ЧЕБЫШЕВСКИЙ СБОРНИК
Том 26. Выпуск 2.

УДК 514.853+517.938.5 DOI 10.22405/2226-8383-2025-26-2-198-217

Топология алгебраически разделимых интегрируемых систем

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Аннотация

Даётся классификация простейших 3-мерных особенностей регулярных алгебраически разделимых интегрируемых систем. Такие системы представляют собой важный класс интегрируемых по Лиувиллю гамильтоновых систем с двумя степенями свободы и встречаются во многих задачах механики и геометрии. Используемая в статье техника основана на анализе некоторой $\mathbb{Z}_2$-матрицы, однозначно определяемой выражениями исходных фазовых переменных через переменные разделения.

Ключевые слова: интегрируемость по Лиувиллю, алгебраически разделимая система, слоение Лиувилля, топологический инвариант, 3-атом.

Библиография: 18 названий.

Для цитирования:
Николаенко С. С. Топология алгебраически разделимых интегрируемых систем // Чебышевский сборник, 2025, т. 26, вып. 2, с. 198–217.

CHEBYSHEVSKII SBORNIK
Vol. 26. No. 2.

UDC 514.853+517.938.5 DOI 10.22405/2226-8383-2025-26-2-198-217

Topology of Algebraically Separable Integrable Systems

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Исследование поддержано Российским научным фондом (проект 24-71-10100) и выполнено в Московском государственном университете имени М. В. Ломоносова.

The work was supported by the Russian Science Foundation (project No. 24-71-10100) and done at the Lomonosov Moscow State University.

Abstract

We classify the simplest 3-dimensional singularities of regular algebraically separable integrable systems. Such systems form an important class of Liouville integrable Hamiltonian systems with two degrees of freedom and occur in many problems of mechanics and geometry. The techniques elaborated in the paper is based on the analysis of a certain $\mathbb{Z}_2$-matrix uniquely determined by the expressions of the initial phase variables via the separating variables.

Keywords: Liouville integrability, algebraically separable system, Liouville foliation, topological invariant, 3-atom.

Bibliography: 18 titles.

For citation:
Nikolaenko, S. S. 2025, "Topology of algebraically separable integrable systems" , *Chebyshevskii sbornik*, vol. 26, no. 2, pp. 198–217.

1. Introduction

The theory of topological classification of (Liouville) integrable Hamiltonian systems with two degrees of freedom created by A. T. Fomenko and his co-authors [1, 2, 3, 4] allows to classify such systems up to different types of equivalence, first of all Liouville equivalence. Two integrable Hamiltonian systems are called *Liouville equivalent* if there exists a diffeomorphism between the phase spaces invariant with respect to the Liouville foliations of these systems. For a typical integrable system its Liouville foliation is defined by closures of integral trajectories and is therefore an important topological characteristics of the system. Within the Fomenko theory, the Liouville equivalence class of an integrable system (restricted to some invariant submanifold) is defined by the appropriate invariant (which in most cases has the form of a graph with some numerical marks). However, explicit calculation of such invariants is not an algorithmic task and, for some concrete systems, may happen to be quite a complicated problem. In this paper, we discuss a remarkable class of integrable systems for which the calculation of topological invariants can be done algorithmically. Following M. P. Kharlamov and his co-authors, we call such systems *algebraically separable* (see Definition 6 below). This notion means that the Hamiltonian equations on each leaf of the Liouville foliation can be reduced to separated equations and, what is crucial for the topological analysis, the initial phase variables are expressed via the separating ones in a "nice" way. In other words, the separating variables deliver a good parametrization for integral submanifolds making clear all interesting topological effects. The systems with such properties occur in many classical problems of the rigid body dynamics, integrable geodesic flows, integrable billiards.

The systematic approach to the study of the topology of algebraically separable systems was suggested by M. P. Kharlamov in [5], though some interesting results were known before (it is worth mentioning the work [6] by O. E. Orel). The M. P. Kharlamov's ideas and methods were successfully applied to many integrable systems arising in mechanics and physics (see for example [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]). However, determining the topological type of singularities within this approach is usually conducted in terms of the phase space and its initial variables and involves some topological arguments like behavior of certain cycles on integral submanifolds. The main idea of the present paper is as follows. Expressions of the phase variables via the variables of separation define a certain Boolean matrix. Once this matrix is written down, there is no need to address the phase space anymore: all the required topological information is already contained in this matrix!

The paper has the following structure. In Section 2 we recall some definitions and introduce necessary notation. In Section 3 we prove some auxiliary statements, which help to discover the topology of integral submanifolds in terms of the variables of separation. Section 4 is devoted to the description of regular Liouville tori in terms of the variables of separation. Section 5 contains the main results of the paper, namely Theorems 3 and 4 classifying the simplest singularities of algebraically separable systems. The proofs of these theorems clarify the dependance of the topological type of singularities on the Boolean matrix mentioned above. Finally, in Conclusion we give several remarks and outline the directions of further investigations.

2. Necessary definitions ans notations

We start with a brief overview of the main concepts arising in the theory of integrable systems.

Definition 1. *A Hamiltonian system is a triple (M^{2n}, ω, H), where (M^{2n}, ω) is a symplectic (hence orientable) manifold with the symplectic structure ω and H is smooth function on M^{2n} called the Hamiltonian function. The Hamiltonian vector field is defined as $v = \omega^{-1}dH$.*

Definition 2. *A Hamiltonian system is called Liouville integrable if it possesses n smooth first integrals $f_1, \ldots, f_n$ such that:*

- *$f_1, \ldots, f_n$ are functionally independent, i.e., their differentials $df_1, \ldots, df_n$ are linearly independent almost everywhere on M^4;*
- *$f_1, \ldots, f_n$ commute with respect to the Poisson bracket defined by the symplectic structure;*
- *the Hamiltonian vector fields $v_i = \omega^{-1}df_i$ are complete, i.e., the natural parameter on their integral trajectories is defined on the whole real axis.*

Remark 1. *In the above definition we may always assume $f_1 = H$.*

Definition 3. *The Liouville foliation corresponding to the given integrable system is the decomposition of the manifold M^{2n} into connected components (leaves) of common level surfaces of the first integrals $f_1, \ldots, f_n$.*

Studying the topology of the Liouville foliation (in particular, its singularities) is an important part of the topological analysis of an integrable system. According to the classical Liouville theorem, any its compact regular leaf L (regularity means that $df_1, \ldots, df_n$ are linearly independent at any $x \in L$) is diffeomorphic to the n-dimensional torus T^n (the *Liouville torus*) and the Liouville foliation is trivial in a small neighborhood of L. Hence the bifurcations of the Liouville foliation may happen only in neighborhoods of singular leaves.

In the sequel, we assume that $n = 2$, i.e., the system has two degrees of freedom. In this case integrability means the existence of only one additional first integral K of the system functionally independent of H.

Definition 4. *The mapping $\mathcal{F} = (H, K)\colon M^4 \to \mathbb{R}^2$ is called the momentum mapping associated with a Liouville integrable system with two degrees of freedom. The image $\Sigma = \{x \in M^4 \mid \operatorname{rank} d\mathcal{F}(x) < 2\}$ of all critical points of $\mathcal{F}$ is called the bifurcation diagram.*

In a typical case the bifurcation diagram is a union of smooth curves (and maybe isolated points) in $\mathbb{R}^2$, which correspond to certain types of bifurcations of the Liouville foliation. More precisely, take any small enough smooth curve γ intersecting Σ transversally at a single point. Its pre-image $\mathcal{F}^{-1}(\gamma)$ is a 3-dimensional invariant submanifold in M^4, which "pictures" the corresponding bifurcation.

Definition 5. *The submanifold $\mathcal{F}^{-1}(\gamma)$ with the structure of the Liouville foliation viewed up to a fiber diffeomorphism is called a 3-atom.*

All compact 3-atoms turn out to be oriented S^1-fibrations (so called *Seifert fibrations*) over 2-atoms ([1, Theorems 3.2 and 3.3]). By definition, a *2-atom* is a small neighborhood of a critical level line of a Morse function f on a smooth 2-dimensional surface foliated into level lines of f and viewed up to a fiber diffeomorphism. Some examples of 2-atoms are shown in Fig. 1. The 3-atoms obtained as direct products of these 2-atoms and the circle S^1 are denoted by the same symbols.

