with the maximal abelian extension of K. The reciprocity map can be explicitly described in terms of the group $\mathcal{F}_\pi$: for any unit $u\in O_K^*$, the automorphism $\theta_K(u)$ acts on torsion points of $\mathcal{F}_\pi$ as $[u^{-1}](X)$. To sum up, the above results provide a construction of the maximal abelian extension of a local field and a description of the reciprocity map in terms of torsion points of Lubin–Tate formal groups. They should be seen as the local analog of the theory of complex multiplication of imaginary quadratic fields.

In the general setting, a formal group $\mathcal{F}$ is said to be p-divisible if the multiplication by p is an isogeny. In that case, the group $\mathcal{F}[p^n]$ of p^n-torsion points of $\mathcal{F}(\overline{\mathfrak{m}})$ is (non-canonically) isomorphic to the direct product of h copies of $\mathbf{Z}/p^n\mathbf{Z}$ for some $h \geqslant 1$ called the height of $\mathcal{F}$. The Tate module $T_{\mathcal{F}} := \varprojlim_n \mathcal{F}[p^n]$ is a free $\mathbf{Z}_p$-module of rank h equipped with a natural action of the absolute Galois group $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. This gives rise to a p-adic representation

$$\rho_{\mathcal{F}} \,:\, G_K \to \mathrm{GL}(T_{\mathcal{F}}).$$

This representation has a number of fine number-theoretic properties. In [86], Tate proved that the Galois module $T_{\mathcal{F}} \otimes_{\mathbf{Q}_p} \mathbf{C}_p$ obtained by tensoring $T_{\mathcal{F}}$ with the p-adic completion of $\overline{\mathbf{Q}}_p$, has a simple structure. Namely, there exists a canonical decomposition

$$T_{\mathcal{F}} \otimes_{\mathbf{Z}_p} \mathbf{C}_p \simeq t_{\mathcal{F}}(\mathbf{C}_p(1)) \oplus t^*_{\mathcal{F}^\vee}(\mathbf{C}_p), \tag{9}$$

where $t_{\mathcal{F}}$ denotes the tangent space of $\mathcal{F}$, $\mathcal{F}^\vee$ is the dual p-divisible group of $\mathcal{F}$ [7], and we write $\mathbf{C}_p(1)$ for the cyclotomic twist of $\mathbf{C}_p$. This isomorphism, called the Hodge–Tate decomposition of $T_{\mathcal{F}}$, is a direct analog of the Hodge decomposition for complex varieties.

The definition of the Hilbert symbol generalizes naturally to formal p-divisible groups. Fix an integer $n \geqslant 1$ and a finite extension L/K containing the coordinates of p^n-torsion points of $\mathcal{F}$. Then we have a well-defined pairing

$$\begin{aligned} &(\ ,\ )_{\mathcal{F},p^n} \,:\, L^* \times \mathcal{F}(\mathfrak{m}_L) \to \mathcal{F}[p^n], \\ &(a,b)_{\mathcal{F},p^n} = \gamma^{\theta_K(a)} -_{\mathcal{F}} \gamma, \qquad \text{where } [p^n](\gamma) = a, \end{aligned} \tag{10}$$

which coincides with the classical Hilbert symbol in the case of the multiplicative formal group $\mathcal{F}_m(X,Y) = (1+X)(1+Y) - 1$. The search of explicit formulas for this pairing was motivated by applications to the Birch and Swinnerton-Dyer conjecture (see Section 6 below).

In [91, 92], Vostokov extended his approach to Lubin–Tate formal groups. In that case, the properties of the Shafarevich basis are more subtle, but the obtained explicit formula has the same structure. Formulas of Artin–Hasse's type for Lubin–Tate groups were proved by Wiles [100] and, in full generality, Destrempes [29]. The proof of Destrempes is a direct generalisation of Sen's method. For the generalization of Coleman's approach to Lubin–Tate groups, see [30].

5.2. In the case of general formal groups the obstacles are twofold:

a) The classification of p-divisible groups is much more complicated over ramified fields [16]. In the unramified case, Honda–Dieudonné theory gives an explicit classification up to isomorphism [35, 51].

b) Classical computations fail for formal groups of dimension $d > 1$.

Vostokov and Demchenko [97] treated the case of one-dimensional groups over absolutely unramified fields. Further results were obtained, by different methods, in the following papers:

a) Formulas of Brückner–Vostokov's type, $\mathcal{F}$ is of arbitrary dimension over an absolutely unramified field K, assuming that the field L contains a primitive p^nth root of unity [2].

[7] The category of formal groups is to small to develop a satisfactory theory, in particular, it is not closed under taking duals. Tate works in the more natural category of p-divisible groups.

b) Formulas of Brückner–Vostokov's type, $\mathcal{F}$ is of arbitrary dimension over an absolutely unramified field K, without any additional assumption [87].

c) Formulas of Artin–Hasse's type, $\mathcal{F}$ is of arbitrary dimension over an arbitrary finite extension K of $\mathbf{Q}_p$ [6]. As in all formulas of this type, it is valid under some restrictions on the second argument.

The work of Abrashkin [2] is a generalization of his proof of the Brückner–Vostokov formula in [1]; he uses the field of norms functor to relate the Hilbert symbol on $\mathcal{F}$ to the Artin–Schreier–Witt symbol. The approaches of [6] and [87] use the interpretation of the Hilbert symbol as the cup product

$$H^1(G_L,\mu_{p^n})\times H^1(G_L,\mathcal{F}[p^n])\xrightarrow{\cup} H^2(G_L,\mu_{p^n}\otimes\mathcal{F}[p^n])\simeq\mathcal{F}[p^n], \tag{11}$$

and compute this pairing using tools of p-adic Hodge theory.

