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Нерегулярность и топологические индексы в деревьях слов Фибоначчи и модифицированный индекс слов Фибоначчи

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Аннотация

В данной статье представлена концепция индекса слов Фибоначчи FWI, нового топологического индекса, полученного на основе индекса Альбертсона, применительно к деревьям, построенным из слов Фибоначчи. Опираясь на классическую последовательность Фибоначчи и ее обобщения, мы исследуем структурные свойства деревьев из слов Фибоначчи и меры их нерегулярности, основанные на степенях. Мы определяем FWI и его варианты, включая полную нерегулярность и модифицированный индекс слов Фибоначчи, где он определяется как

$$\mathrm{FWI}^*(\mathscr{T}) = \sum_{n,m\in E(\mathscr{T})} [\deg F_n^2 - \deg F_m^2],$$

и устанавливаем фундаментальные неравенства, связывающие эти индексы с максимальной степенью нижележащих деревьев. Наши результаты распространяют известные инварианты графов на комбинаторику слов Фибоначчи, позволяя по-новому взглянуть на их алгебраические и топологические характеристики. Кроме того, мы приводим аналитические выражения для чисел Фибоначчи и их порождающих функций, подкрепленные формулой Бине, чтобы облегчить вычисление этих индексов. Теоретические разработки иллюстрируются примерами, включая подробные конструкции словесных деревьев Фибоначчи и их степенных распределений. Данная работа открывает возможности для дальнейшего изучения инвариантов графов на основе слов и их применения в комбинаторике и теоретической информатике.

Ключевые слова: Фибоначчи, слова, деревья, топология, индексы, иррегулярность.

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Irregularity and topological indices in Fibonacci word trees and modified Fibonacci word index

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Abstract

This paper introduces the concept of the Fibonacci Word Index FWI, a novel topological index derived from the Albertson index, applied to trees constructed from Fibonacci words. Building upon the classical Fibonacci sequence and its generalizations, we explore the structural properties of Fibonacci word trees and their degree-based irregularity measures. We define the FWI and its variants, including the total irregularity and modified Fibonacci Word Index where it defined as

$$\mathrm{FWI}^*(\mathscr{T}) = \sum_{n,m \in E(\mathscr{T})} [\deg F_n^2 - \deg F_m^2],$$

and establish foundational inequalities relating these indices to the maximum degree of the underlying trees. Our results extend known graph invariants to the combinatorial setting of Fibonacci words, providing new insights into their algebraic and topological characteristics. Additionally, we present analytical expressions involving Fibonacci numbers and their generating functions, supported by Binet's formula, to facilitate computation of these indices. The theoretical developments are illustrated with examples, including detailed constructions of Fibonacci word trees and their degree distributions. This work opens avenues for further investigation of word-based graph invariants and their applications in combinatorics and theoretical computer science.

Keywords: Fibonacci, Word, Trees, Topological, Indices, Irregularity.

Bibliography: 27 titles.

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1. Introduction

Throughout this paper,[1] let $\mathscr{T}$ be a tree of Fibonacci word F, the Fibonacci sequence, recursively defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$, has been generalised in several ways. In 2000, Divakar Viswanath in [8] studied random Fibonacci sequences given by $t_1 = t_2 = 1$ and

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$t_n = \pm t_{n-1} \pm t_{n-2}$ for all $n \geq 3$. Here each $\pm$ is chosen to be $+$ or with probability $1/2$, and are chosen independently. Viswanath proved that

$$\lim_{n\to\infty} \sqrt[n]{|t_n|} = 1.13198824\ldots$$

The Fibonacci word is a sequence of binary strings over the alphabet $\{0,1\}$, constructed recursively in a manner analogous to the Fibonacci sequence but using string concatenation instead of numerical addition. It is defined as $S_0 = 0$, $S_1 = 01$, and $S_n = S_{n-1}S_{n-2}$ for $n \geq 2$, where concatenation combines the strings, furthermore see [1, 3, 5]. In 1997, M. O. Albertson in [19] mention to the imbalance of an edge uv by $imb(uv) = |d_u - d_v|$, we considered graph G regular if all of its vertices have the exact same degree, then irregularity measure define in [19, 10, 17] as:

$$\mathrm{irr}(G) = \sum_{uv\in E(G)} |d_u(G) - d_v(G)|.$$

Let $M_1(\mathscr{T})$ and $M_2(\mathscr{T})$ be the first and the second Zagreb index of Fibonacci word are defined (furthermore see [16, 13, 19]) as:

$$M_1(\mathscr{T}) = \sum_{i=1}^{n} \deg F_i^2, \quad \text{and} \quad M_2(G) = \sum_{n,m\in\mathbb{N}} \deg_{F_n}(\mathscr{T}) \deg_{F_m}(\mathscr{T}).$$

Definition 1. *The length of the word $|\mathbf{w}|$ is the number of letters contained in it. The empty word is denoted by ε.*

Definition 2. *Infinite words as functions $\mathbf{w} : \mathbb{N} \to \Sigma$. The set of all finite words over Σ is denoted by Σ^*, and $\Sigma^+ = \Sigma^*\backslash\{\varepsilon\}$; the set of all infinite words is denoted by $\Sigma^{\mathbb{N}}$.*

In 2013, Ramirez, J.L, et al. In [11] mention to k-Fibonacci Words are an extension of the Fibonacci word notion that generalises Fibonacci word features to higher dimensions. These words were investigated for their distinct curves and patterns. Recently, in 2023 Rigo, M., Stipulanti, M., & Whiteland, M.A. in [2] mentioned the Thue-Morse sequence, which is the fixed point of the substitution $0 \to 01, 1 \to 10$, has unbounded 1-gap k-binomial complexity for $k \geq 2$. Also, we want to mention for a Sturmian sequence and $g \geq 1$, all of its long enough factors are always pairwise $g - gapk$-binomially inequivalent for any $k \geq 2$. Furthermore, for Fibonacci sequence with trees see in [14, 2, 6].

This paper is organized as follows. In Section 1, we observe the important concepts to our work including literature view of most related papers, in Section 2 We have provided a preface through some of the important theories and properties we have utilised in understanding the work, in Section 3 we introduced the main result of our paper and the goal of this paper about Fibonacci word, in Section 4 we computing the advanced result in density of Fibonacci word.

2. Preliminaries

The infinite Fibonacci word, obtained as the limit $n \to \infty$, is a Sturmian word with a slope related to the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. This construction makes the Fibonacci word a significant object in combinatorics, with applications in coding theory and the study of non-repetitive sequences. The inclusion of negative indices is meant to simplify the computation of $F_{p,n}$ for $n = 2, 3, \ldots, p$.