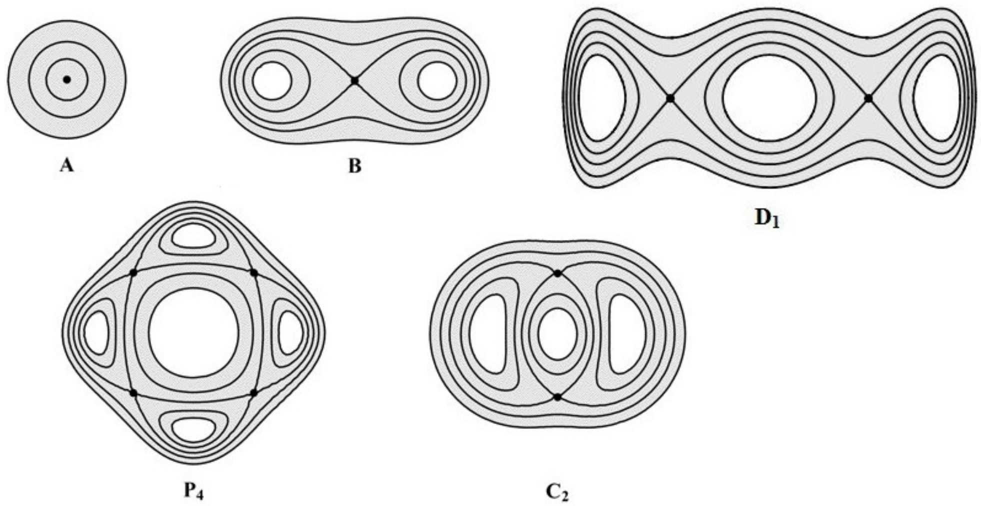

Рис. 1: 2-atoms A, B, D_1, P_4, C_2

For more details about topological invariants in the theory of Liouville integrable systems we address the reader to [1].

Now we define the main notion of this paper (which may be generalized to arbitrary many degrees of freedom).

Definition 6. *We say that a Liouville integrable system with 2 degrees of freedom on a symplectic manifold M^4 (or on its invariant submanifold N) is algebraically separable if the following conditions hold.*

(1) *There exist real smooth functions u_1, u_2 on M^4 (on N) called the variables of separation in which the Hamiltonian equations separate in the form*

$$\dot{u_1} = \sqrt{P_1(u_1)}, \qquad \dot{u_2} = \sqrt{P_2(u_2)} \tag{1}$$

or in the form of the Abel equations

$$\dot{u_1} = \frac{\sqrt{P(u_1)}}{u_1 - u_2}, \qquad \dot{u_2} = \frac{\sqrt{P(u_2)}}{u_1 - u_2}. \tag{2}$$

Here P_1 and P_2 are (maybe coinciding) real polynomials of degree 3 or 4, P is a polynomial of degree 5 or 6. The coefficients of these polynomials depend smoothly on the values h, k of the first integrals H, K of the system.

(2) *The initial phase variables on M^4 (on N) can be expressed via u_1, u_2 as multivalued rational functions on two-valued radicals of the form $\sqrt{u_i - \alpha_j}$, $i = 1, 2$, where α_j are the (complex) roots of P_1, P_2 (or P). The coefficients of these rational functions are smooth functions on u_1, u_2, h, k.*

Remark 2. *This definition is given according to [7]. In [5] such systems are called algebraically solvable.*

Remark 3. *In the classical integrable cases of the rigid body dynamics the phase space M^4 appears as a symplectic leaf in $\mathbb{R}^6$ endowed with an appropriate Poisson structure. In this case the second condition of the above definition implies that the coordinates in $\mathbb{R}^6$ are expressed via u_1, u_2 as indicated above.*

Remark 4. *The second condition of the above definition is crucial for studying the topology of the Liouville foliation of the underlying integrable system. It allows one to describe explicitly the Liouville tori and their bifurcations in terms of the variables u_1, u_2. This distinguishes algebraically separable systems from other types of systems with separating variables.*

Let $R_1, \dots, R_n$ be the monomials in the rational expressions of the phase variables via u_1, u_2:

$$R_m = \sqrt{\pm(u_1 - \alpha^{(1)}_{m,1}) \dots (u_1 - \alpha^{(1)}_{m,p(m)})(u_2 - \alpha^{(2)}_{m,1}) \dots (u_2 - \alpha^{(2)}_{m,q(m)})}, \tag{3}$$

where $\alpha^{(i)}_{m,j}$ are some (maybe complex) roots of the polynomial P (in the case (1) here and further it is convenient to denote by P the least common multiple of P_1 and P_2). It may happen that $\alpha^{(i_1)}_{m_1,j_1} = \alpha^{(i_2)}_{m_2,j_2}$ for some triples $(i_1, m_1, j_1) \neq (i_2, m_2, j_2)$, but $\alpha^{(i)}_{m,j_1} \neq \alpha^{(i)}_{m,j_2}$ for fixed i, m and $j_1 \neq j_2$. Moreover, the set of all the numbers $\alpha^{(i)}_{m,j}$ may be smaller than the set of all the roots of P. Note that the expressions under the radicals in (3) must be real and non-negative which influences the range of u_1, u_2.

Denote by $\pi_{h,k}$ the projection of a certain leaf $L \subset \{x \in M^4 \mid H(x) = h,\ K(x) = k\}$ of the Liouville foliation to the plane $\mathbb{R}^2(u_1, u_2)$. Then $\pi_{h,k}(L) = [\alpha_l, \alpha_r] \times [\alpha_b, \alpha_t]$, where $\alpha_l, \alpha_r, \alpha_b, \alpha_t$ are real roots of P (the indices stand for *left*, *right*, *bottom*, *top*).

Following [5], introduce the sign function $\operatorname{bsgn}\colon \mathbb{R} \to \mathbb{Z}_2$:

$$\operatorname{bsgn}(\theta) = \begin{cases} 0, & \theta \geq 0, \\ 1, & \theta < 0. \end{cases}$$

Obviously, $\operatorname{bsgn}(\theta_1\theta_2) = \operatorname{bsgn}(\theta_1) \oplus \operatorname{bsgn}(\theta_2)$ if $\theta_1, \theta_2 \neq 0$ ($\oplus$ is the sum modulo 2).
Set $S_m = \operatorname{bsgn} R_m$, $s^{(i)}_j = \operatorname{bsgn} \sqrt{\pm(u_i - \alpha_j)}$ if $\alpha_j \in \mathbb{R}$, and $s^{(i)}_j = \operatorname{bsgn} \sqrt{(u_i - \alpha_j)(u_i - \overline{\alpha_j})}$ if $\alpha_j \notin \mathbb{R}$ (in the latter case the multipliers $(u_i - \alpha_j)$ and $(u_i - \overline{\alpha_j})$ are both contained or not contained in each radical R_m). Thus we obtain a $\mathbb{Z}_2$-linear mapping $\mathcal{A}\colon \mathbb{Z}_2^{2\theta}(s^{(1)}_1, \dots, s^{(1)}_\theta, s^{(2)}_1, \dots, s^{(2)}_\theta) \to$ $\to \mathbb{Z}_2^n(S_1, \dots, S_n)$, $\theta \leq \deg P$, defined by (3):

$$S_m = s^{(1)}_{m,1} \oplus \dots \oplus s^{(1)}_{m,p(m)} \oplus s^{(2)}_{m,1} \oplus \dots \oplus s^{(2)}_{m,q(m)}, \qquad m = \overline{1, n}. \tag{4}$$

Definition 7. *We shall say that an algebraically separable system is regular if the set $(S_1, \dots, S_n)$ is uniquely determined by a point $x \in M^4$, i. e., different signs of the radicals $R_1, \dots, R_n$ cannot define (under the same values of u_1, u_2, h, k) the same point in the phase space.*

Let A be the matrix of the linear mapping $\mathcal{A}$. The main idea of the method discussed in this paper is following: for a regular algebraically separable system the matrix A “knows” everything about its Liouville foliation. Topology and singularities of this foliation can be deduced from the matrix A directly.

