ЧЕБЫШЕВСКИЙ СБОРНИК
Том 26. Выпуск 2.

УДК 517 DOI 10.22405/2226-8383-2025-26-2-160-175

Неприводимые представления колчанов, ассоциированных с кольцами

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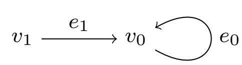
Аннотация

В этой статье мы представляем текущие исследования по классификации неприводимых представлений следующего колчана или, скорее, диграфа (который в этой статье мы обозначаем через $\mathbb{A}$):



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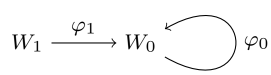
Каждое представление $\mathbb{A}$ задается двумя векторными пространствами W_0 и W_1 и двумя гомоморфизмами $\varphi_0 : W_0 \to W_0$ и $\varphi_1 : W_1 \to W_0$:



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Обозначим предыдущее представление через $(W_1, W_0, \varphi_1, \varphi_0)$. Если $\dim(W_0) = n$ и $\dim(W_1) = m$, то можно определить $W_0 = K^n$ и $W_1 = K^m$, и тогда φ_0 и φ_1 отождествляются соответственно с $n \times n$ и $n \times m$ матрицами M_0 и M_1, так что указанное представление определяется четырехкратным (m, n, M_1, M_0). Вычислим неприводимые представления для некоторого m.

Ключевые слова: конечные кольца, направленные графы, колчанные представления

Библиография: 3 названия.

Для цитирования:
Матович Е. Неприводимые представления колчанов, ассоциированных с кольцами // Чебышевский сборник, 2025, т. 26, вып. 2, с. 160–175.

CHEBYSHEVSKII SBORNIK
Vol. 26. No. 2.

UDC 517 DOI 10.22405/2226-8383-2025-26-2-160-175

Irreducible representations of quivers associated to rings

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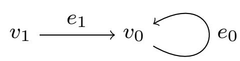
Abstract

In this paper we present the ongoing research on classifying irreducible representations of the following quiver, or rather the digraph (which throughout this paper we denote by $\mathbb{A}$):



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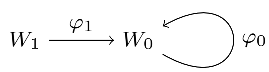
Every representation of $\mathbb{A}$ is given by two vector spaces W_0 and W_1, and two homomorphisms $\varphi_0 : W_0 \to W_0$ and $\varphi_1 : W_1 \to W_0$:



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We denote the previous representation by $(W_1, W_0, \varphi_1, \varphi_0)$. If $\dim(W_0) = n$ and $\dim(W_1) = m$, we may identify $W_0 = K^n$ and $W_1 = K^m$, and then φ_0 and φ_1 are identified respectively with $n \times n$ and $n \times m$ matrices M_0 and M_1, so the above representation is determined by the quadruple (m, n, M_1, M_0). We calculate irreducible representations for some m.

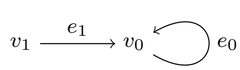
Keywords: finite rings, directed graphs, quiver representations

Bibliography: 3 titles.

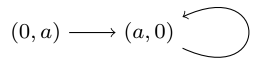
For citation:
Matović, J. 2025, "Irreducible representations of quivers associated to rings" , *Chebyshevskii sbornik*, vol. 26, no. 2, pp. 160–175.

1. Introduction

We are interested in classifying irreducible representations of the following quiver, or rather the digraph (which throughout this paper we denote by $\mathbb{A}$):



This digraph appears as a subdigraph of the digraphs associated with commutative rings in the following way (see [2] and [3] for details): For a ring R we define $G_R = (R^2, E)$, where E is given by $E = \{(a, b) \to (a + b, ab) \mid a, b \in R\}$. Now, the digraph $\mathbb{A}$ appears in the following way: For $a \in R$, $a \neq 0$, we always have:



The present work is merely a beginning of a research project of understanding irreducible representations of digraphs G_R.

2. Preliminaries

Throughout, K will *always* be an algebraically closed field.