All these papers rely heavily on the theory of p-adic integration on formal groups [24, 25, 35], which we review below. A differential form

$$\omega=\sum_{i=1}^{d}a_i(X_1,\dots,X_d)dX_i,\qquad a_i(X_1,\dots,X_d)\in K[[X_1,\dots,X_d]]$$

will be called closed if there exists a power series $\lambda_\omega\in K[[X_1,\dots,X_d]]$ such that $\lambda_\omega(0,\dots,0)=0$ and $d\lambda_\omega=\omega$. By analogy with the integration theory on complex abelian varieties, we will say that a closed form ω is

- *invariant,* if $\lambda_\omega(X+_{\mathcal{F}}Y)=\lambda_\omega(X)+\lambda_\omega(Y)$; one says that λ_ω is a *logarithm* of $\mathcal{F}$.
- *of the second kind,* if there exists $r\geqslant 0$ such that
$$\lambda_\omega(X+_{\mathcal{F}}Y)-\lambda_\omega(X)-\lambda_\omega(Y)\in p^{-r}O_K[[X,Y]];$$
- *exact,* if there exists $r\geqslant 0$ such that $\lambda_\omega\in p^{-r}O_K[[X]]$.

The quotient

$$H^1_{\mathrm{dR}}(\mathcal{F})=\frac{\{\text{differential forms of the second kind}\}}{\{\text{ exact forms}\}}$$

is a K-vector space of dimension h; it is the analog of the first de Rham cohomology group of $\mathcal{F}$. The space $\Omega^1_{\mathcal{F}}$ of invariant forms on $\mathcal{F}$ is of dimension d over K and injects into $H^1_{\mathrm{dR}}(\mathcal{F})$. p-adic integration theory provides a non-degenerate bilinear pairing

$$T_{\mathcal{F}}\times H^1_{\mathrm{dR}}(\mathcal{F})\to\mathbf{B}_{\mathrm{dR}} \tag{12}$$

with values in the field of p-adic periods $\mathbf{B}_{\mathrm{dR}}$, compatible with the filtration and the Galois action[8]. This pairing refines the Hodge–Tate decomposition (9) and should be seen as the p-adic analog of the complex period map

$$H_1(A(\mathbf{C}),\mathbf{Z})\times H^1_{\mathrm{dR}}(A)\to\mathbf{C}.$$

5.3. Now we explain the method of [6]. As in the multiplicative case, there exists a commutative diagram

$$\begin{tikzcd}
L^*\times\mathscr{F}(\mathfrak{m}_L) \arrow[rr, "{(\ ,\ ]_n}"] \arrow[d, "{(\delta_{p^n},\delta_{\mathscr{F},p^n})}"] & & \mathscr{F}[p^n] \\
H^1\big(G_L,\mu_{p^n}\big)\times H^1\big(G_L,\mathscr{F}[p^n]\big) \arrow[r, "\cup"] & H^2\big(G_L,\mu_{p^n}\big)\otimes\mathscr{F}[p^n], \arrow[ur] &
\end{tikzcd}$$

[8]The construction of the field $\mathbf{B}_{\mathrm{dR}}$ is technically involved. We refer the reader to [40, 41] for a detailed exposition and to [10] for a survey of p-adic integration on formal groups in connection with explicit formulas.

where δ_{p^n} and $\delta_{\mathscr{F},p^n}$ denote the coboundary maps induced by the exact sequences

$$0 \to \mu_{p^n} \to \overline{L}^* \xrightarrow{p^n} \overline{L}^* \to 0, \qquad 0 \to \mathscr{F}[p^n] \to \mathscr{F}(\mathfrak{m}_{\overline{L}}) \xrightarrow{p^n} \mathscr{F}(\mathfrak{m}_{\overline{L}}) \to 0.$$

The map $\delta_{\mathscr{F},p^n}$ can be described in a purely algebraic way in terms of the p-adic integration. This description involves the matrix

$$\Theta_{L,n} = p^n \begin{pmatrix} \lambda'_{\omega_1}(\xi_1)\,\dfrac{d\xi_1}{d\pi_L} & \lambda'_{\omega_1}(\xi_2)\,\dfrac{d\xi_2}{d\pi_L} & \cdots & \lambda'_{\omega_1}(\xi_h)\,\dfrac{d\xi_h}{d\pi_L} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda'_{\omega_d}(\xi_1)\,\dfrac{d\xi_1}{d\pi_L} & \lambda'_{\omega_d}(\xi_2)\,\dfrac{d\xi_2}{d\pi_L} & \cdots & \lambda'_{\omega_d}(\xi_h)\,\dfrac{d\xi_h}{d\pi_L} \\ \lambda_{\omega_{d+1}}(\xi_1) & \lambda_{\omega_{d+1}}(\xi_2) & \cdots & \lambda_{\omega_{d+1}}(\xi_h) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{\omega_h}(\xi_1) & \lambda_{\omega_h}(\xi_2) & \cdots & \lambda_{\omega_h}(\xi_h) \end{pmatrix},$$

where $\{\omega_j\}_{j=1}^d$ is a fixed basis $\Omega^1_{\mathscr{F}}$ completed to a basis $\{\omega_j\}_{i=j}^h$ of $H^1_{\mathrm{dR}}(\mathscr{F})$, and $\{\xi_i\}_{i=1}^h$ is a fixed basis of the p^n-torsion group $\mathscr{F}[p^n]$. The horizontal cup-product can be computed using the fundamental exact sequence of p-adic Hodge theory (see Section 6). The final result states as follows:

Theorem 5. *Let $\mathscr{F}$ be a p-divisible formal group over a p-adic local field K. Let $X = (X_{ij})_{1\leqslant i,j\leqslant h}$ denote the inverse matrix of $\Theta_{L,n}$. For all $a \in L^*$ and $b \in \mathscr{F}(\mathfrak{m}_L)$ such that $v_L(b) > \dfrac{2v_L(p)}{p-1}$, one has:*

$$(a,b)_{\mathscr{F},p^n} = \sum_{i=1}^{h}\sum_{j=1}^{d} \left[\mathrm{Tr}_{L/\mathbf{Q}_p}\left(X_{ij}\frac{d\log(a)}{d\pi_L}\lambda_{\omega_j}(b)\right)\right](\xi_j).$$

This is the main result of [6]. See also [43] for the impovement of conditions on the second argument and generalization to higher-dimensional fields. The above formula includes the results of Sen and Destrempes as particular cases.