For given h, k the variables $s^{(i)}_j$ can be divided into two groups: the first one contains the signs which do not change on a fixed leaf L of the Liouville foliation whereas the second group contains the signs changing on L. Accordingly, $A = (B \,|\, C)$, where the columns of the submatrices B and C correspond respectively to the variables of the first and the second groups. Roughly speaking, for given h, k the matrix B influences the number of connected components in $\mathcal{F}^{-1}(h, k)$ (see [5]) whereas the matrix C determines the topological structure of the pre-image $\pi^{-1}_{h,k}([\alpha_l, \alpha_r] \times [\alpha_b, \alpha_t])$

for a fixed leaf. It is convenient to consider $\pi_{h,k}^{-1}$ as a multi-valued mapping of $[\alpha_l,\alpha_r]\times[\alpha_b,\alpha_t]$. We shall call it the *lifting mapping*. In what follows, we study the matrix C and its influence on the topological properties of $\pi_{h,k}^{-1}$ separately for regular and singular values h,k. Note that if the polynomial $P=P_{h,k}$ has no multiple roots, the variables of the second group are solely $s_l^{(1)}, s_r^{(1)}, s_b^{(2)}, s_t^{(2)}$ and the matrix C has 4 columns. In this case the lifting mapping $\pi_{h,k}^{-1}$ defines the structure of a square tiled surface [18] on the leaves in $\pi_{h,k}^{-1}([\alpha_l,\alpha_r]\times[\alpha_b,\alpha_t])$.

At the end of this section we give a formal proof of the well-known principle stating that singularities of the Liouville foliation correspond to multiple roots of the polynomial P.

Theorem 1. *The bifurcation diagram $\Sigma\subset\mathbb{R}^2(h,k)$ of the momentum mapping $\mathcal{F}=(H,K)$ of an algebraically separable system is contained in the discriminant set Δ of the polynomial P, i. e., the set of all points $(h,k)\in\mathbb{R}^2$ such that $P=P_{h,k}$ has multiple roots.*

Доказательство. For $x_0\in M^4$ put $h_0=H(x_0)$, $k_0=K(x_0)$. We shall prove that, if all the roots of the polynomial P_{h_0,k_0} are simple, x_0 is a regular point of the momentum mapping $\mathcal{F}$, i. e., $\operatorname{rank} d\mathcal{F}|_{x_0}=2$.

Let L be the leaf of the Liouville foliation containing x_0, and let $\pi_{h_0,k_0}(L)=[\alpha_l,\alpha_r]\times[\alpha_b,\alpha_t]$. Since $u_1\in[\alpha_l,\alpha_r]$ and $u_2\in[\alpha_b,\alpha_t]$ on L, we may set

$$u_1=\alpha_l\cos^2\varphi+\alpha_r\sin^2\varphi,\qquad u_2=\alpha_b\cos^2\psi+\alpha_t\sin^2\psi,\qquad \varphi,\psi\in[0,2\pi).$$

Then

$$\sqrt{u_1-\alpha_l}=\sqrt{\alpha_r-\alpha_l}\,\sin\varphi,\qquad \sqrt{\alpha_r-u_1}=\sqrt{\alpha_r-\alpha_l}\,\cos\varphi, \tag{5}$$
$$\sqrt{u_2-\alpha_b}=\sqrt{\alpha_t-\alpha_b}\,\sin\psi,\qquad \sqrt{\alpha_t-u_2}=\sqrt{\alpha_t-\alpha_b}\,\cos\psi, \tag{6}$$

where the radicals in the right-hand sides are the non-negative arithmetic square roots. This way, we can take into account the signs $s_l^{(1)}, s_r^{(1)}, s_b^{(2)}, s_t^{(2)}$.

Take any φ_0,ψ_0 such that the equalities (5), (6) are true for $u_1(x_0), u_2(x_0)$. Consider the mapping $\xi\colon U(h_0,k_0)\to M^4$ obtained by substitung (5), (6) with $\varphi=\varphi_0$, $\psi=\psi_0$ in the expressions of the phase variables via u_1,u_2. Here $U(h_0,k_0)$ is a neighborhood of the point (h_0,k_0) in $\mathbb{R}^2(h,k)$. Note that ξ is well defined: the signs of the radicals from the first group (which do not change on the leaf L) are taken the same as on L, and signs from the second group are uniquely determined by $\sin\varphi_0$, $\cos\varphi_0$, $\sin\psi_0$, $\cos\psi_0$. The neighborhood $U(h_0,k_0)\subset\mathbb{R}^2$ is taken small enough so that for any $(h,k)\in U(h_0,k_0)$ all the roots of the polynomial $P=P_{h,k}$ are simple. It is easy to see that ξ is smooth since α_j are smooth functions on h,k. The latter is true due to the implicit function theorem since locally α_j are simple roots of P and $\frac{\partial P}{\partial\alpha_j}\neq 0$.

Now notice that $\mathcal{F}\circ\xi=\operatorname{id}|_{U(h_0,k_0)}$. Taking differentials at (h_0,k_0), we obtain $d\mathcal{F}|_{x_0}\circ d\xi|_{(h_0,k_0)}=$ $=\operatorname{id}|_{\mathbb{R}^2}$ which yields $\operatorname{rank} d\mathcal{F}|_{x_0}\geq\operatorname{rank}(d\mathcal{F}|_{x_0}\circ d\xi|_{(h_0,k_0)})=2$. Hence $\operatorname{rank} d\mathcal{F}|_{x_0}=2$. □

3. Topological properties of the lifting mapping $\pi_{h,k}^{-1}$

In this section we prove auxiliary statements which will help us to study the image of the lifting mapping $\pi_{h,k}^{-1}$ defined on the rectangle $\Pi=[\alpha_l,\alpha_r]\times[\alpha_b,\alpha_t]$. If the signs of the first group (see previous section) are fixed, $\pi_{h,k}^{-1}(\Pi)$ is a single leaf of the Liouville foliation, otherwise we can obtain several leaves. It is easy to see that for each point $y\in\operatorname{int}\Pi$ its image $\pi_{h,k}^{-1}(y)$ consists of $2^{\operatorname{rank}C}$ points in the first case and $2^{\operatorname{rank}A}$ in the second one, where A and C are the $\mathbb{Z}_2$-matrices defined in the previous section.

By $A_j^{(i)}$ denote the column of the matrix A corresponding to the sign $s_j^{(i)}$, $i=1,2$, and by $\hat{A}_j^{(i)}$ the matrix obtained from A by the following procedure: eliminate all the rows of A with entries in

$A_j^{(i)}$ equal to 1 and then eliminate the column $A_j^{(i)}$. By $\hat{A}_{j_1j_2}^{(i_1i_2)}$ denote the matrix obtained from A by the same procedure applied twice (with respect to the columns $A_{j_1}^{(i_1)}$ and $A_{j_2}^{(i_2)}$). Obviously, $\hat{A}_j^{(i)}$ is the matrix of the reduced linear mapping $\mathcal{A}$ (obtained when $\sqrt{u_i-\alpha_j}=0$) and $\hat{A}_{j_1j_2}^{(i_1i_2)}$ is the matrix of the linear mapping obtained from $\mathcal{A}$ when $\sqrt{u_{i_1}-\alpha_{j_1}}=\sqrt{u_{i_2}-\alpha_{j_2}}=0$ (it is natural to assume that $i_1\neq i_2$). Similar notation will be used for the matrix C.

Lemma 9. *Suppose that for given h,k all the numbers α_j are pairwise distinct. Then*

1) $\operatorname{rank}\hat{A}_j^{(i)}=\operatorname{rank}A-1$ *for* $(i,j)\in\{(1,l),(1,r),(2,b),(2,t)\}$;

2) $\operatorname{rank}\hat{A}_{j_1j_2}^{(12)}\geq\operatorname{rank}A-2$ *for* $(j_1,j_2)\in\{(l,b),(l,t),(r,b),(r,t)\}$.

Доказательство. The image $\pi_{h,k}^{-1}(\Pi)$ consists of $2^{\operatorname{rank}A}$ sheets which are somehow glued together along their boundaries (when some of the radicals vanish). In view of Theorem 1, the result of this gluing is a 2-manifold (one or several Liouville tori), hence the sheets must be glued pairwise over each side of Π. Take for instance the side $\Pi_l=\{(\alpha_l,u_2)\mid\alpha_b<u_2<\alpha_t\}$. Its image $\pi_{h,k}^{-1}(\Pi_l)$ consists of $2^{\operatorname{rank}\hat{A}_l^{(1)}}$ connected components which must be twice less than $2^{\operatorname{rank}A}$. This exactly means that $\operatorname{rank}\hat{A}_l^{(1)}=\operatorname{rank}A-1$.