Proposition 1. *[25] For $1 < n \leq p$, the term $F_{p,n}$ is given by 2^{n-2}.*

Definition 3 (Standard Fibonacci words [12]). *Let Σ be an alphabet with $|\Sigma| \geq 2$ and let $u, v \in \Sigma^+$. The n^{th} standard Fibonacci words are defined recursively as:*

$$f_1(u,v) = u, f_2(u,v) = v,$$
$$f_n(u,v) = f_{n-1}(u,v) \cdot f_{n-2}(u,v), n \geq 3.$$

A natural way to generalize this sequence is to consider a recursion where each term is the sum of the previous p terms. For a positive integer p, the *p-generalized Fibonacci numbers*, denoted by $F_{p,n}$, are defined by the recursion:

$$F_{p,n} = \sum_{i=1}^{p} F_{p,n-i}, \quad n > 1,$$

with the initial values $F_{p,n} = 0$ for $n = -p+2, -p+1, \ldots, 0$, and $F_{p,n} = 1$ for $n = 1$.

Теорема 1 (Binet's Formula [18]). *The n-th Fibonacci number F_n is given by:*

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio), $\psi = \frac{1-\sqrt{5}}{2} = -\phi^{-1}$, and n is a non-negative integer.

Definition 4 (Correlation [21]). *For every pair of words $(u,v) \in A^n \times A^m$, the correlation of u over v is the word $C_{u,v} \in A^n$ such that for all $k \in [n]$,*

$$C_{u,v}[k] = \begin{cases} 1 & \text{if } \forall i \in [n],\ j \in [m],\ \text{with } i = j+k,\ \text{when } u[i] = v[j], \\ 0 & \text{otherwise.} \end{cases}$$

S. A. Hosseini et al. in [27] was introduced the formal definition of a Kragujevac tree. Since all the researches [4, 7, 9] were done at the University of Kragujevac. Let $B_1, \ldots, B_k$ be a branches of tree given in [27].

Definition 5. *A proper Kragujevac tree is a tree possessing a central vertex of degree at least 3, to which branches of the form B_1 and/or B_2 and/or B_3 and/or …are attached.*

Proposition 2. *Let F_n be a Fibonacci word, and let $k > 0$ an integer number, then*

$$\sum_{k=1} \frac{F_{2n+k}F_{2n-k}}{k!.2^n} = \frac{L_{4n}(e-1) - (e^{-\phi^2} - 1) - (e^{-\phi^{-2}} - 1)}{5 \cdot 2^n}$$

where $L_{4n} = F_{4n-2} + F_{4n+2}, \phi = \frac{1+\sqrt{5}}{2}$.

Доказательство. Let F_n be the n-th Fibonacci word. Typically refers to the n-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Fibonacci words are sequences over an alphabet (e.g., 0,1) generated by a substitution rule, but they are less commonly used in such analytical sums. Given the structure of the sum, involving factorials and numerical operations, it's reasonable to interpret F_n as the Fibonacci number sequence. Thus, we proceed assuming F_n is the Fibonacci number. The Fibonacci numbers by using Binet's formula: $F_m = \frac{\phi^m - \psi^m}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2} = -\frac{1}{\phi}$, with $\phi + \psi = 1$, $\phi - \psi = \sqrt{5}$, and $\phi\psi = -1$.
Using Binet's formula, we write:

$$F_{2n+k} = \frac{\phi^{2n+k} - \psi^{2n+k}}{\sqrt{5}}, \quad F_{2n-k} = \frac{\phi^{2n-k} - \psi^{2n-k}}{\sqrt{5}}.$$

Thus,

$$F_{2n+k}F_{2n-k} = \frac{(\phi^{2n+k} - \psi^{2n+k})(\phi^{2n-k} - \psi^{2n-k})}{5}.$$

Expanding the numerator as $(\phi^{2n+k}-\psi^{2n+k})(\phi^{2n-k}-\psi^{2n-k})=\phi^{4n}-\phi^{2n+k}\psi^{2n-k}-\phi^{2n-k}\psi^{2n+k}+\psi^{4n}$. Since $\phi^{2n+k}\psi^{2n-k} = (\phi\psi)^{2n-k}\phi^{2k} = (-1)^{2n-k}\phi^{2k} = (-1)^{k+1}\phi^{2k}$ (because $2n-k$ is odd when k is odd and even when k is even), and similarly for the other term, we need to handle the sum carefully. Instead, let's compute the term:

$$\frac{F_{2n+k}F_{2n-k}}{k!\cdot 2^n} = \frac{1}{5\cdot k!\cdot 2^n}(\phi^{2n+k} - \psi^{2n+k})(\phi^{2n-k} - \psi^{2n-k}).$$

The sum becomes:

$$\sum_{k=1}^{\infty}\frac{F_{2n+k}F_{2n-k}}{k!\cdot 2^n} = \frac{1}{5\cdot 2^n}\sum_{k=1}^{\infty}\frac{(\phi^{2n+k} - \psi^{2n+k})(\phi^{2n-k} - \psi^{2n-k})}{k!}.$$

Expand the product inside the sum: $(\phi^{2n+k} - \psi^{2n+k})(\phi^{2n-k} - \psi^{2n-k}) = \phi^{4n} - \phi^{2n+k}\psi^{2n-k} - {}$ $- \phi^{2n-k}\psi^{2n+k} + \psi^{4n}$. So the summand is: $\frac{1}{k!}\left(\phi^{4n} - \phi^{2n+k}\psi^{2n-k} - \phi^{2n-k}\psi^{2n+k} + \psi^{4n}\right)$. We split the sum: $\sum_{k=1}^{\infty}\frac{\phi^{4n}-\phi^{2n+k}\psi^{2n-k}-\phi^{2n-k}\psi^{2n+k}+\psi^{4n}}{k!}$. Then

$$\sum_{k=1}^{\infty}\frac{\phi^{4n} - (\phi/\psi)^k - (\psi/\phi)^k + \psi^{4n}}{k!} = \phi^{4n}(e-1) - (e^{-\phi^2} - 1) - (e^{-\phi^{-2}} - 1) + \psi^{4n}(e-1).$$

Asb desire. □

3. Main Result

In this section, Fibonacci Word Index FWI is defined in Definition 6, a novel topological index derived from the Albertson index, applied to trees constructed from Fibonacci words. Building upon the classical Fibonacci sequence and its generalizations. The maximum degree Δ of FWI in Proposition 3 defined as: $\Delta(\mathscr{T}) = \{\max(\deg V(\mathscr{T}) \mid V \in \mathscr{T}\}$.

3.1. Fibonacci Word Index

In this subsection, we will introduce a “Fibonacci Word Index” according to the important topological index, it known a “Albertson Index” and its variants, including the total irregularity and modified Fibonacci Word Index, and establish foundational inequalities relating these indices to the maximum degree of the underlying trees.