5.4. The method of Tavares Ribeiro [87] is different and is based on the theory of (φ,Γ)-modules [39]. As we already noticed in Section 3.3, the classical theory of (φ,Γ)-modules is not well adapted to the study of arithmetic properties of non-cyclotomic extensions because of a very complicated action of the cyclotomic Frobenius operator φ and the cyclotomic Galois group on the ring O_{cyc}. On the other hand, this theory can be developed in the context of an arbitrary p-adic Lie extension M/L, where the cyclotomic Galois group Γ_L is replaced by $\mathcal{G} = \mathrm{Gal}(M/L)$. The price to pay is the necessity to work with a rather complicated coefficient ring A [9]. Since Brückner–Vostokov's type formulas are closely related to the field of norms of the Kummer extension $L(\sqrt[p^\infty]{\pi_L})$, it is natural to take $M = L[\mu_{p^\infty}, \sqrt[p^\infty]{\pi_L}]$, the Galois closure of $L(\sqrt[p^\infty]{\pi_L})$. In that case, the ring A contains the lifting O_{Kum} of the Kummerian field of norms. Applying these constructions to a field L containing p^n-torsion points of a formal group $\mathscr{F}$, Tavares Ribeiro computed the cup product (11) in terms of $(\varphi,\mathscr{G})$-modules and obtained Brückner–Vostokov's type formulas in full generality.

6. The dual exponential map

6.1. The connection between explicit formulas and special values of L-functions was probably first observed by Iwasawa [54, 55]. Following Kato [59], it can be formulated in terms of the dual

[9] We should consider $(\varphi,\mathcal{G})$-modules over the ring of Witt vectors with coefficients in the perfectization of O_M/pO_M in the sense of Fontaine.

exponential map. The p-adic period rings $\mathbf{B}_{\mathrm{dR}}$ and $\mathbf{B}_{\mathrm{cris}}$ are related by the fundamental exact sequence

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^0 \to 0,$$

where the middle map is induced by the inclusion $\mathbf{B}_{\mathrm{cris}} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$. Let V be a de Rham representation of G_K. The de Rham module $\mathbf{D}_{\mathrm{dR}}(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_K}$ is a filtered K-vector space of dimension $\dim(V)$. Bloch and Kato [15] defined the exponential map

$$\exp_V\,:\,\mathbf{D}_{\mathrm{dR}}(V)/\mathbf{D}_{\mathrm{dR}}^0(V) \to H^1(G_K, V)$$

as the coboundary map associated to the short exact sequence of Galois modules

$$0 \to V \to V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^0 \to 0.$$

Assume that $V_{\mathscr{F}} = T_{\mathscr{F}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is the p-adic representation arising from a formal group $\mathscr{F}$. From the theory of p-adic integration it follows that $V_{\mathscr{F}}$ is de Rham [10] and $\mathbf{D}_{\mathrm{dR}}(V_{\mathscr{F}})$ is the dual of $H^1_{\mathrm{dR}}(\mathscr{F})$. The invariant differential forms on $\mathscr{F}$ provide us with the logarithm map $\log_{\mathscr{F}}\,:\,\mathscr{F}(\mathfrak{m}_K) \to t_{\mathscr{F}}(K)$ from the group of points to the tangent space of $\mathscr{F}$; this map is a local isomorphism.

Looking at Galois invariants in the Hodge–Tate decomposition (9), we see that

$$t_{\mathscr{F}}(K) \simeq \big(V_{\mathscr{F}} \otimes \mathbf{C}_p(-1)\big)^{G_K} \simeq \mathbf{D}_{\mathrm{dR}}(V_{\mathscr{F}})/\mathbf{D}_{\mathrm{dR}}^0(V_{\mathscr{F}}).$$

The resulting isomorphism describes the tangent space entirely in terms of the de Rham module. Taking the composition with the logarithm map, we obtain a morphism

$$F(\mathfrak{m}_K) \to \mathbf{D}_{\mathrm{dR}}(V_{\mathscr{F}})/\mathbf{D}_{\mathrm{dR}}^0(V_{\mathscr{F}}).$$

Theorem 6 (Bloch–Kato)**.** *The following diagram is commutative:*

$$\begin{tikzcd}
\mathscr{F}(\mathfrak{m}_K) \arrow[r, "\delta_{\mathscr{F}}"] \arrow[d] & H^1(G_K, T_{\mathscr{F}}) \arrow[d] \\
\mathbf{D}_{\mathrm{dR}}(V_{\mathscr{F}})/\mathbf{D}_{\mathrm{dR}}^0(V_{\mathscr{F}}) \arrow[r, "\exp_{V_{\mathscr{F}}}"] & H^1(G_K, V_{\mathscr{F}}),
\end{tikzcd}$$

Here $\delta_{\mathscr{F}}$ is defined as the projective limit of coboundary maps $\delta_{\mathscr{F},p^n}$.

This theorem gives an algebraic interpretation of the usual exponential map for the group $\mathscr{F}$ in terms of p-adic periods. Therefore, the map $\exp_V$ can be seen as an analog of the exponential map for general de Rham representations.

Now we define the dual exponential map. Let $V^* = \mathrm{Hom}_{\mathbf{Q}_p}(V, \mathbf{Q}_p)$ denote the dual p-adic representation. The pairing $V \times V^*(1) \to \mathbf{Q}_p(1)$ induces the local duality on cohomology

$$(\ ,\ )_V\,:\,H^1(G_K, V) \times H^1(G_K, V^*(1)) \to \mathbf{Q}_p$$

and a natural duality between de Rham modules:

$$[\,,\,]_V\,:\,\mathbf{D}_{\mathrm{dR}}(V) \times \mathbf{D}_{\mathrm{dR}}(V^*(1)) \to \mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p(1)) \xrightarrow{\mathrm{Tr}_{K/\mathbf{Q}_p}} \mathbf{Q}_p.$$

The dual exponential map $\exp^*_{V^*(1)}\,:\,H^1(G_K, V^*(1)) \to \mathbf{D}_{\mathrm{dR}}^0(V^*(1))$ is defined as the adjoint map:

$$[x, \exp^*_{V^*(1)}(y)]_V = (\exp_V(x), y)_V, \quad x \in \mathbf{D}_{\mathrm{dR}}(V), \quad y \in H^1(G_K, V^*(1)).$$

[10]In fact, $V_{\mathscr{F}}$ is even crystalline, but we don't use it here.