Now consider $\pi_{h,k}^{-1}$ in a neighborhood of a corner of Π, say $\Pi_{lb}=\{(\alpha_l,\alpha_b)\}$. Suppose $A_l^{(1)}\neq A_b^{(2)}$ and fix the signs of all the radicals except $s_l^{(1)}$ and $s_b^{(2)}$. Then we obtain four sheets which differ by the values of $s_l^{(1)}$ and $s_b^{(2)}$. These sheets are glued together pairwise along their boundaries and have a common corner point Z (this is similar to the "corner" of a sheet of paper folded in half twice, Fig. 2). The neighborhood of the point Z in $\pi_{h,k}^{-1}(\Pi)$ is readily homeomorphic to the 2-disk, therefore it should not be glued with any other similar point in $\pi_{h,k}^{-1}(\Pi_{lb})$. This means that the number of different points in $\pi_{h,k}^{-1}(\Pi_{lb})$ (which equals $2^{\operatorname{rank}\hat{A}_{lb}^{(12)}}$) is four times smaller than $2^{\operatorname{rank}A}$ implying $\operatorname{rank}\hat{A}_{lb}^{(12)}=\operatorname{rank}A-2$.

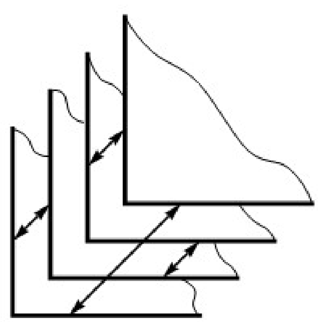

Рис. 2: Gluing of sheets in a neighborhood of a corner point

The case $A_l^{(1)}=A_b^{(2)}$ is treated in a similar way. Here only two sheets are glued along their boundaries resulting in $\operatorname{rank}\hat{A}_{lb}^{(12)}=\operatorname{rank}A-1$. However, as we shall see below, this case is actually impossible. □

Corollary 1. *Under the conditions of Lemma 9 we also have:*

1) $\operatorname{rank}\hat{C}_j^{(i)}=\operatorname{rank}C-1$ *for* $(i,j)\in\{(1,l),(1,r),(2,b),(2,t)\}$;

2) $\operatorname{rank}\hat{C}_{j_1j_2}^{(12)}\geq\operatorname{rank}C-2$ *for* $(j_1,j_2)\in\{(l,b),(l,t),(r,b),(r,t)\}$.

Remark 5. *Since the matrices A and C are constant, the (in)equalities stated in Lemma 9 and Corollary 1 remain true when some of the numbers α_j coincide.*

Lemma 10.

1) *If α_j is not a multiple root of the polynomial $P_{h,k}$, the equality*

$$\operatorname{rank} \hat{A}_j^{(i)} = \operatorname{rank} A - 1, \qquad (i,j) \in \{(1,l),(1,r),(2,b),(2,t)\},$$

is sufficient for each point in $\pi_{h,k}^{-1}(\Pi_j)$ to have a neighborhood in $\pi_{h,k}^{-1}(\Pi)$ homeomorphic to the 2-disk. Here $\Pi_j = \{(\alpha_j, u_2) \mid \alpha_b < u_2 < \alpha_t\}$ if $j \in \{l, r\}$ and $\Pi_j = \{(u_1, \alpha_j) \mid \alpha_l < u_1 < \alpha_r\}$ if $j \in \{b, t\}$.

2) *If $\alpha_{j_1}, \alpha_{j_2}$ are not multiple roots of the polynomial $P_{h,k}$, the equality*

$$\operatorname{rank} \hat{A}_{j_1 j_2}^{(12)} = \operatorname{rank} A - 2, \qquad (j_1, j_2) \in \{(l,b),(l,t),(r,b),(r,t)\},$$

is sufficient for each point in $\pi_{h,k}^{-1}(\Pi_{j_1 j_2})$, where $\Pi_{j_1 j_2} = \{(\alpha_{j_1}, \alpha_{j_2})\}$, to have a neighborhood in $\pi_{h,k}^{-1}(\Pi)$ homeomorphic to the 2-disk.

Доказательство. The equality in the first statement means that the sheets in $\pi_{h,k}^{-1}(\Pi)$ are glued together pairwise over Π_j. The sheets in each pair differ by the value of $s_j^{(i)}$.

The second statement is readily seen from the proof of Lemma 9. □

Now fix all the signs from the first group. Then $\pi_{h,k}^{-1}(\Pi)$ is a single leaf of the Liouville foliation. Put $\eta = \operatorname{rank} C$. Let $C_{j_1}^{(i_1)}, \dots, C_{j_\eta}^{(i_\eta)}$ be $\mathbb{Z}_2$-linearly independent columns of C (thus forming the basis in the span of the columns of C). Each sheet in $\pi_{h,k}^{-1}(\operatorname{int}\Pi)$ can be encoded by the values of $s_{j_1}^{(i_1)}, \dots, s_{j_\eta}^{(i_\eta)}$ if we assign fixed (for instance, zero) values to all the other variables $s_j^{(i)}$ from the second group. The following lemma provides the rule indicating which of these sheets must be glued together over the boundary of Π.

Lemma 11. *For $(i_0, j_0) \in \{(1,l),(1,r),(2,b),(2,t)\}$ let $C_{j_0}^{(i_0)} = C_{j_1}^{(i_1)} \oplus \dots \oplus C_{j_\nu}^{(i_\nu)}$ ($\nu \le \eta$) be the decomposition of the column $C_{j_0}^{(i_0)}$ into the sum (modulo 2) of some basic columns. Suppose* $\operatorname{rank} \hat{C}_{j_0}^{(i_0)} = \operatorname{rank} C - 1$. *Then the pairs of sheets that must be glued together over the corresponding side of Π are defined by the following rule: the signs $s_{j_1}^{(i_1)}, \dots, s_{j_\nu}^{(i_\nu)}$ are different whereas the signs $s_{j_{\nu+1}}^{(i_{\nu+1})}, \dots, s_{j_\eta}^{(i_\eta)}$ are the same in each pair.*

Доказательство. Suppose $(i_0, j_0) = (1, l)$. It is sufficient to prove that the signs of all non-zero radicals (3) are the same in each pair of sheets provided that $u_1 = \alpha_l$. For each such radical R_m its sign is given by (4). Denote by $a_{mj}^{(i)}$ the element of the matrix A (or C) in the intersection of the m's row and the column $A_j^{(i)}$ (or $C_j^{(i)}$). The variable $s_j^{(i)}$ is present in the right-hand side of (4) iff $a_{mj}^{(i)} = 1$. Since $R_m \ne 0$, $a_{ml}^{(1)} = 0$ which yields $a_{mj_1}^{(i_1)} \oplus \dots \oplus a_{mj_\nu}^{(i_\nu)} = a_{ml}^{(1)} = 0$. This means that the number of non-zero elements $a_{mj_\beta}^{(i_\beta)}$ ($1 \le \beta \le \nu$) and, equivalently, the number of variables $s_{j_\beta}^{(i_\beta)}$ present in the right-hand side of (4) are even. Therefore, if we change the values of $s_{j_1}^{(i_1)}, \dots, s_{j_\nu}^{(i_\nu)}$, the sum in (4) remains the same. □

Remark 6. *In Lemma 11 we do not require α_{j_0} to be a simple root of $P_{h,k}$.*

4. Topology of regular leaves in terms of the lifting mapping $\pi_{h,k}^{-1}$

We are now ready to classify regular Liouville tori through the framework of the lifting mapping $\pi_{h,k}^{-1}$. Let all the numbers α_j be different and all the signs from the first group be fixed, thus $\pi_{h,k}^{-1}(\Pi) = T_{h,k}^2$ is a Liouville torus.

Fix the value of u_2 together with the signs $s_b^{(2)}$, $s_t^{(2)}$ or, equivalently, fix the value of $\psi \in [0, 2\pi)$ in (6). We obtain a closed curve γ_φ in $T_{h,k}^2$ parametrized by φ. The value of φ changes by $\pi/2$ as u_1 changes from α_l to α_r. If the columns $C_l^{(1)}$ and $C_r^{(1)}$ of the matrix C are equal, the radicals $\sqrt{u_1 - \alpha_l}$ and $\sqrt{\alpha_r - u_1}$ always appear in (3) in pair, so φ only appears in (3) as $\cos\varphi \sin\varphi$. Therefore, in the case $C_l^{(1)} = C_r^{(1)}$ the natural range of φ is $[0, \pi)$ and $\pi_{h,k}$ restricted to γ_φ is a 2-fold branched covering of the line segment $\{(u_1, u_2) \mid \alpha_l \le u_1 \le \alpha_r\}$ (Fig. 3). We shall use the notation γ_φ^2 for such curve γ_φ. If $C_l^{(1)} \ne C_r^{(1)}$, the natural range of φ is $[0, 2\pi)$ and $\pi_{h,k}$ restricted to γ_φ is a 4-fold branched covering of the line segment $\{(u_1, u_2) \mid \alpha_l \le u_1 \le \alpha_r\}$ (Fig. 4). In this case we write $\gamma_\varphi = \gamma_\varphi^4$. The curves γ_ψ^2, γ_ψ^4 are defined in a similar way.