Definition 6. *Let F_n, F_m be a two Fibonacci word, let $\mathscr{T}$ be a tree of Fibonacci word, then Albertson index of Fibonacci word given us a Fibonacci Word Index, define by:*

$$\mathrm{FWI}(\mathscr{T}) = \sum_{n,m\in E(\mathscr{T})}|\deg F_n - \deg F_m|.$$

The total irregularity of Fibonacci Word Index $\mathrm{FWI}_t(\mathscr{T})$ is given by:

$$\mathrm{FWI}_t(\mathscr{T}) = \sum_{\{n,m\}\subseteq E(\mathscr{T})}|\deg F_n - \deg F_m|.$$

The *modified Fibonacci word index* $\mathrm{FWI}^*(\mathscr{T})$ is define as:

$$\mathrm{FWI}^*(\mathscr{T}) = \sum_{n,m\in E(\mathscr{T})}[\deg F_n^2 - \deg F_m^2].$$

Furthermore, see [26] for understanding clearly both of Proposition 3,4 and Lemma 1. The star S_n graph of Fibonacci word defined by:

$$\mathrm{FWI}^*(S_n) = \sum_{u,v \in E(S_n)} \left(\deg(F_u)^2 - \deg(F_v)^2\right) = \sum_{(u,v) \in E(S_n)} \left(\deg(u)^2 - \deg(v)^2\right).$$

Proposition 3. *Let $\mathscr{T}$ be a tree with maximum degree Δ then* $\mathrm{FWI}^*(\mathscr{T}) \geqslant \Delta(\Delta^2 - 1)$ *with equality if and only if $\mathscr{T}$ is isomorphic to either a path or a tree containing only one vertex of* $\deg(V(\mathscr{T})) > 2$.

Доказательство. Assume $\Delta \leqslant 2$, in this case, the result is clear and hence we assume that $\Delta \geqslant 3$. If $v \in V(\mathscr{T})$ has maximum degree then there are P pendant vertices namely $w_1, w_2, \cdots, w_{d_v}$ in $\mathscr{T}$ such that the paths $v - w_1, v - w_2, \cdots, v - w_P$ are pairwise internally disjoint. Then,

$$\begin{aligned}
&\left|(P_v)^2 - (P_{w_{1,1}})^2\right| + \left|(P_{w_{1,1}})^2 - (P_{w_{1,2}})^2\right| + \cdots + \left|(P_{w_{1,r}})^2 - (P_{w_1})^2\right| \\
&\geqslant \left[(P_v)^2 - (P_{w_{1,1}})^2\right] + \left[(P_{w_{1,1}})^2 - (P_{w_{1,2}})^2\right] + \cdots + \left[(P_{w_{1,r}})^2 - (P_{w_1})^2\right] = \Delta^2 - 1\,.
\end{aligned}$$

We note that the equality

$$\left|(P_v)^2 - (P_{w_{1,1}})^2\right| + \left|(P_{w_{1,1}})^2 - (P_{w_{1,2}})^2\right| + \cdots + \left|(P_{w_{1,r}})^2 - (P_{w_1})^2\right| = \Delta^2 - 1$$

holds if and only if the degrees of successive vertices along the path from v to w_1 decrease monotonously. $\square$ Actually, Proposition 3 provided us a new results shown in Proposition 4 for the path P_n and the star S_n of tree $\mathscr{T}$. This results will be employee in Lemma 1. In fact, for extended more we show in Figure 1 Fibonacci word F_{10} as an example of tree of Fibonacci word.

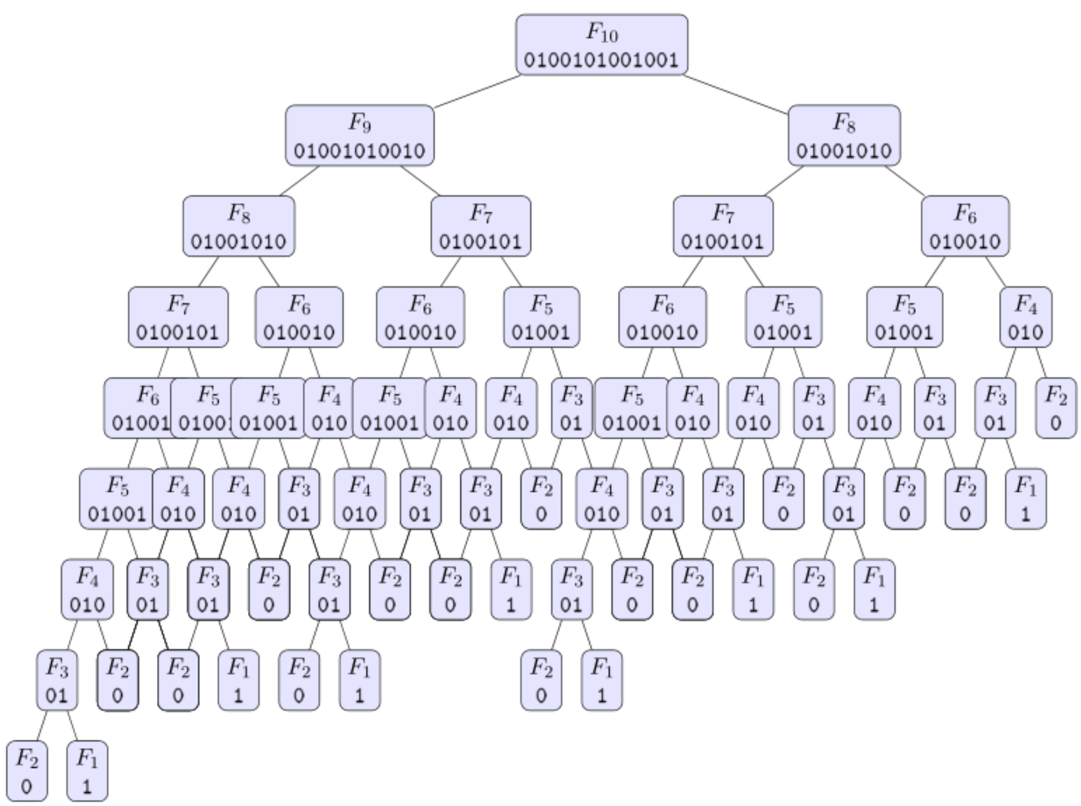

Рис. 1: An example of Tree $\mathscr{T}$ of Fibonacci word for base F_{10}.

Proposition 4. *Let $P_n(\mathscr{T})$ be the path of a tree $\mathscr{T}$ of n-vertex where $n \geq 5$, let $S_n(\mathscr{T})$ be the star of $\mathscr{T}$, then* $\mathrm{FWI}^*(P_n) < \mathrm{FWI}^*(\mathscr{T}) < \mathrm{FWI}^*(S_n)$.

Let α be a fixed vertex of $\mathscr{T}$. Let $N(\alpha)$, be a set of fixed number let $L(\alpha)$, $E(\alpha)$, $R(\alpha)$ be a sets of fixed vertices in $\mathscr{T}$ according to base $N(\alpha)$, where

$$\begin{cases} L(\alpha) = \{\beta \in N(\alpha) : \ \deg(F_\beta) < \deg(F_\alpha)\}, \\ E(\alpha) = \{\beta \in N(\alpha) : \ \deg(F_\beta) = \deg(F_\alpha)\}, \\ R(\alpha) = \{\beta \in N(\alpha) : \ \deg(F_\beta) > \deg(F_\alpha)\}. \end{cases}$$

Proposition 5. *The number of elements in $L(\alpha)$, $E(\alpha)$ and $R(\alpha)$ are denoted by l_α, e_α and r_α, respectively. Clearly,* $\deg(F_\alpha) = l_\alpha + e_\alpha + r_\alpha$.