6.2. Let K_∞/K be an infinite Galois extension of K, such that $\mathcal{G}=\mathrm{Gal}(K_\infty/K)$ is a p-adic Lie group. Then there exists a tower of finite Galois extensions

$$K=K_0\subset K_1\subset\ldots\subset K_n\subset\ldots,\quad \text{such that } K_\infty=\bigcup_{n=0}^{\infty}K_n.$$

Let V be a p-adic representation of G_K. Fix a $\mathbf{Z}_p$-lattice T in V stable under the action of G_K. The information about cohomology groups $H^1(G_{K_n},T)$ is encoded in the *Iwasawa cohomology* of T, which is defined as the projective limit

$$H^1_{\mathrm{Iw}}(K_\infty/K,T)=\varprojlim_n H^1(G_{K_n},T/p^nT)\simeq\varprojlim_n H^1(G_{K_n},T)$$

taken with respect to the norm maps. For each $m\geqslant 0$, we denote by

$$\mathrm{pr}_m\,:\,H^1_{\mathrm{Iw}}(K_\infty/K,T)\to H^1(G_{K_m},T)$$

the canonical projection. The behavior of the exponential map in the tower K_∞/K is a difficult problem even in the classical case of the cyclotomic p-extension. We will discuss it in Section 7.

6.3. Explicit formulas for the generalized Hilbert symbol can be interpreted in terms of dual exponential maps

$$\exp^*_{V^*_{\mathscr{F}}(1),K_m}\,:\,H^1(G_{K_m},V^*_{\mathscr{F}}(1))\to\mathbf{D}_{\mathrm{dR}}(V^*_{\mathscr{F}}(1))\otimes_K K_m,\qquad m\geqslant 0.$$

For a formal p-divisible group $\mathscr{F}$ over K, let K_n denote the extension of K generated by p^n-torsion points of $\mathscr{F}$ and $K_\infty=\bigcup\limits_{n=0}^{\infty}K_n$. Fix a basis $\{\boldsymbol{\xi}^{(i)}\}_{i=1}^h$ of the Tate module $T_{\mathscr{F}}$ and write $\boldsymbol{\xi}^{(i)}$ as a collection $\boldsymbol{\xi}^{(i)}=(\xi_n^{(i)})_{n\geqslant 1}$ of p^n-torsion points. Let $T^*_{\mathscr{F}}$ denote the dual module with the dual basis $\{\boldsymbol{\xi}^{(i)*}\}_{i=1}^h$. For any $1\leqslant i\leqslant h$, the group G_{K_n} acts trivially on $\xi_n^{(i)*}$, and the cup products

$$K_n^*\xrightarrow{\delta_{p^n}\cup\xi_n^{(i)*}}H^1(G_{K_n},\mathscr{F}[p^n]^*(1))$$

are well defined. Passing to the projective limit, we obtain a map

$$\varprojlim_n K_n^*\to H^1_{\mathrm{Iw}}(K_\infty/K,T^*_{\mathscr{F}}(1)).$$

Consider the composition

$$s_m^{(i)}\,:\,\varprojlim_n K_n^*\to H^1_{\mathrm{Iw}}(K_\infty/K,T^*_{\mathscr{F}}(1))\xrightarrow{\mathrm{pr}_m}H^1(G_{K_m},T^*_{\mathscr{F}}(1)).\tag{13}$$

Then Theorem 5 implies:

$$\exp^*_{V^*_{\mathscr{F}}(1)}\left(s_m^{(i)}\left((a_n)_{n\geqslant 1}\right)\right)=\lim_{n\to\infty}\sum_{j=1}^{d}\mathrm{Tr}_{K_n/K_m}\left(X_{ij}^{(n)}\frac{d\log(a_n)}{d\pi_{K_n}}\right)\omega_j.\tag{14}$$

It is clear that other explicit formulas can be interpreted in the same manner.

6.4. The disadvantage of formula (14) is the necessity of the limit process in the right hand side. If $\mathscr{F}$ is a Lubin–Tate formal group, Coleman's series allow to write it in a compact form. Namely, let $\boldsymbol{\xi}=(\xi_n)_{n\geqslant 0}$ be a fixed generator of $T_{\mathscr{F}}$ viewed as a free O_K-module of rank one. Fix an unramified extension L_0/K of degree f and set $L_{\pi,n}=L_0[\xi_n]$. Consider the composition

$$s_m^{(\boldsymbol{\xi})}\,:\,\varprojlim_n L^*_{\pi,n}\to H^1_{\mathrm{Iw}}(L_{\pi,\infty}/L,T^*_{\mathscr{F}}(1))\xrightarrow{\mathrm{pr}_m}H^1(G_{L_{\pi,m}},T^*_{\mathscr{F}}(1)),$$

defined analogously to (13) replacing K by L and p^n-torsion by π^n-torsion. In the Lubin–Tate case, Theorem 4 says that for each $a = (a_n)_{n\geqslant 1} \in \varprojlim_n L^*_{\pi,n}$ there exists a unique power series $f_a \in O_{L_0}((X))$ such that

$$a_n = f_a^{\sigma_K^{-n}}(\xi_n), \qquad n \geqslant 1, \quad \text{where } \sigma_K \text{ denotes the relative Frobenius.}$$

Then we have:

$$\exp^*_{V^*_{\mathscr{F}}(1)}\left(s_m^{(\boldsymbol{\xi})}(a)\right) = \frac{1}{\pi^m}\left(\left(\frac{d\log f_a^{\sigma_K^{-n}}}{\omega}\right)(\xi_n)\right)\omega, \tag{15}$$

where ω is a fixed differential form on $\mathscr{F}$. Note that $\omega = d\lambda_\omega$. This is the reciprocity law of Wiles [100] (formulated using Coleman series).