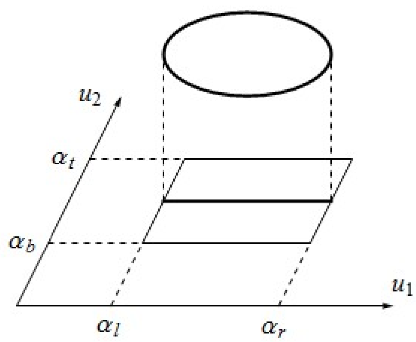

Рис. 3: Curve γ_φ^2

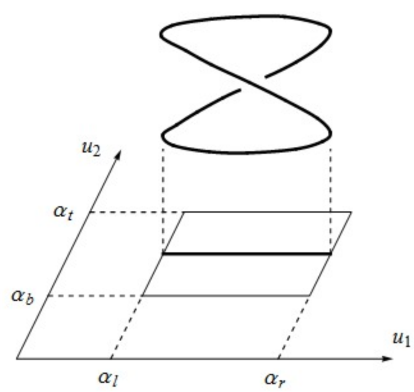

Рис. 4: Curve γ_φ^4

Theorem 2. *In terms of the curves γ_φ, γ_ψ, the Liouville torus $T_{h,k}^2$ can be described in one of the following ways:*

1) $\gamma_\varphi^2 \times \gamma_\psi^2$ *(00-torus);*

2) $\gamma_\varphi^2 \times \gamma_\psi^4$ *(08-torus);*

3) $\gamma_\varphi^4 \times \gamma_\psi^2$ *(80-torus);*

4) $\gamma_\varphi^4 \times \gamma_\psi^4$ *(88-torus);*

5) $\gamma_\varphi^2 \times \gamma_\psi^2/\tau$*, where* τ *is the involution taking* (φ, ψ) *to* $(\varphi+\pi, \psi+\pi)$ *(88/2-torus).*

Доказательство. Consider all principally different (up to symmetries of indices $l \leftrightarrow r$, $b \leftrightarrow t$) cases depending on the rank of the matrix $C = (C_l^{(1)}\, C_r^{(1)}\, C_b^{(2)}\, C_t^{(2)})$, which obviously does not exceed 4 (the number of columns of C). In each case we have 2, 4, 8, or 16 sheets, which are somehow glued together along their boundaries to form a torus. The rules for the gluing are determined by Lemma 11. If the result of a gluing is not a torus, this means that the corresponding case is impossible. In most figures below we supply each sheet with the values of variables $s_j^{(i)}$ corresponding to basic columns in $\langle C_l^{(1)}, C_r^{(1)}, C_b^{(2)}, C_t^{(2)}\rangle$.

1) rank $C = 1$. We have two sheets, which are glued together along their boundaries forming the 2-sphere S^2 (Fig. 5).

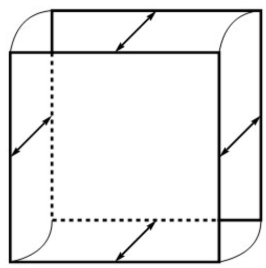

Рис. 5: Gluing of sheets: rank $C = 1$

2) rank $C = 2$.

2.1) $C_l^{(1)} = C_r^{(1)}$, $C_b^{(2)} = C_t^{(2)}$. We obtain the 00-torus (Fig. 6).

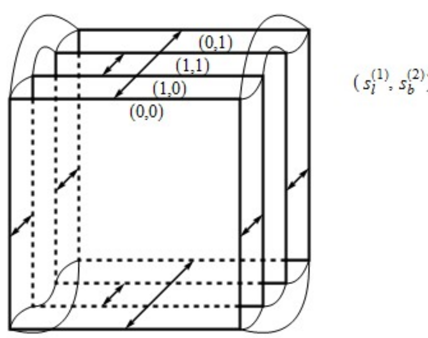

Рис. 6: Gluing of sheets: rank $C = 2$, $C_l^{(1)} = C_r^{(1)}$, $C_b^{(2)} = C_t^{(2)}$

2.2) $C_l^{(1)} \neq C_r^{(1)}$, $C_b^{(2)} = C_t^{(2)}$. We may choose the columns $C_l^{(1)}, C_r^{(1)}$ as basic.

2.2.1) $C_b^{(2)} = C_t^{(2)} = C_l^{(1)}$. We obtain the sphere S^2 (Fig. 7).

2.2.2) $C_b^{(2)} = C_t^{(2)} = C_l^{(1)} \oplus C_r^{(1)}$. We obtain the Klein bottle KL (Fig. 8).

2.3) $C_l^{(1)} \neq C_r^{(1)}$, $C_b^{(2)} \neq C_t^{(2)}$. Again, the columns $C_l^{(1)}, C_r^{(1)}$ are basic.

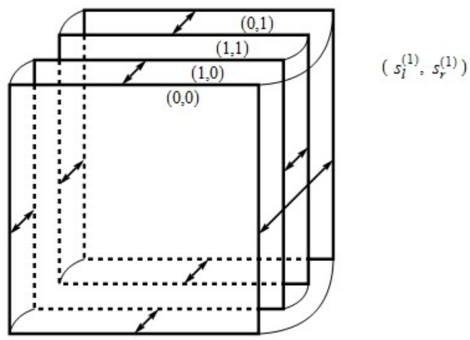

Рис. 7: Gluing of sheets: $\operatorname{rank} C = 2$, $C_l^{(1)} \neq C_r^{(1)}$, $C_b^{(2)} = C_t^{(2)} = C_l^{(1)}$

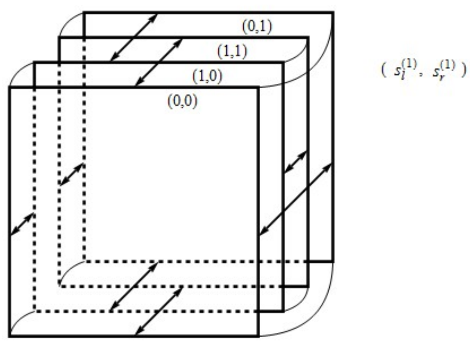

Рис. 8: Gluing of sheets: $\operatorname{rank} C = 2$, $C_l^{(1)} \neq C_r^{(1)}$, $C_b^{(2)} = C_t^{(2)} = C_l^{(1)} \oplus C_r^{(1)}$

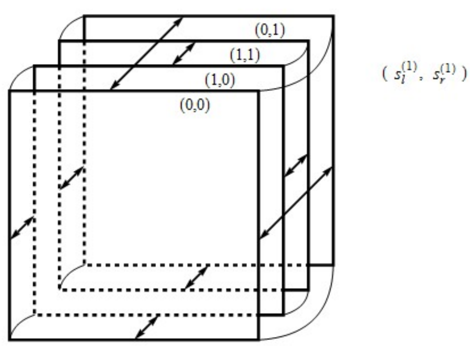

Рис. 9: Gluing of sheets: $\operatorname{rank} C = 2$, $C_l^{(1)} \neq C_r^{(1)}$, $C_b^{(2)} = C_l^{(1)}$, $C_t^{(2)} = C_r^{(1)}$

2.3.1) $\underline{C_b^{(2)} = C_l^{(1)},\ C_t^{(2)} = C_r^{(1)}}$. We have the sphere S^2 (Fig. 9).

2.3.2) $\underline{C_b^{(2)} = C_l^{(1)},\ C_t^{(2)} = C_l^{(1)} \oplus C_r^{(1)}}$. We have the projective plane $\mathbb{R}P^2$ (Fig. 10).

3) $\underline{\operatorname{rank} C = 3.}$ Assume that the columns $C_l^{(1)}, C_r^{(1)}, C_b^{(2)}$ are linearly independent.

3.1) $\underline{C_t^{(2)} = C_b^{(2)}}$. Similarly to the case 2.1), we have the 80-torus (Fig. 11). The 08-torus is obtained in the symmetric case $C_l^{(1)} = C_r^{(1)}$ with linearly independent columns $C_l^{(1)}, C_b^{(2)}, C_t^{(2)}$.

3.2) $\underline{C_t^{(2)} = C_l^{(1)}}$. We obtain the sphere S^2 (Fig. 12).