Лемма 1. *Let $\mathscr{T}$ be a tree of Fibonacci word, let α and β are non-adjacent vertices in $\mathscr{T}$ where* $\deg(F_\alpha \geqslant \deg(F_\beta)$ *then*

$$\mathrm{FWI}^*(\mathscr{T}) = \mathrm{FWI}^*(\mathscr{T} - \alpha\beta) + 2[(2\deg(F_\alpha) + 1)r_\alpha + (2\deg(F_\beta) + 1)r_\beta] - \Gamma(\mathscr{T}),$$

where $\Gamma(\mathscr{T}) = 3\deg(F_\alpha)(\deg(F_\alpha) + 1) - \deg(F_\beta)(\deg(F_\beta) + 1)$.

Доказательство. Firstly, we provided a motivation to stabilise the Fibonacci word by Proposition 3,4 as $\mathrm{FWI}^*(\mathscr{T}) \geqslant \Delta(\Delta^2 - 1)$, then we have:

$$\begin{aligned} \mathrm{FWI}^*(\mathscr{T} - \alpha\beta) - \mathrm{FWI}^*(\mathscr{T}) = {} & (\deg(F_\alpha) + 1)^2 - (\deg(F_\beta) + 1)^2 \\ & + \sum_{x \in N(\alpha)} \Big(\big|(\deg(F_\alpha) + 1)^2 - (\deg(F_x))^2\big| - \big|(\deg(F_\alpha))^2 - (\deg(F_x))^2\big| \Big) \\ & + \sum_{y \in N(\beta)} \Big(\big|(\deg(F_\beta) + 1)^2 - (\deg(F_y))^2\big| - \big|(\deg(F_\beta))^2 - (\deg(F_y))^2\big| \Big). \end{aligned}$$

Now, consider $N(\alpha) = L(\alpha) \cup E(\alpha) \cup R(\alpha)$. We aim to determine the union of the sets $L(\alpha)$, $E(\alpha)$, and $R(\alpha)$, defined as follows:

$$\begin{cases} L(\alpha) = \{\beta \in N(\alpha) : \deg(F_\beta) < \deg(F_\alpha)\}, \\ E(\alpha) = \{\beta \in N(\alpha) : \deg(F_\beta) = \deg(F_\alpha)\}, \\ R(\alpha) = \{\beta \in N(\alpha) : \deg(F_\beta) > \deg(F_\alpha)\}, \end{cases}$$

where $N(\alpha)$ is a set containing elements β, and F_β and F_α are polynomials associated with elements β and α, respectively, with $\deg(\cdot)$ denoting the degree of a polynomial. The objective is to compute $L(\alpha) \cup E(\alpha) \cup R(\alpha)$. The sets $L(\alpha)$, $E(\alpha)$, and $R(\alpha)$ are subsets of $N(\alpha)$, defined based on the degree of the polynomial F_β relative to the degree of F_α. Specifically:

- $L(\alpha)$ contains all $\beta \in N(\alpha)$ such that $\deg(F_\beta) < \deg(F_\alpha)$.
- $E(\alpha)$ contains all $\beta \in N(\alpha)$ such that $\deg(F_\beta) = \deg(F_\alpha)$.
- $R(\alpha)$ contains all $\beta \in N(\alpha)$ such that $\deg(F_\beta) > \deg(F_\alpha)$.

For any $\beta \in N(\alpha)$, the value of $\deg(F_\beta)$ must satisfy exactly one of the following mutually exclusive conditions relative to $\deg(F_\alpha)$:

1. $\deg(F_\beta) < \deg(F_\alpha)$, placing β in $L(\alpha)$,
2. $\deg(F_\beta) = \deg(F_\alpha)$, placing β in $E(\alpha)$,

3. $\deg(F_\beta) > \deg(F_\alpha)$, placing β in $R(\alpha)$.

Same results if we consider $N(\alpha) = L(\beta) \cup E(\beta) \cup R(\beta)$, then we have:

$$\begin{aligned}\mathrm{FWI}^*(\mathscr{T} - \alpha\beta) - \mathrm{FWI}^*(\mathscr{T}) &= (\deg F_\alpha - \deg F_\beta)(\deg F_\alpha + \deg F_\beta + 2) \\ &\quad + (2\deg F_\alpha + 1)(e_\alpha + l_\alpha - r_\alpha) \\ &\quad + (2\deg F_\beta + 1)(e_\beta + l_\beta - r_\beta)\,,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathrm{FWI}^*(\mathscr{T} - \alpha\beta) - \mathrm{FWI}^*(\mathscr{T}) &= 3\deg F_\alpha(\deg F_\alpha + 1) + \deg F_\beta(\deg F_\beta - 1) \\ &\quad - 2[(2\deg F_\alpha + 1)r_\alpha + (2\deg F_\beta + 1)r_\beta]\,.\end{aligned}$$

As desire. □

Лемма 2. *Let $\mathscr{T} = (V, E)$ be a tree of Fibonacci word, let $\mathrm{FWI}(\mathscr{T})$ be a Fibonacci word index, let $m = |E(\mathscr{T})|$ then*

$$\mathrm{FWI}(\mathscr{T}) \leqslant \sqrt{m\,(\mathrm{FWI}^*(\mathscr{T}) - 2M_2(\mathscr{T}))}.$$

Теорема 2. *According to Lemma 2, let $k \geq 1$ is an integer, then we have:*

$$\mathrm{FWI}(\mathscr{T}) \leqslant (m-1)^{1-\frac{1}{k}}\,(\mathrm{FWI}^*(\mathscr{T}))^{\frac{1}{k}}\,.$$

Доказательство. In fact, from Lemma 2 we have $\mathrm{FWI}(\mathscr{T}) \leqslant \sqrt{m\,(\mathrm{FWI}^*(\mathscr{T}) - 2M_2(\mathscr{T}))}$. Thus, we have:

$$\begin{aligned}\mathrm{FWI}(\mathscr{T}) &= \sum_{n,m\in E(\mathscr{T})} |\deg F_n - \deg F_m| \\ &\leq \left(\sum_{n,m\in E(\mathscr{T})} [\deg F_n^2 - \deg F_m^2]\right)^{\frac{1}{k}} \leq (m-1)\,(\mathrm{FWI}^*(\mathscr{T}))^{\frac{1}{k}}\,.\end{aligned}$$