We cannot expect a straightforward generalization of Coleman series to general formal groups [13].

6.5. We are now in a position to discuss connections between explicit formulas and L-functions. First we consider the case of Dirichlet characters. Fix an integer $N \geqslant 2$ coprime with p and set $F = \mathbf{Q}(\mu_N)$. Fix a primitive Dirichlet character

$$\eta \,:\, (\mathbf{Z}/N\mathbf{Z})^* \to \overline{\mathbf{Q}}^*$$

and a finite extension $E/\mathbf{Q}_p$ containing the values of η. Fix a primitive Nth root of unity ζ_N and a compatible system $(\zeta_{p^n})_{n\geqslant 1}$ of p^nth roots of unity. Recall how to construct a special element in cohomology groups $H^1(G_{\mathbf{Q}}, E(\eta))$ using cyclotomic units $\mathbf{u} = (1 - \zeta_N^{p^{-n}}\zeta_{p^n})_{n\geqslant 1}$. Consider Iwasawa cohomology

$$H^1_{\mathrm{Iw}}(G_F, \mathbf{Z}_p(1)) = \varprojlim_n H^1(G_{F(\mu_{p^n})}, \mu_{p^n}),$$

where the projective limit is taken under the corestriction map. Since the family $\mathbf{u}$ is compatible under the norm map, the Kummer map send it to some element $\mathbf{z} \in H^1_{\mathrm{Iw}}(G_F, \mathbf{Z}_p(\chi_F))$. Twisting it by the cyclotomic character χ_F^{-1} we obtain an element $\widetilde{\mathbf{z}} \in H^1_{\mathrm{Iw}}(G_F, \mathbf{Z}_p)$. The Galois group $G = \mathrm{Gal}(F/\mathbf{Q})$ acts on $H^1_{\mathrm{Iw}}(G_F, \mathbf{Z}_p)$; extending coefficients and decomposing $H^1_{\mathrm{Iw}}(G_F, E)$ characterwisely, we define $\widetilde{\mathbf{z}}^{(\eta)}$ as the image of $\widetilde{\mathbf{z}}$ under the canonical projection $H^1_{\mathrm{Iw}}(G_F, E) \to H^1_{\mathrm{Iw}}(G_{\mathbf{Q}}, E(\eta))$. Finally we denote by $\widetilde{\mathbf{z}}^{(\eta)} \in H^1(G_{\mathbf{Q}}, E(\eta))$ the image of $\widetilde{\mathbf{z}}^{(\eta)}$ under $H^1_{\mathrm{Iw}}(G_{\mathbf{Q}}, E(\eta)) \to H^1(G_{\mathbf{Q}}, E(\eta))$. The following result is implicitly contained in [54].

Theorem 7 (Iwasawa)**.** *The dual exponential map*

$$\exp^*_{E(\eta)} \,:\, H^1(G_{\mathbf{Q}_p}, E(\eta)) \to E$$

sends $\widetilde{\mathbf{z}}^{(\eta)}$ *to* $\left(1 - \dfrac{\eta^{-1}(p)}{p}\right) L(\eta, 0)$, *where* $L(\eta, s)$ *is the Dirichlet* L*-function attached to* η.

The proof of this theorem is the result of two independent computations. Using (14), one computes the image of $\widetilde{\mathbf{z}}^{(\eta)}$ under the dual exponential map and compare it with an explicit formula for $L(\eta, 0)$.

A similar argument was used in the celebrated paper of Coates–Wiles [21] on the conjecture of Birch and Swinnerton-Dyer: here the dual exponential map relates elliptic units to the special value $L(E, 1)$ of the L-function of a CM-elliptic curve. These examples are only fragments of a general, mostly conjectural picture relating compatible systems of global cohomology classes (Euler systems) to special values of complex L-functions [59, 74]. From this point of view, explicit formulas should be seen as an explicit description of the dual exponential map. Below we discuss some results obtained in this direction. Note that $(c) \Rightarrow (b) \Rightarrow (a)$. All these results are particular cases of Kato's generalized reciprocity law for higher dimensional fields (see Section 8.3 below).

(a) *Bloch–Kato explicit reciprocity law for the dual exponential map*

$$\exp^* : H^1(G_{\mathbf{Q}_p}, \mathbf{Q}_p(1-r)) \to \mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p(1-r)), \quad r \geqslant 1,$$

(see [15]). The original proof of Bloch and Kato is based on syntomic cohomology, already used by Kato in the classical setting [58]. In [42], Fontaine deduces Bloch–Kato's formula from the formula of Artin–Schreier–Witt in characteristic p using the field-of-norms functor.

(b) *Kato's reciprocity law for the dual exponential map*

$$\exp^* : H^1(G_{L_n}, V_{\mathscr{F}}^{\otimes(-r)}(1)) \to \mathbf{D}_{\mathrm{dR}}(V_{\mathscr{F}}^{\otimes(-r)}(1)) \otimes_K L_n, \quad r \geqslant 1.$$

Here $\mathscr{F}$ is a Lubin–Tate formal group and the notation is the same as in Section 6.1. For each $r \geqslant 1$, consider the composition

$$s_{m,r}^{(\boldsymbol{\xi})} : \varprojlim_n K_n^* \xrightarrow{\cup \boldsymbol{\xi}^{\otimes(-r)}} H^1_{\mathrm{Iw}}(K_\infty/K, T_{\mathscr{F}}^{\otimes(-r)}(1)) \xrightarrow{\mathrm{pr}_m} H^1(G_{K_m}, T_{\mathscr{F}}^{\otimes(-r)}(1)).$$

Then (see [59]):

$$s_{m,r}^{(\boldsymbol{\xi})}(a) = \frac{1}{\pi^{mr}(r-1)!} \left(\left(\left(\frac{d}{d\omega} \right)^r \log f_a^{\sigma_K^{-n}} \right) (\xi_n) \right) \omega^{\otimes r} \tag{16}$$

Kato proves this formula using techniques of crystalline cohomology. It should be possible to prove it using the method of [6], however we didn't check the details. See also [80]. If $r = 1$, we recover Wiles' formula (15).