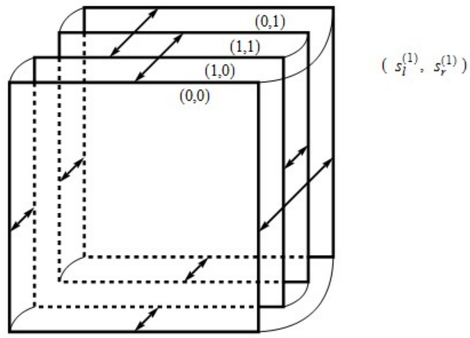

Рис. 10: Gluing of sheets: $\operatorname{rank} C = 2$, $C_l^{(1)} \neq C_r^{(1)}$, $C_b^{(2)} = C_l^{(1)}$, $C_t^{(2)} = C_l^{(1)} \oplus C_r^{(1)}$

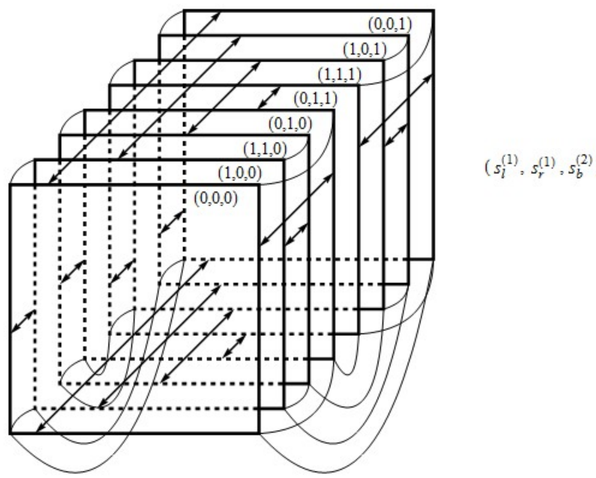

Рис. 11: Gluing of sheets: $\operatorname{rank} C = 3$, $C_t^{(2)} = C_b^{(2)}$

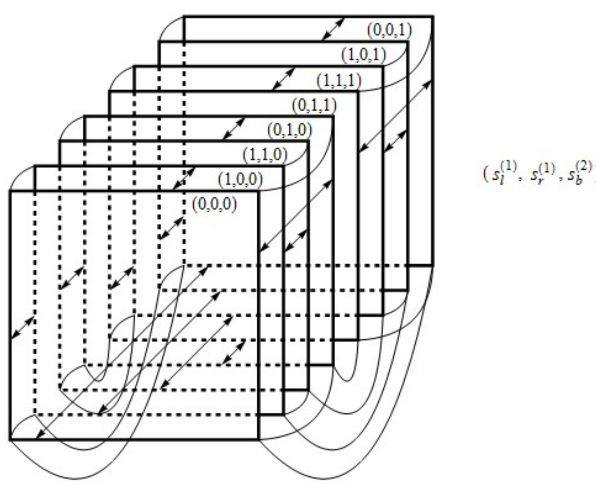

Рис. 12: Gluing of sheets: $\operatorname{rank} C = 3$, $C_t^{(2)} = C_l^{(1)}$

3.3) $\underline{C_t^{(2)} = C_l^{(1)} \oplus C_r^{(1)}}$. We have the Klein bottle KL (Fig. 13).

3.4) $\underline{C_t^{(2)} = C_r^{(1)} \oplus C_b^{(2)}}$. Again the Klein bottle (Fig. 14).

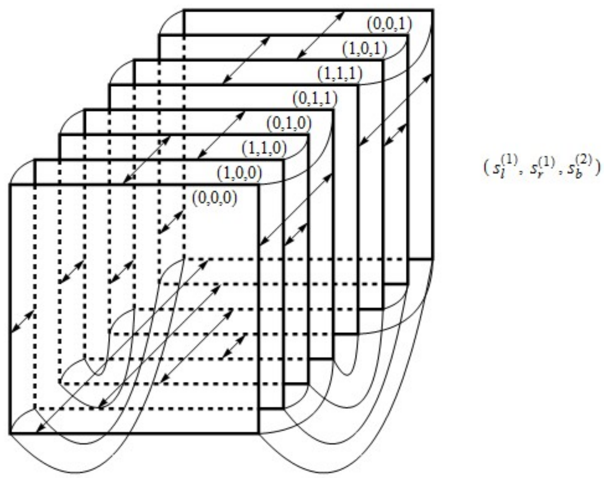

Рис. 13: Gluing of sheets: $\operatorname{rank} C = 3$, $C_t^{(2)} = C_l^{(1)} \oplus C_r^{(1)}$

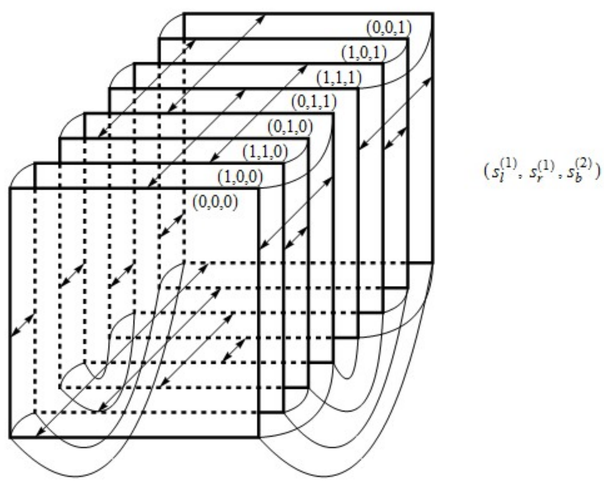

Рис. 14: Gluing of sheets: $\operatorname{rank} C = 3$, $C_t^{(2)} = C_r^{(1)} \oplus C_b^{(2)}$

3.5) $\underline{C_t^{(2)} = C_l^{(1)} \oplus C_r^{(1)} \oplus C_b^{(2)}}$. In this case we obtain the 88/2-torus (Fig. 15).

4) $\underline{\operatorname{rank} C = 4}$. Similarly to the cases 2.1) and 3.1), we obtain a 88-torus.

□

Corollary 2. *If $\pi_{h,k}^{-1}(\Pi)$ is a Liouville torus, the matrix $C = (C_l^{(1)}\, C_r^{(1)}\, C_b^{(2)}\, C_t^{(2)})$ satisfies one of the following conditions:*

1) $\operatorname{rank} C = 2$, $C_l^{(1)} = C_r^{(1)}$, $C_b^{(2)} = C_t^{(2)}$ *(00-torus);*

2) $\operatorname{rank} C = 3$, $C_b^{(2)} = C_t^{(2)}$ *(80-torus);*

3) $\operatorname{rank} C = 3$, $C_l^{(1)} = C_r^{(1)}$ *(08-torus);*

4) $\operatorname{rank} C = 3$, $C_l^{(1)} \oplus C_r^{(1)} \oplus C_b^{(2)} \oplus C_t^{(2)} = \bar{0}$ *(88/2-torus);*

5) $\operatorname{rank} C = 4$ *(88-torus).*

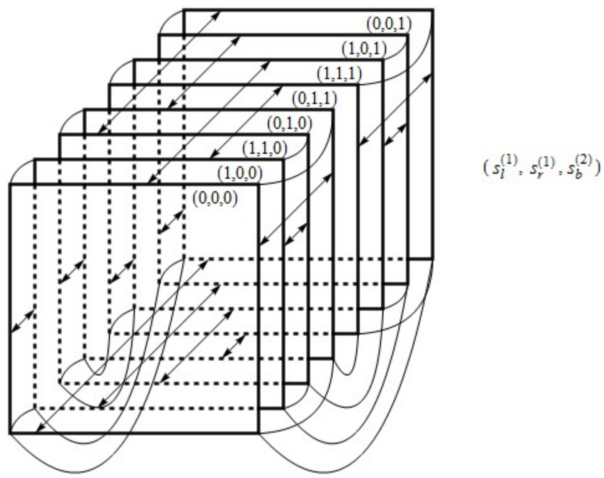

Рис. 15: Gluing of sheets: $\operatorname{rank} C = 3$, $C_t^{(2)} = C_l^{(1)} \oplus C_r^{(1)} \oplus C_b^{(2)}$

5. Classification of bifurcations

Our following aim is to classify the most simple 3-dimensional bifurcations which happen with the Liouville foliations of regular algebraically separable systems.