□

Теорема 3. *Let $\mathscr{T}$ be a tree of Fibonacci word, let $n \geq 1$, $k \geq 2$ are an integer, let S_n the star graph of Fibonacci word, then we have:*

$$\mathrm{FWI}^*(S_n) \geqslant \mathrm{irr}(S_n) \geqslant 2^{\frac{1}{k}}.$$

Доказательство. It is clear to see $\mathrm{FWI}^*(S_n) \geqslant 2^{\frac{1}{k}}$ according to Theorem 2 and Lemma 2, then for $n > 2$ we have $\mathrm{irr}(S_n) = (n-1)(n-2)$, when $n > k$ we have $\mathrm{irr}(S_n) \geqslant 2^{\frac{1}{k}}$. Now, in both cases we have $2^{\frac{1}{k}}$ at last, we need to prove $\mathrm{FWI}^*(S_n) \geqslant \mathrm{irr}(S_n)$. Recall that for the star graph S_n, the degree sequence consists of one central vertex c with degree $\deg(c) = n-1$, and $n-1$ leaves each with degree 1. By definition,

$$\mathrm{FWI}^*(S_n) = \sum_{(u,v)\in E(S_n)} \left(\deg(u)^2 - \deg(v)^2\right).$$

Since every edge connects the central vertex c to a leaf v, we have

$$\mathrm{FWI}^*(S_n) = \sum_{v\in V(S_n)\setminus\{c\}} \left(\deg(c)^2 - \deg(v)^2\right) = (n-1)\left((n-1)^2 - 1^2\right).$$

Simplifying the expression,

$$\mathrm{FWI}^*(S_n) = (n-1)\left((n-1)^2 - 1\right) = (n-1)(n^2-2n) = n(n-1)(n-2).$$

Recall the irregularity $\mathrm{irr}(S_n)$. It is known that

$$\mathrm{irr}(S_n) = \sum_{(u,v)\in E(S_n)} |\deg(u) - \deg(v)| = (n-1)(n-2) \quad \text{when } n > k, \mathrm{irr}(S_n) \geqslant 2^{\frac{1}{k}}.$$

Thus

$$\begin{aligned}\mathrm{FWI}^*(S_n) - \mathrm{irr}(S_n) &= \sum_{n,m\in E(\mathscr{T})} [F_n^2 - F_m^2] - (n-1)(n-2)\\ &= \sum_{n,m\in E(\mathscr{T})} [\deg F_n^2 - \deg F_m^2] - (n-1)(n-2).\\ &= n(n-1)(n-2) - (n-1)(n-2)\\ &= (n-1)^2(n-2).\end{aligned}$$

Since $n \geq 2$, we have $(n-1)^2 \geq 0$ and $n-2 \geq 0$ for $n \geq 2$, hence

$$\mathrm{FWI}^*(S_n) - \mathrm{irr}(S_n) \geq 0,$$

which implies

$$\mathrm{FWI}^*(S_n) \geqslant \mathrm{irr}(S_n).$$

As desire. □

Следствие 1. *Let* $\mathrm{KT}_{F_n}(\mathscr{T})$ *be a Kragujevac tree of Fibonacci word with* n *vertices, let then*

$$\mathrm{FWI}^*(\mathrm{KT}_{F_n}(\mathscr{T})) = \left(\frac{n-k+1}{2} + \sum_{i=1}^{k}\left(d_i(d_i-1)^k + |d_i - k + 1|\right) + \sum_{i=1}^{k}((2d_i+1) + d_i + 3)\right)^{\frac{1}{k}}.$$

Следствие 2. *According to Corollary* 1, let p be a pendent vertices of a Kragujevac tree of Fibonacci word, then we have:

$$\mathrm{FWI}^*(\mathrm{KT}_{F_n}(\mathscr{T})) = \left(\frac{n-k+1}{2} + (k(k-1)^p + |k-p+1|) + ((2k+1) + k + 3)\right)^{\frac{1}{p}}.$$

Actually, bot of corollary 1,2 had based on results in [27] but we development it for Fibonacci word index. In Table 1 we show the length of Fibonacci word up to F_{10}

F_n	Length	F_n (Fibonacci word)
F_1	1	1
F_2	1	0
F_3	2	01
F_4	3	010
F_5	5	01001
F_6	8	01001010
F_7	13	0100101001001
F_8	21	010010100100101001010
F_9	34	0100101001001010010100100101001001
F_{10}	55	0100101001001010010100100101001001010010100100101001001

Таблица 1: Fibonacci words F_n up to F_{10}, their values, and lengths.

For extended this results, the code of generate all Fibonacci words up to the user-specified order n, and saves the results to a text file named by the user, had given by:

```
def fibonacci_words_up_to(n):
    """
    Generate all Fibonacci words from F_1 up to F_n.

    Parameters:
    n (int): The maximum order of Fibonacci words to generate (n >= 1)

    Returns:
    list of tuples: Each tuple contains (order, fibonacci_word)
    """
    if n < 1:
        raise ValueError("Order n must be a positive integer (>=1).")

    fib_words = []
    if n >= 1:
        fib_words.append((1, "1"))
    if n >= 2:
        fib_words.append((2, "0"))

    for i in range(3, n + 1):
        prev1 = fib_words[-1][1]
        prev2 = fib_words[-2][1]
        fib_words.append((i, prev1 + prev2))

    return fib_words

def main():
    try:
        n = int(input("Enter the maximum order n of Fibonacci words
        to generate (n >= 1): "))
        filename = input("Enter the filename to save the results
        (e.g., output.txt): ").strip()

        fib_words = fibonacci_words_up_to(n)

        with open(filename, 'w') as file:
            file.write(f"Fibonacci words from F_1 to F_{n}:\n\n")
            for order, word in fib_words:
                file.write(f"F_{order} (length {len(word)}): {word}\n")

        print(f"\nResults successfully saved to '{filename}'.")

    except ValueError as e:
        print(f"Error: {e}")
    except IOError as e:
        print(f"File error: {e}")
```

```
if __name__ == "__main__":
    main()
```

4. Advanced Results in Density of Fibonacci Word

Actually, according to Lemma 4 and Lemma 3 we notice that for the concept of density. As we show that in Theorem 4.

Теорема 4. *Let F_a, F_b be the Fibonacci word with $a \le b$, then we have:*

$$\mathrm{DF}(F_a, F_b) = \mathrm{DF}(\mathrm{F_c}).$$

where $c = \gcd(a, b)$.