(c) *Tsuji's explicit reciprocity law for twisted Lubin–Tate characters* [88]. For a Lubin–Tate group $\mathscr{F}$ over an unramified local field K of degree h over $\mathbf{Q}_p$, Tsuji gives a description of the dual exponential map of the representation

$$\left(\bigotimes_{0 \leqslant i \leqslant h-1} \left(V_{\mathscr{F}}^{\sigma^{-i}} \right)^{\otimes(-r_i)} \right) (1).$$

Here $V_{\mathscr{F}}^{\sigma^{-i}}$ denotes the Galois module $V_{\mathscr{F}}$ endowed with the twisted action of K:

$$a \star v = \sigma^{-i}(a)v, \qquad a \in K, \quad v \in V_{\mathscr{F}},$$

where σ denotes the absolute Frobenius. If $r_0 = r$ and $r_1 = \ldots = r_{h-1} = 0$, his formula gives the reciprocity law (b).

(d) *Kato's reciprocity law for modular curves* [61]. Let $Y(M,N)$ be the modular curve classifying isomorphism classes of elliptic curves E with a level structure $\mathbf{Z}/M\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z} \to E$. Set $Y_1(N) = Y(1,N)$ and $Y(N) = Y(N,N)$. Consider the Galois representation

$$V = H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}_p}, \mathrm{Sym}^k \mathrm{R}^1 \mathrm{p}_*(\mathbf{Q}_p)), \qquad k \geqslant 0,$$

where $\mathrm{p} : \mathscr{E} \to Y_1(N)$ denotes the universal elliptic curve over $Y_1(N)$. For each $n \geqslant 1$, the composition of the cup product

$$H^1_{\text{ét}}(Y(Lp^n), \mu_{p^n}) \times H^1_{\text{ét}}(Y(Lp^n), \mu_{p^n}) \xrightarrow{\cup} H^2_{\text{ét}}(Y(Lp^n), \mu_{p^n}^{\otimes 2})$$

with the coboundary map $\delta_{p^n} : \mathscr{O}(Lp^n)^* \to H^1_{\text{ét}}(Y(Lp^n), \mu_{p^n})$ gives a morphism

$$\mathrm{K}_2(\mathscr{O}(Lp^n)) \to H^2_{\text{ét}}(Y(Lp^n), \mu_{p^n}^{\otimes 2}).$$

Fix an integer $M \geqslant 1$. Choose L such that $M \mid L$ and $N \mid L$. Using the Iwasawa theoretic twist, under some additional technical condition on L, one constructs morphisms

$$s_{j,r} \,:\, \varprojlim_n \mathrm{K}_2(\mathscr{O}(Y(Lp^n)) \to H^1(\mathbf{Q}[\zeta_{Mp^m}], V_k(r)) \qquad 1 \leqslant r \leqslant k+1.$$

(Here j parametrizes twists by canonical generators of the universal elliptic curve.) Kato gives an explicit formula for the image of the localization of $s_{j,r}(a)$ under the dual exponential map. Note that $\mathbf{D}^0_{\mathrm{dR}}(V_k(r))$ can be identified with the space of modular forms of weight $k+2$.

This result plays a key role in Kato's fundamental work on Iwasawa main conjecture for modular forms [61]. Kato deduces it from his reciprocity law for two-dimensional local fields [60], that in turn can be seen as a highly nontrivial generalization of the Vostokov–Kirillov formula (see Section 8). See also the paper of Wang [98] for another approach. The case $k=r=1$ is discussed in the paper of Scholl [82].

(e) *Lemma–Ochiai's explicit formula for the dual exponential map*

$$\exp^* \,:\, H^1\big(G_{L_n}, H^3_{\text{ét}}(S(N)_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p(2))\big) \to \mathrm{Fil}^2 H^3_{\mathrm{dR}}(S(N)) \otimes L_n, \quad (p,N)=1,$$

for the Siegel threefold $S(N)$ [67]. Here $L_n = \mathbf{Q}_p(\mu_{Np^n})$. Lemma and Ochiai compute the image of diagonal cohomology classes constructed using the embedding of the direct product of two modular curves in $S(N)$.

7. Iwasawa theory of p-adic representations

7.1. In this section, we discuss Iwasawa theory of p-adic representations of local fields in the classical setting of the p-cyclotomic extension K_∞/K. Set $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ and $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$. Let V be a p-adic representation of the Galois group G_K of a p-adic local field K. Fix a $\mathbf{Z}_p$-lattice T of V stable under the action of G_K. The projective limits

$$H^i_{\mathrm{Iw}}(K_\infty/K, T) = \varprojlim_n H^i(G_{K_n}, T), \qquad K_n = K(\mu_{p^n})$$

are finitely generated modules over the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[\Gamma_K]]$. It it easy to see that $H^0_{\mathrm{Iw}}(K_\infty/K, T) = 0$. From local duality if follows that $H^2_{\mathrm{Iw}}(K_\infty/K,T)$ is the Pontriagin dual of $((V^*(1)/T^*(1))^{H_K}$. The first Iwasawa cohomology $H^1_{\mathrm{Iw}}(K_\infty/K,T)$ is a Λ-module of rank $\dim(V)$ [77]. The (φ,Γ_K)-module $\mathbf{D}(T)$ is equipped with a left inverse ψ of the Frobenius operator [11] .

Theorem 8 (Fontaine)**.** *The Iwasawa cohomology of T is isomorphic to the cohomology of the complex*

$$\mathbf{D}(T) \xrightarrow{\psi-1} \mathbf{D}(T).$$

We refer to [20] for the proof of this result.

Local duality induces a pairing

$$H^1_{\mathrm{Iw}}(K_\infty/K,T) \times H^1_{\mathrm{Iw}}(K_\infty/K, T^*(1)) \to \Lambda,$$

which can be explicitly described in terms of (φ,Γ_K)-modules [28, Chapitre VIII].