Consider the isoenergy surface $Q^3_{h_0} = \{x \in M^4 \mid H(x) = h_0\}$. Suppose $(h_0, k_0) \in \Sigma$ is a critical value of the momentum mapping $\mathcal{F} = (H, K)$ and $L_0 \subset \mathcal{F}^{-1}(h_0, k_0)$ is the corresponding singular leaf of the Liouville foliation. Let $U_\varepsilon(L_0)$ be a small invariant 3-dimensional neighborhood of L_0 in $Q^3_{h_0}$ defined by the inequalities $k_0 - \varepsilon \le K \le k_0 + \varepsilon$. Here we assume that (h_0, k_0) is a unique intersection point of the bifurcation diagram Σ with the curve $\{(h_0, k) \mid k_0 - \varepsilon \le k \le k_0 + \varepsilon\}$. Then we may treat $U_\varepsilon(L_0)$ as a 3-dimensional bifurcation of the Liouville foliation (3-atom).

As follows from Theorem 1, the polynomial P_{h_0,k_0} has multiple roots.

Definition 8. *We shall call the bifurcation defined by $U_\varepsilon(L_0)$ simple if it corresponds to a unique multiple root α_j of P_{h_0,k_0}, which has multiplicity 2, and $(\alpha_j, \alpha_j) \notin \pi_{h_0,k_0}(L_0)$, where π_{h_0,k_0} is the projection defined in Section 2.*

The last requirement in this definition means that the bifurcation happens with only one of the cycles $\gamma_\varphi, \gamma_\psi$. In what follows, we assume that it happens with γ_φ, i.e., the line $u_1 = \alpha_j$ intersects the rectangle $\pi_{h_0,k_0}(L_0)$ and the line $u_2 = \alpha_j$ does not.

Remark 7. *The given definition of a simple bifurcation has nothing in common with that of a simple atom given in [1] (Definition 2.4).*

Let $\alpha_{j_1} = \alpha_{j_1}(h, k)$ and $\alpha_{j_2} = \alpha_{j_2}(h, k)$ be two roots of the polynomial $P_{h,k}$ coinciding at (h_0, k_0): $\alpha_{j_1}(h_0, k_0) = \alpha_{j_2}(h_0, k_0)$. There exist two possibilities:

1) $\alpha_{j_1}(h_0, k) = \overline{\alpha_{j_2}(h_0, k)} \in \mathbb{C}\backslash\mathbb{R}$ for $k \in [k_0 - \varepsilon, k_0)$ and $\alpha_{j_1}(h_0, k), \alpha_{j_2}(h_0, k) \in \mathbb{R}$ for $k \in (k_0, k_0 + \varepsilon]$ or vice versa;

2) $\alpha_{j_1}(h_0, k), \alpha_{j_2}(h_0, k) \in \mathbb{R}$ for $k \in [k_0 - \varepsilon, k_0 + \varepsilon]$.

Definition 9. *We shall say that a simple bifurcation is of the first type in the first case and of the second type in the second one.*

The rest of the paper is devoted to the classification of simple bifurcations of the first type that occur in regular algebraically separable systems. Here we list all the required assumptions for this.

(1) The given algebraically separable integrable system is regular.

(2) The isoenergy surface $Q^3_{h_0}$ is regular (i.e., $dH(x) \neq 0$ for any $x \in Q^3_{h_0}$).

(3) The surface $U_\varepsilon(L_0)$ is connected and compact (hence all the leaves $L \subset U_\varepsilon(L_0)$ are compact).

(4) The set $\mathfrak{K}$ of critical points of $\mathcal{F}$ in L is diffeomorphic to a disjoint union of circles and K is a Bott function on $U_\varepsilon(L_0)$, i.e., K is a Morse function on small 2-disks intersecting $\mathfrak{K}$ transversally at each point of $\mathfrak{K}$.

(5) The bifurcation defined by $U_\varepsilon(L_0)$ is simple.

For simple bifurcations of the first type we have again two possibilities.

(1) *(Dis)appearance case.* For any leaf $L \subset \{x \in U_\varepsilon(L_0) \mid K(x) = k\}$ its projection $\pi_{h_0,k}(L)$ lies between the lines $\{u_1 = \alpha_{j_1}(h_0, k)\}$ and $\{u_1 = \alpha_{j_2}(h_0, k)\}$ (Fig. 16).

(2) *Splitting case.* For any leaf $L \subset \{x \in U_\varepsilon(L_0) \mid K(x) = k\}$ its projection $\pi_{h_0,k}(L)$ lies on the left and on the right of the lines $\{u_1 = \alpha_{j_1}(h_0, k)\}$ and $\{u_1 = \alpha_{j_2}(h_0, k)\}$ (Fig. 17).

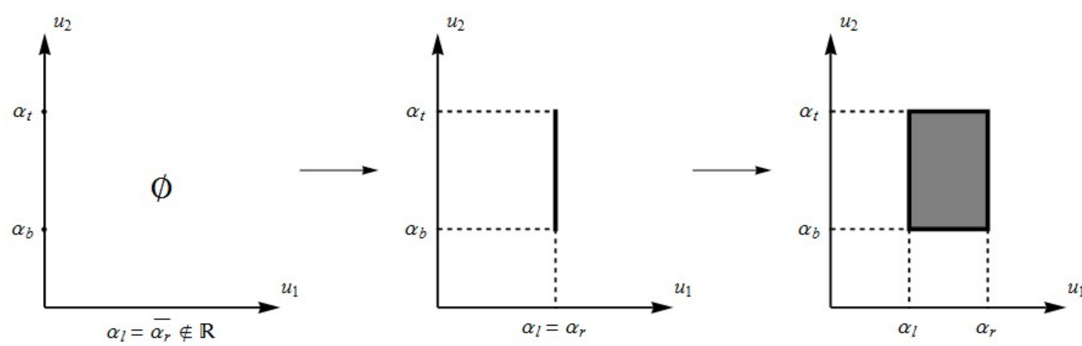

Рис. 16: (Dis)appearance case

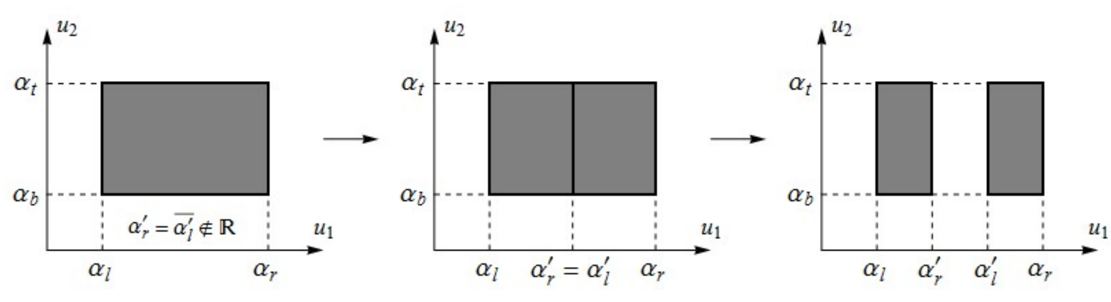

Рис. 17: Splitting case

Consider each of these possibilities separately.

5.1. (Dis)appearance case

Theorem 3. *In the (dis)appearance case, any bifurcation satisfying the above five conditions has the type of the atom A (Fig. 1).*

Доказательство. Since for some values of k α_l is the complex conjugate of α_r, the radicals $\sqrt{u_1-\alpha_l}$ and $\sqrt{\alpha_r-u_1}$ always appear in the expressions (3) in pair. Hence the columns $C_l^{(1)}$ and $C_r^{(1)}$ of the matrix C coincide and $\pi^{-1}_{h_0,k}(\Pi)$ is the 00- or 08-torus when $\alpha_l,\alpha_r\in\mathbb{R}$ and $\alpha_l<\alpha_r$ (here $\Pi=[\alpha_l,\alpha_r]\times[\alpha_b,\alpha_t]$ as above). It follows that the type of bifurcation is totally determined by the evolution of the cycle γ_φ (Fig. 18) and we obviously obtain the atom A.