Доказательство. According to [15] we have The gcd of F_m and F_n is F_k, where k is the gcd of m and n, we have in [23, 24] the density given by:

$$\mathrm{DF}(m) = \lim_{\lambda\to\infty} \frac{|\{F(n) \bmod m^\lambda : n \ge 0\}|}{m^\lambda}.$$

Now, this matches the classic density of Fibonacci word as:

$$\begin{aligned}\mathrm{DF}(F_n, F_m) &= \gcd\left(\lim_{\lambda\to\infty} \frac{|\{F(n) \bmod m^\lambda : n \ge 0\}|}{m^\lambda}, \lim_{\lambda\to\infty} \frac{|\{F(m) \bmod n^\lambda : m \ge 0\}|}{n^\lambda}\right)\\ &= \mathrm{DF}(F_{\gcd(n,m)})\\ &= \mathrm{DF}(F_r).\end{aligned}$$

we have for r be number of 1 's where $r = \gcd(n, m)$, for any subwords the density satisfying:

$$\begin{cases} |\frac{r\phi^2 - n}{n\phi^2}| \le \frac{1}{n} & \text{of length } n,\\ |\frac{r\phi^2 - m}{m\phi^2}| \le \frac{1}{n} & \text{of length } m.\end{cases}$$

□ In Table 2 we analyse these results in Theorem 4 where $n \in \{1, \dots, 10\}$, $m \in \{1, \dots, 10\}$ and $r = \gcd(n, m)$ when $n > m$ as we show that, if we cancelling the term $n > m$ the result must be equals 0, thus the rang is $[0.0477, 0.3819]$, the term $n > m$ is important:

(n,m)	DF	**(n,m)**	DF	**(n,m)**	DF	**(n,m)**	DF
(1,1)	0.3819	(1,2)	0.3819	(1,3)	0.3819	(7,4)	0.3819
(1,4)	0.3819	(1,5)	0.3819	(1,6)	0.3819	(7,5)	0.3819
(2,3)	0.3819	(2,4)	0.1909	(2,5)	0.3819	(7,6)	0.3819
(2,6)	0.1909	(2,7)	0.3819	(2,8)	0.1909	(8,3)	0.3819
(3,2)	0.3819	(3,3)	0.1273	(3,4)	0.3819	(8,4)	0.0954
(3,5)	0.3819	(3,6)	0.1273	(3,7)	0.3819	(8,5)	0.3819
(4,1)	0.3819	(4,2)	0.1909	(4,3)	0.3819	(8,6)	0.1909
(4,4)	0.0954	(4,5)	0.3819	(4,6)	0.1909	(8,7)	0.3819
(5,3)	0.3819	(5,4)	0.3819	(5,5)	0.0764	(8,8)	0.0477
(5,6)	0.3819	(5,7)	0.3819	(5,8)	0.3819	(9,2)	0.3819
(6,2)	0.1909	(6,3)	0.1273	(6,4)	0.1909	(9,3)	0.1273
(6,5)	0.3819	(6,6)	0.0636	(6,7)	0.3819	(9,4)	0.3819
(7,1)	0.3819	(7,2)	0.3819	(7,3)	0.3819	(10,7)	0.3819
(9,5)	0.3819	(9,6)	0.1273	(9,7)	0.3819	(10,8)	0.1909

Таблица 2: Density of subwords

Замечание 9. *The sequence of standard Fibonacci words in Definition* 3 is defined as $F(u,v) = \{f_n(u,v)\}_{n\geq 1}$, that is, $F(u,v) = \{u, v, vu, vuv$, vuvvu, vuvvuvuv, vuvvuvuvvuvvu,...$\}$. Similarly, the n^{th} reverse Fibonacci words are defined recursively as:

$$f'_1(u,v) = u, f'_2(u,v) = v,$$
$$f'_n(u,v) = f'_{n-2}(u,v) \cdot f'_{n-1}(u,v), \quad n \geq 3,$$

and the sequence of reverse Fibonacci words is defined as $F'(u,v) = \{f'_n(u,v)\}_{n\geq 1}$, that is, $F'(u,v) = \{u, v, uv$, vuv, uvvuv, vuvuvvuv, uvvuvvuvuvvuv, ...$\}$.

Лемма 3. *Let $\mathscr{T}(u,v) \in A^n \times A^m$ be pair of words, where $n, m \in \mathbb{N}$, then we define Fibonacci word by:*

$$\mathscr{T}(u,v) = \mathscr{T}(u-1,v)\mathscr{T}(u,v-1).$$

Доказательство. Assume: $\mathscr{T}(0,v) = (a,a)$ for all v, $\mathscr{T}(u,0) = (b,b)$ for all u. Then, compute $\mathscr{T}(1,1)$ we have

$$\mathscr{T}(1,1) = \mathscr{T}(0,1) \cdot \mathscr{T}(1,0) = (a,a) \cdot (b,b) = (ab,ab).$$

Actually, word $\mathscr{T}(u,v)$ consists of a pair of words and in order to realise the Fibonacci word, we need to return to the Fibonacci sequence-based recursive function from which we observe the given condition. □

Лемма 4. *According to Lemma 3. The asymptotic growth rate of $n(u,v)$ and $m(u,v)$ are:*

$$n(u,v), m(u,v) \sim C \cdot \varphi^{u+v}.$$

Доказательство. Let $n(u,v)$ and $m(u,v)$ denote the lengths of the words generated by the two-dimensional Fibonacci-type recurrence:

$$n(u,v) = n(u-1,v) + n(u,v-1).$$

with suitable initial conditions (e.g., $n(0,v)$ and $n(u,0)$ specified). We need to answer on question: What is the asymptotic growth rate of $n(u,v)$ and $m(u,v)$?. In this case, this recurrence is equivalent to the one satisfied by Delannoy numbers or central binomial coefficients (depending on the boundary values). The solution grows exponentially with $u+v$. In Figure 2, we show lattice grid as:

$v=4$	1	5	15	35	70
$v=3$	1	4	10	20	35
$v=2$	1	3	6	10	15
$v=1$	1	2	3	4	5
$v=0$	1	1	1	1	1
	$u=0$	$u=1$	$u=2$	$u=3$	$u=4$

Рис. 2: Lattice grid

Specifically, for large u and v, the dominant term is:

$$n(u,v) \sim C \cdot \lambda^{u+v}$$

where λ is the unique positive root of $x^2 = x + 1$, i.e., the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$. Thus,

$$\lim_{N\to\infty} \frac{1}{N} \ln n(N,N) = \ln\varphi \approx 0.4812$$

This matches the classical Fibonacci sequence's exponential growth rate. The same argument applies to $m(u,v)$ if it satisfies the same recurrence and initial conditions. The asymptotic growth rate of $n(u,v)$ and $m(u,v)$ is exponential in $u+v$, with base equal to the golden ratio φ, so

$$n(u,v), m(u,v) \sim C \cdot \varphi^{u+v},$$

for some constant C depending on the initial conditions. □

Лемма 5. *[21] Let $u \in A^n$, $v \in A^m$, with $n \leq m$. Then $C_{u,v}[k] = 1$ if and only if $u[k, k+1, \cdots, n-1] = v[0, 1, \cdots, n-k-1]$; i.e. the overlapping blocks are the suffix of u and the prefix of v of length $n-k$.*

Теорема 5. *Let $\mathscr{T}(u,v) \in A^n \times A^m$ be pair of words, where $n, m \in \mathbb{N}$, then the density of $\mathscr{T}(u,v)$ given by:*

$$\mathrm{DF}(\mathscr{T}(u,v)) = \log\varphi.$$

Proposition 6. [2] *Let F_n be a Fibonacci number, then we have:*

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{F_n}{23^n} = \sum_{n=0}^{\infty} \left[\binom{-1/2}{n} (-4)^n\right] \frac{F_n}{23^n}$$

where the generation function is: $\dfrac{1}{1-z-z^2}$.