7.2. Fix a generator ε of the Tate module $\mathbf{Z}_p(1)$. Then we have a well-defined twist map

$$\mathrm{Tw}_k \,:\, H^1_{\mathrm{Iw}}(G_K,T) \xrightarrow{\sim} H^1_{\mathrm{Iw}}(G_K, T(k)), \qquad x \mapsto x \otimes \varepsilon^{\otimes k},$$

[11] ψ acts on the ring $\mathbf{A} := \widehat{O}^{\mathrm{ur}}_{\mathrm{cyc}}$ as $\psi(x) = \frac{1}{p}\mathrm{Tr}_{\mathbf{A}/\varphi(\mathbf{A})}(x)$. Since $\mathbf{D}(T) = (\mathbf{A}\otimes T)^{H_K}$, this action extends to $\mathbf{D}(T)$.

which is a non Galois equivariant isomorphism of $\mathbf{Z}_p$-modules.

The conjectural relation between Euler systems and special values of L-functions suggests that the Bloch–Kato exponential maps of cyclotomic twists $V(k)$ $(k \gg 0)$ should have nice p-adic interpolation properties. In [78], Perrin-Riou proved that this is indeed the case for crystalline representations of absolutely unramified local fields. To state her result we should introduce the large Iwasawa algebra. For any $n \geqslant 1$ set $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$. Write Γ_K as the direct sum of its pro-p-part Γ_1 and a cyclic group Δ_K of order prime to p. Choose a generator γ_1 of Γ_1 and define

$$\mathscr{H}(\Gamma_1) := \{f(\gamma_1 - 1) \mid f(z) \in \mathbf{Q}_p[[z]] \\ \text{and converges on the open unit } p\text{-adic disk } |z|_p < 1\}.$$

Let $\mathscr{H}(\Gamma) = \mathbf{Z}_p[\Delta_K] \otimes_{\mathbf{Z}_p} \mathscr{H}(\Gamma_1)$. For each $n \geqslant 1$, the natural projection

$$H^1_{\mathrm{Iw}}(G_K, T) \to H^1(G_{K_n}, T)$$

extends to a map $\mathrm{pr}_n : \mathscr{H}(\Gamma) \otimes_\Lambda H^1_{\mathrm{Iw}}(G_K, T) \to H^1(G_{K_n}, V)$. Note that $\mathscr{H}(\Gamma)$ is isomorphic to the algebra of distributions on Γ and the projecton pr_n can be interpreted as integration.

The precise formulation of Perrin-Riou's result is very technical by nature. For any crystalline representation V of G_K she constructs a map

$$\mathrm{Exp}_V : \mathbf{D}_{\mathrm{cris}}(V) \otimes \Lambda \to \mathscr{H}(\Gamma) \otimes_\Lambda H^1_{\mathrm{Iw}}(G_K, T)$$

interpolating Bloch–Kato exponentials in the following sense: for each $k \gg 0$ and $n \geqslant 0$ there exists a commutative diagram

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{cris}}(V) \otimes \Lambda \arrow[r, "\mathrm{Tw}_k \circ \mathrm{Exp}_V"] \arrow[d] & \mathscr{H}(\Gamma) \otimes_\Lambda H^1_{\mathrm{Iw}}(G_K, T(k)) \arrow[d, "\mathrm{pr_n}"] \\
\mathbf{D}_{\mathrm{cris}}(V) \otimes_K K_n \arrow[r, "(-1)^k (k-1)! \exp_{V(k), K_n}"] & H^1(G_{K_n}, V(k))
\end{tikzcd}$$

with an explicit left vertical map. To sum up, the map Exp_V interpolates both the exponential maps along the cyclotomic tower K_n and for different twists $V(k)$. The condition $k \gg 0$ is crucial because the exponential map $\exp_{V(k)}$ vanishes if $k \in \mathbf{Z}$ is small enough for trivial reason.

7.3. Perrin-Riou conjectured an explicit formula for the local Iwasawa pairing

$$\mathscr{H}(\Gamma) \otimes_\Lambda H^1_{\mathrm{Iw}}(G_K, T) \times \mathscr{H}(\Gamma) \otimes_\Lambda H^1_{\mathrm{Iw}}(G_K, T^*(1)) \to \mathscr{H}(\Gamma)$$

in terms of the exponentials Exp_V and $\mathrm{Exp}_{V^*(1)}$ (Perrin-Riou's explicit reciprocity law). Roughly speaking, it says that for $k \ll 0$, Exp_V interpolates the maps $(\exp^*_{V(k)})^{-1}$. We refer the reader to [7, 11, 26] for different proofs of this conjecture.

The approaches of [7] and [11] are based on the theory of (φ, Γ)-modules. In [7], we construct a family of maps $\Sigma_{T,k,n} : \mathbf{D}_{\mathrm{cris}}(T) \otimes \Lambda \to H^1(G_{K_n}, T(k))$ for almost all $k \in \mathbf{Z}$ and all $n \geqslant 1$ satisfying some interpolation properties. For $k \gg 0$, these maps coincide with the projections of $\mathrm{Tw}_k \circ \mathrm{Exp}_V$ on $H^1(G_{K_n}, V(k))$. Perrin-Riou's reciprocity law is deduced from the computation of cup products $H^1(G_{K_n}, T(1-k)) \times H^1(G_{K_n}, T(k)) \to \mathbf{Z}_p$ using the theory of (φ, Γ)-modules. This description of the map Exp_V can be useful for the study of congruences in Iwasawa theory.

Berger uses the theory of (φ, Γ)-modules over the Robba ring $\mathscr{R}$ [27], which we don't recall here. There exists an isomorphism

$$\mathscr{H}(\Gamma) \otimes_\Lambda H^1_{\mathrm{Iw}}(G_K, T) \simeq \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1},$$

which can be viewed as an "analytic" version of Theorem 8 [62]. (Here $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ denotes the (φ, Γ_K)-module over $\mathscr{R}$ attached to V.) Berger constructs Exp_V explicitly as a map $\mathbf{D}_{\mathrm{cris}}(V) \otimes \Lambda \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\psi=1}$. The explicit reciprocity law is deduced from this construction.