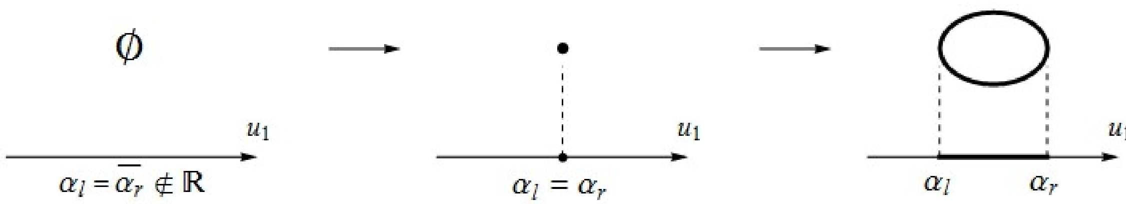

Рис. 18: Evolution of the cycle γ_φ: (dis)appearance case

□

5.2. Splitting case

Theorem 4. *In the splitting case, any bifurcation satisfying the above five conditions has the type of one of the atoms B, C_2, D_1, P_4 (Fig. 1).*

Доказательство. Put $\Pi_1=[\alpha_l,\alpha'_r]\times[\alpha_b,\alpha_t]$ and $\Pi_2=[\alpha'_l,\alpha_r]\times[\alpha_b,\alpha_t]$ as in Fig. 17. Suppose $\alpha'_r=\overline{\alpha'_l}\in\mathbb{C}\backslash\mathbb{R}$ for $k\in[k_0-\varepsilon,k_0)$ and $\alpha'_r,\alpha'_l\in\mathbb{R}$ for $k\in(k_0,k_0+\varepsilon]$. For simplicity, from now on we shall omit the upper indices (1), (2), which stand for the order number of a separation variable, as they are clear from the lower ones. Thus $s'_r=\operatorname{bsgn}(\alpha'_r-u_1)$, A'_r is the column of the matrix A corresponding to s'_r etc.

Put $C=(C_l\,C_r\,C_b\,C_t)$, $C'=(C_l\,C'_r\,C'_l\,C_r\,C_b\,C_t)$, $C_1=(C_l\,C'_r\,C_b\,C_t)$, $C_2=(C'_l\,C_r\,C_b\,C_t)$. Note that the variables $s'_l=\operatorname{bsgn}\sqrt{u_1-\alpha'_l}$, $s'_r=\operatorname{bsgn}\sqrt{u_1-\alpha'_r}$ and hence the columns C'_l,C'_r are only well-defined for $k\in[k_0,k_0+\varepsilon]$. As in the previous theorem, we have $C'_l=C'_r$. So instead of s'_r and s'_l it is convenient to introduce the variable $s'_{rl}=\operatorname{bsgn}\sqrt{(u_1-\alpha'_r)(u_1-\alpha'_l)}$ which is well-defined for any $k\in[k_0-\varepsilon,k_0+\varepsilon]$.

Similar to the proof of Theorem 2, we consider all principally different cases depending on the ranks of the matrices C and C' (the columns $C'_l=C'_r$ of C' may be treated as corresponding to the variable s'_{rl}). Note that the matrices C, C_1, and C_2 corresponding to the rectangles Π, Π_1, and Π_2 satisfy Corollary 2.

1) $\operatorname{rank}C'=\operatorname{rank}C+1$.

- **1.1)** $\operatorname{rank}C=2$, $\operatorname{rank}C'=3$. Two 00-tori differing by the value of s'_{rl} transform into two 80-tori. This corresponds to the atom C_2.
- **1.2)** $\operatorname{rank}C=3$, $\operatorname{rank}C'=4$.
  - **1.2.1)** $C_l=C_r$. This case is similar to the case 1.1): two 08-tori differing by the value of s'_{rl} transform into two 88-tori. Again the atom C_2.
  - **1.2.2)** $C_b=C_t$. Two 80-tori differing by value of s'_{rl} transform into four 80-tori. This corresponds to the atom P_4.
  - **1.2.3)** $C_l\oplus C_r\oplus C_b\oplus C_t=\bar{0}$. A 88/2-torus transforms into two 88-tori. The corresponding atom is the result of the factorization of the atom P_4 from the case 1.3) by the involution acting by central symmetry on the 2-atom P_4 and on the circle S^1. We obtain the atom C_2.

1.3) rank $C = 4$, rank $C' = 5$. This case is similar to the case 1.2.2): two 88-tori differing by the value of s'_{rl} transform into four 88-tori. We obtain the atom P_4.

2) rank $C' =$ rank C.

2.1) rank $C' =$ rank $C = 2$. We have $C_l = A'_r = A'_l = C_r$ and $C_b = C_t$, hence a 00-torus transforms into two 00-tori. This corresponds to the atom B.

2.2) rank $C' =$ rank $C = 3$.

2.2.1) $C_l = C_r$.

2.2.1.1) $A'_l = C_r$. This case is similar to the case 2.1): a 08-torus transforms into two 08-tori. Again the atom B.

2.2.1.2) $A'_l = C_r \oplus C_b \oplus C_t$. A 08-torus transforms into two 88/2-tori. This corresponds to the atom B.

2.2.2) $C_b = C_t$.

2.2.2.1) $A'_l = C_r$. A 80-torus transforms into two 00-tori and a 80-torus. This corresponds to the atom D_1. In the symmetric case $A'_l = C_l$ we also have the atom D_1.

2.2.2.2) $A'_l = C_l \oplus C_r$. The resulting 3-surface is a direct product of a non-orientable 2-atom and the circle. Hence it is non-orientable and does not correspond to a 3-atom. So this case is impossible under our assumptions.

2.2.2.3) $A'_l = C_l \oplus C_r \oplus C_b$. It is easy to see that the critical trajectories on the singular leaf cannot be oriented in the same way. This contradicts the existence of the oriented S^1-fibration in a neighborhood of the singular leaf ([1, Theorems 3.2 and 3.3]). Hence this case is also impossible.

2.2.3) $C_l \oplus C_r \oplus C_b \oplus C_t = \bar{0}$. A 88/2-torus transforms into a 08-torus and a 88-torus. This corresponds to the atom B.

2.3) rank $C' =$ rank $C = 4$.

2.3.1) $A'_l = C_r$. This case is similar to the case 2.2.2.1): a 88-torus transforms into two 08-tori and a 88-torus. We obtain the atom D_1. In the symmetric case $A'_l = C_l$ we also have the atom D_1.

2.3.2) $A'_l = C_l \oplus C_r$. This case is similar to the case 2.2.2.2) and is therefore impossible.

2.3.3) $A'_l = C_l \oplus C_r \oplus C_b$. This case is similar to the case 2.2.2.3) and is also impossible.

2.3.4) $A'_l = C_l \oplus C_r \oplus C_b \oplus C_t$. This case is similar to the cases 2.2.2.2) and 2.3.2). Hence it is impossible.

2.3.5) $A'_l = C_r \oplus C_b \oplus C_t$. A 88-torus transforms into two 88/2-tori and a 88-torus. We obtain the atom D_1.

□

6. Conclusion

As follows from Theorems 3 and 4, the only simple bifurcations of the first type that may occur (and actually do) in regular algebraically separable integrable systems under the five conditions listed above have the type of the 3-atoms A, B, C_2, D_1, and P_4. For instance, all these atoms occur in elliptical billiards with a polynomial potential [14].

Our result was obtained by the direct analysis of the gluing of sheets over the boundaries of the rectangles in the plane $\mathbb{R}^2(u_1, u_2)$, so the techniques demonstrated here can be easily applied for the simple 3-atoms of the second type (which will be the subject of the next paper). Moreover, we may generalize these results to the non-compact case or non-simple 3-atoms.

What is remarkable, for the topological analysis of a concrete regular algebraically separable system, there is no need to parametrize the leaves of the Liouville foliation in terms of the initial phase variables. Given the formulae for the expressions of these variables via the variables of separation, one can write down the corresponding $\mathbb{Z}_2$-matrix and just analyze this matrix for different domains in $\mathcal{F}(M^4) \setminus \Delta$, where $\mathcal{F}$ is the momentum mapping and Δ is the discriminant set of the polynomial P.

It should be emphasized that singularities of algebraically separable systems often occur not due to the coincidence of roots of the polynomial P, but because of degeneration of the Hamiltonian equations written down in the separating variables. In (2) this happens whenever $u_1 = u_2$. The corresponding singularities are much more complicated than those described above. Their topological classification is the subject for future studies.

The author is grateful to his teacher Prof. A. T. Fomenko for constant support and attention to this work; to all the participants of the Seminar "Modern geometry methods", especially Prof. E. A. Kudryavtseva, for fruitful discussions and valuable comments; to Profs. M. P. Kharlamov and P. E. Ryabov for encouraging to develop the Boolean functions method; to the reviewer of this paper for carefully reading the text and pointing out the connection of the problems discussed here with the theory of square tiled surfaces.

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Получено: 07.12.2024
Принято в печать: 07.04.2025