Доказательство. we have:

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(-\frac{1}{2}\right)^n F_n = \sum_{n=0}^{\infty} \left[\oint_{|z|=1} \frac{(1+z)^{-1/2}}{z^{1/2-n}} \frac{dz}{2\pi i}\right] \left(-\frac{1}{2}\right)^n F_n$$

then

$$\begin{aligned}
&= \oint_{|z|=1} \frac{(1+z)^{-1/2}}{z^{1/2}} \sum_{n=0}^{\infty} F_n \left(-\frac{z}{2}\right)^n \frac{dz}{2\pi i} \\
&= 2\oint_{|z|=1} \frac{z^{1/2}}{\sqrt{1+z}\,(z^2-2z-4)} \frac{dz}{2\pi i} \\
&= -2\int_{-1}^{0} \frac{(-x)^{1/2} i}{\sqrt{1+x}\,(x^2-2x-4)} \frac{dx}{2\pi i} - 2\int_{0}^{-1} \frac{(-x)^{1/2}(-i)}{\sqrt{1+x}\,(x^2-2x-4)} \frac{dx}{2\pi i} \\
&= -\frac{2}{\pi} \int_0^1 \frac{x^{1/2}}{1-x} \frac{1}{x^2+2x-4} dx \\
&= z^{-1(1+t)^2} \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{t^2}{t^4+6t^2+4} dt \\
&= \frac{\sqrt{10}}{5} \approx 0.6325
\end{aligned}$$

The last integral can be straightforward evaluated in the upper half complex plane (UHPC). The poles in the UHPC at $\sqrt{3} \pm \sqrt{5}\,i$. □

[2] By Felix Marin, 10Mar2025 at https://x.com/seriesintegral/status/1899008778740101586?s=61.

4.1. Open questions

The question that should be at the centre of any research of which this research is the basis is: What is the maximum value of the Fibonacci word index and what is the minimum value of the Fibonacci word index?

Through the Albertson index we can test this on the Fibonacci word index and compare these results.

5. Conclusion

The theoretical developments are illustrated with examples, including detailed constructions of Fibonacci word trees and their degree distributions. This work opens avenues for further investigation of word-based graph invariants and their applications in combinatorics and theoretical computer science.

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Берстель Ж. Фибоначчиевы слова - обзор // Книга L. - 1986. - С. 13-27.

2. Риго М. Формальные языки, автоматы и системы нумерации 1: Введение в комбинаторику слов. - 2014. - DOI: 10.1002/9781119008200.

3. Чжи-Сюнь В., Чжи-Инь В. Некоторые свойства сингулярных слов фибоначчиева слова // Европейский журнал комбинаторики. - 1994. - Т. 15, № 6. - С. 587-598.

4. Гутман И., Фуртула Б. Деревья с минимальным индексом атом-связь // MATCH Commun. Math. Comput. Chem. - 2012. - Т. 68. - С. 131-136.

5. Моннеро-Дюмен А. Фрактал фибоначчиева слова. - 2009.

6. Глен А. О штурмианских и эпистурмианских словах и связанных темах // Bulletin of the Australian Mathematical Society. - 2006. - Т. 74, № 1. - С. 155-160.

7. Фуртула Б., Гутман И., Иванович М., Вукичевич Д. Компьютерный поиск деревьев с минимальным индексом ABC // Applied Mathematics and Computation. - 2012. - Т. 219, № 2. - С. 767-772.

8. Висванат Д. Случайные последовательности Фибоначчи и число 1.13198824... // Mathematics of Computation. - 2000. - Т. 69, № 231. - С. 1131-1155.

9. Гутман И., Фуртула Б., Иванович М. Заметки о деревьях с минимальным индексом атом-связь // MATCH Commun. Math. Comput. Chem. - 2012. - Т. 67. - С. 467-482.

10. Гутман И., Хансен П., Мело Х. Поиск экстремальных графов с переменным окрестностным поиском 10. Сравнение индексов неравномерности для химических деревьев // Journal of Chemical Information and Modeling. - 2005. - Т. 45, № 2. - С. 222-230.

11. Рамирес Х.Л., Рубиано Г.Н. О k-фибоначчиевых словах // Acta Universitatis Sapientiae, Informatica. - 2013. - Т. 5. - С. 212-226. - DOI: 10.2478/ausi-2014-0011.

12. Кари Л., Кулкарни М.С., Махалингам К., Ван З. Инволютивные фибоначчиевы слова // J. Autom. Lang. Comb. - 2021. - Т. 26. - С. 255-280.

13. Гутман И., Тринајстич Н., Уилкокс К.Ф. Теория графов и молекулярные орбитали. XII. Ациклические полиены // Journal of Chemical Physics. - 1975. - Т. 62, № 9. - С. 3399-3405.

14. Хамуд Ж., Курносов А. Индекс сигма в деревьях с заданными последовательностями степеней // arXiv e-prints. - 2024. - arXiv:2405.05300. - DOI: 10.48550/arXiv.2405.05300.

15. Калман Д., Мена Р. Числа Фибоначчи - раскрыты // Mathematics Magazine. - 2003. - Т. 76, № 3. - С. 167-181. - DOI: 10.1080/0025570X.2003.11953176.

16. Гутман И., Тринајстич Н. Теория графов и молекулярные орбитали. Полная ϕ-электронная энергия альтернативных углеводородов // Chemical Physics Letters. - 1972. - Т. 17, № 4. - С. 535-538.

17. Абдо Х., Брандт С., Димитров Д. Полная неравномерность графа // Discrete Mathematics & Theoretical Computer Science. - 2014. - Т. 16 (Graph Theory). - С. 201-206.

18. Кнут Д.Э. Искусство программирования. Т. 1: Основные алгоритмы. - 3-е изд. - Addison-Wesley, 1997.

19. Альбертсон М.О. Неравномерность графа // Ars Combinatoria. - 1997. - Т. 46. - С. 219-226.

20. Васильева Е.А. О коэффициентах связи Джека и их вычислении // Чебышёвский сборник. - 2014. - Т. 15, № 1. - С. 65-76. - DOI: 10.22405/2226-8383-2014-15-1-65-76.

21. Рамперсад Н., Вибе М. Корреляции минимальных запрещённых факторов фибоначчиева слова // arXiv preprint. - 2023. - arXiv:2309.07070.

22. Аванесов Е.Т., Гусев В.А. Единицы и линейные рекуррентные последовательности // Чебышёвский сборник. - 2014. - Т. 15, № 3. - С. 4-11. - DOI: 10.22405/2226-8383-2014-15-3-4-11.