If $V = \mathbf{Q}_p(1)$, we recover the explicit formula of Coleman [23], and the reciprocity law of Bloch and Kato for $\mathbf{Q}_p(1-r)$ can be deduced from this case using interpolation properties of $\mathrm{Exp}_{\mathbf{Q}_p(1)}$.

The determinant of the large exponential map is computed in [9] (Perrin-Riou's $\delta_{\mathbf{Z}_p}(V)$-conjecture). This implies a duality formula for Tamagawa numbers of crystalline representations in cyclotomic extensions. It is expected that such a formula holds for all de Rham representations (Kato's ε-conjecture). See also [69].

7.4. Perrin-Riou's theory [79] provides a coherent approach to construction of p-adic L-functions from Euler systems using the large exponential map Exp_V, and it would be desirable to extend the above results to a more general setting. Let us mention two important directions:

a) Nakamura generalized Berger's construction of the large exponential map to (φ, Γ)-modules over arbitrary local fields with applications to Kato's ε-conjecture [72, 73].

b) Berger–Fourquaux and Schneider–Venjakob developed Iwasawa theory of analytic p-adic representations over Lubin–Tate towers [14, 81].

These developments are discussed in the recent survey [89] of Venjakob, and we refer the reader to his paper for more detail.

8. Higher dimensional local fields

8.1. Classical local fields are local fields of dimension one. A d-dimensional local field is a complete discrete valuation field whose residue field is a local field of dimension $d-1$. In this hierarchy, finite fields are local fields of dimension 0. Archetypical examples of local fields of dimension 2 are $\mathbf{F}_p((t_1))((t_2))$ and the field of fractions of the p-adic completion of $\mathbf{Z}_p[[t]][1/t]$, usually denoted as $\mathbf{Q}_p\{\{t\}\}$.

Parshin [75, 76] (in characterstic p case) and Kato [56, 57] (in the general case) developed class field theory for higher dimensional fields. For a local field F of dimension d, they constructed a functorial reciprocity homomorphism

$$\theta_F \,:\, \mathrm{K}_d(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F),$$

where $\mathrm{K}_d(F)$ denotes the dth Milnor K-group of F. This map satisfies the fundamental property:

$$\theta_{L/F} \,:\, \mathrm{K_d}(F)/N_{L/F}(\mathrm{K_d}(F)) \xrightarrow{\sim} \mathrm{Gal}(L/F) \quad \text{if } L/F \text{ is finite abelian.}$$

If $d=0$, one has $\mathrm{K_d}(F) = \mathbf{Z}$, and the map θ_F coincides with the map $\mathbf{Z} \to \mathrm{Gal}(F^{\mathrm{sep}}/F)$ which sends 1 to the Frobenius automorphism over F. If $d=1$, one has $\mathrm{K}_1(F) = F^*$, and θ_F coincides with the classical reciprocity map. We refer the reader to the notes [71] for a detailed introduction to higher dimensional local fields.

Assume that $\mathrm{char}(F) = 0$. If F contains the group of nth roots of unity, the multilinear map

$$(\,\cdot\,,\,\cdot\,,\ldots,\,\cdot\,)_n \,:\, \underbrace{F^* \times F^* \times \cdots \times F^*}_{d+1} \to \mu_n, \qquad (a_1,\ldots,a_d,a_{d+1}) \mapsto \frac{\sqrt[n]{a_{d+1}}^{\,\theta_F(a_1,\ldots,a_d)}}{\sqrt[n]{a_{d+1}}}.$$

is the higher dimensional analog of the Hilbert symbol. It can be alternatively defined as the composition of the cup product

$$H^1(G_F,\mu_n) \times H^1(G_F,\mu_n) \times \cdots \times H^1(G_F,\mu_n) \to H^{d+1}(G_F,\mu_n^{\otimes d+1}) \simeq \mu_n$$

with the coboundary map $F^* \to H^1(G_F, \mu_n)$. In particular, it satisfies the symbol property and factorizes through $\mathrm{K}_{d+1}(F)$.

In [94, 95], Vostokov generalized his explicit formulas to higher dimensional local fields of unequal characteristic. Let p denote the characteristic of the first residue field of F. As in the one-dimensional case, Vostokov constructed an explicit multilinear map

$$\langle \cdot, \cdot, \dots, \cdot \rangle_{p^n} \,:\, \underbrace{F^* \times F^* \times \cdots \times F^*}_{d+1} \to \mu_{p^n}$$

in terms of expansions of $a_1, \dots, a_{d+1}$ in power series of a fixed uniformizer and proved that it coincides with the generalized Hilbert symbol on the higher dimensional analog of the Shafarevich basis. The construction of this pairing does not depend on the reciprocity map. This fact was used by Fesenko in his independent approach to higher local class field theory [31, 32]. For the more general case of higher dimensional fields with arbitrary perfect last residue field, we refer to the papers [33] and [96].

8.2. Kato's paper [58] covers the case of higher dimensional local fields and gives an alternative proof of Vostokov formulas based on syntomic cohomology under the additional assumption $p > d + 1$. Abrashkin and Jenni [4] showed how explicit formulas for the generalized Hilbert symbol can be deduced from the Artin–Schreier–Witt formula using the fields of norms of higher dimensional local fields [3] and obtained Vostokov's formula as a special case of their construction. Their approach is a direct generalization of [1]. The higher dimensional version of Coleman's formula was obtained in [103] following the method discussed in Section 4.3.

We also remark that Fukaya [43] generalized the method of our paper [6] to formal groups over higher dimensional local fields.

8.3. Kato's explicit reciprocity law [60] can be very roughly seen as a generalization of (16) to higher dimensional setting. We can not discuss it in more detail here because even its formulation is very technical and involves constructions of relative p-adic Hodge theory. Fukaya [44] built an analog of Coleman power series for the group K_2 of a two dimensional local field.

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