23. Хамуд Ж., Абдулла Д. Улучшение эргодической теории для бесконечного слова $\mathfrak{F} = \mathfrak{F}_b := ({}_bf_n)_{n\geqslant 0}$ по плотности Фибоначчи // arXiv preprint. - 2025. - arXiv:2504.05901. - DOI: 10.48550/arXiv.2504.05901.

24. Is the density of 1s in the Fibonacci word uniform? [Электронный ресурс]. - URL: https://mathoverflow.net/questions/323614/is-the-density-of-1s-in-the-fibonacci-word-uniform.

25. Пулидо Х.Ф., Рамирес Х.Л., Виндас-Мелендес А.Р. Генерирующие деревья и фибоначчиевы полиомино // arXiv e-prints. - 2024. - arXiv:2411.

26. Юсаф С., Бхатти А.А., Али А. Заметка об изменённом индексе Альбертсона // arXiv e-prints. - 2019. - arXiv:1902.01809. - DOI: 10.48550/arXiv.1902.01809.

27. Хоссейни С.А., Ахмади М.Б., Гутман И. Деревья Крагуеваца с минимальным индексом атом-связь // MATCH Commun. Math. Comput. Chem. - 2014. - Т. 71, № 20. - С. 5-20.

REFERENCES

1. Berstel, J. 1986, "Fibonacci words—a survey", in Rozenberg, G. and Salomaa, A. (eds.) *The Book of L*. Berlin: Springer, pp. 13–27.

2. Rigo, M. 2014, *Formal Languages, Automata and Numeration Systems 1: Introduction to Combinatorics on Words*. Hoboken, NJ: Wiley. Available at: https://doi.org/10.1002/9781119008200

3. Zhi-Xiong, W., Zhi-Ying, W. 1994, “Some properties of the singular words of the Fibonacci word”, *European Journal of Combinatorics*, 15(6), pp. 587–598.

4. Gutman, I., Furtula, B. 2012, “Trees with smallest atom–bond connectivity index”, *MATCH Communications in Mathematical and in Computer Chemistry*, 68, pp. 131–136.

5. Monnerot-Dumaine, A. 2009, *The Fibonacci Word Fractal* [Preprint]. Available at: https://hal.archives-ouvertes.fr/hal-00367972

6. Glen, A. 2006, “On Sturmian and episturmian words, and related topics”, *Bulletin of the Australian Mathematical Society*, 74(1), pp. 155–160.

7. Furtula B., Gutman I., Ivanovich M., Vukichevich D. 2012, “Computer search for trees with minimal ABC index”, *Applied Mathematics and Computation*, 219(2), pp. 767–772.

8. Viswanath, D. 2000, “Random Fibonacci sequences and the number 1.13198824...”, *Mathematics of Computation*, 69(231), pp. 1131–1155.

9. Gutman, I., Furtula, B., Ivanović, M. 2012, “Notes on trees with minimal atom-bond connectivity index”, *MATCH Communications in Mathematical and in Computer Chemistry*, 67, pp. 467–482.

10. Gutman I., Hansen P., Melo H. 2005, “Variable neighborhood search for extremal graphs. 10. Comparison of irregularity indices for chemical trees”, *Journal of Chemical Information and Modeling*, 45(2), pp. 222–230.

11. Ramírez, J.L., Rubiano, G.N. 2013, “On the k-Fibonacci words”, *Acta Universitatis Sapientiae, Informatica*, 5, pp. 212–226. https://doi.org/10.2478/ausi-2014-0011

12. Kari L., Kulkarni M.S., Mahalingam K., Wang Z. 2021, “Involutive Fibonacci words”, *Journal of Automata, Languages and Combinatorics*, 26, pp. 255–280.

13. Gutman, I., Trinajstić, N., Wilcox, C.F. 1975, “Graph theory and molecular orbitals. XII. Acyclic polyenes”, *Journal of Chemical Physics*, 62(9), pp. 3399–3405.

14. Hamoud, J., Kurnosov, A. 2024, “Sigma index in trees with given degree sequences”, *arXiv* [Preprint]. Available at: https://doi.org/10.48550/arXiv.2405.05300

15. Kalman, D., Mena, R. 2003, “The Fibonacci numbers—exposed”, *Mathematics Magazine*, 76(3), pp. 167–181. https://doi.org/10.1080/0025570X.2003.11953176

16. Gutman, I., Trinajstić, N. 1972, “Graph theory and molecular orbitals. Total ϕ-electron energy of alternant hydrocarbons”, *Chemical Physics Letters*, 17(4), pp. 535–538.

17. Abdo, H., Brandt, S., Dimitrov, D. 2014, “The total irregularity of a graph”, *Discrete Mathematics & Theoretical Computer Science*, 16(Graph Theory), pp. 201–206.

18. Knuth, D.E. 1997, *The Art of Computer Programming, Volume 1: Fundamental Algorithms*, 3rd edn. Boston: Addison-Wesley.

19. Albertson, M.O. 1997, “The irregularity of a graph”, *Ars Combinatoria*, 46, pp. 219–226.

20. Vassilieva, E.A. 2014, “On Jack’s connection coefficients and their computation”, *Chebyshevskii Sbornik*, 15(1), pp. 65–76 (in Russian). https://doi.org/10.22405/2226-8383-2014-15-1-65-76

21. Rampersad, N., Wiebe, M. 2023, “Correlations of minimal forbidden factors of the Fibonacci word”, *arXiv* [Preprint]. Available at: arXiv:2309.07070

22. Avanesov, E.T., Gusev, V.A. 2014, “Units and linear recurrent sequences”, *Chebyshevskii Sbornik*, 15(3), pp. 4–11 (in Russian). https://doi.org/10.22405/2226-8383-2014-15-3-4-11

23. Hamoud, J., Abdullah, D. 2025, “Improvement ergodic theory for the infinite word $\mathfrak{F} = \mathfrak{F}_b := ({}_b f_n)_{n \geqslant 0}$ on Fibonacci density”, *arXiv* [Preprint]. Available at: https://doi.org/10.48550/arXiv.2504.05901

24. MathOverflow (n.d.) “Is the density of 1s in the Fibonacci word uniform?”. Available at: https://mathoverflow.net/questions/323614/is-the-density-of-1s-in-the-fibonacci-word-uniform.

25. Pulido H.F., Ramirez H.L., Vindas-Melendez A.R. 2024, “Generating trees and Fibonacci polyominoes”, *arXiv* [Preprint]. Available at: arXiv:2411

26. Yousaf S., Bhatti A.A., Ali A. 2019, “A note on the modified Albertson index”, *arXiv* [Preprint]. Available at: https://doi.org/10.48550/arXiv.1902.01809

27. Hosseini S.A., Ahmadi M.B., Gutman I. 2014, “Kragujevac trees with minimal atom-bond connectivity index”, *MATCH Communications in Mathematical and in Computer Chemistry*, 71(20), pp. 5–20.

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