Definition 1. *Let $G = (V, E)$ be a digraph. For $e \in E$ denote by $s(e) \in V$ and $t(e) \in V$ the starting and the target vertex of the edge e respectively (i.e. $e = (s(e), t(e))$).*

(a) A representation *of the graph G is a collection $\{W_v \mid v \in V\}$ of vector spaces over a field K together with a collection of linear mappings (i.e. vector space-homomorphisms) $\{\varphi_e : W_{s(e)} \to W_{t(e)} \mid e \in E\}$.*

(b) The representation $(\{W_v \mid v \in V\}, \{\varphi_e \mid e \in E\})$ with $W_v = 0$ for all $v \in V$ (and so $\varphi_e = 0$ for all $e \in E$, too) is said to be the zero-representation *of G.*

(c) Two representations $(\{W_v \mid v \in V\}, \{\varphi_e \mid e \in E\})$ and $(\{W'_v \mid v \in V\}, \{\varphi'_e \mid e \in E\})$ are said to be isomorphic *if there is a collection of vector space-isomorphisms $\{\theta_v : W_v \to W'_v \mid v \in V\}$ such that for each $e \in E$ the following diagram commutes:*

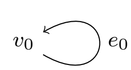
$$\begin{array}{ccc} W_{s(e)} & \xrightarrow{\varphi_e} & W_{t(e)} \\ \theta_{s(e)} \downarrow & & \downarrow \theta_{t(e)} \\ W'_{s(e)} & \xrightarrow[\varphi'_e]{} & W'_{t(e)} \end{array}$$

i.e. for each $e \in E$, $\varphi'_e \circ \theta_{s(e)} = \theta_{t(e)} \circ \varphi_e$ holds.

(d) The sum *of two representations $(\{W_v \mid v \in V\}, \{\varphi_e \mid e \in E\})$ and $(\{W'_v \mid v \in V\}, \{\varphi'_e \mid e \in E\})$ is the representation given by $W_v \oplus W'_v$ for all $v \in V$ and $\varphi_e \oplus \varphi'_e$ for all $e \in E$.*

(e) A representation $(\{W_v \mid v \in V\}, \{\varphi_e \mid e \in E\})$ is irreducible *if it is not isomorphic to a sum of two non-zero-representations. (see [1] for details)*

Утверждение 5. *(i) Consider the loop-digraph $\mathbb{L} = (\{v_0\}, \{e_0 = v_0 \to v_0\})$:*



Every irreducible representation of $\mathbb{L}$ is isomorphic to a representation given by $W_{v_0} = K^n$ and $\varphi_{e_0} = J_{n,a}$, where $n \geqslant 1$, $\alpha \neq 0$ and $J_{n,\alpha}$ is the Jordan $(n \times n)$-block matrix:

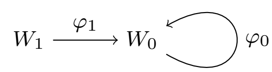
$$J_{n,\alpha} = \begin{bmatrix} \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & \alpha & 1 & \dots & 0 & 0 \\ 0 & 0 & \alpha & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha & 1 \\ 0 & 0 & 0 & \dots & 0 & \alpha \end{bmatrix}$$

Moreover, these representations are mutually non-isomorphic.

(ii) *A matrix $A \in GL_n(K)$ commutes with $J_{n,\alpha}$ if and only if A is of the form:*

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1 \end{bmatrix}, \quad a_1 \neq 0.$$

We now return to the digraph $\mathbb{A}$. Every representation of $\mathbb{A}$ is given by two vector spaces W_0 and W_1, and two homomorphisms $\varphi_0 : W_0 \to W_0$ and $\varphi_1 : W_1 \to W_0$:



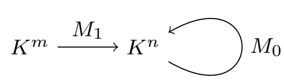
We denote the previous representation by $(W_1, W_0, \varphi_1, \varphi_0)$. If $\dim(W_0) = n$ and $\dim(W_1) = m$, we may identify $W_0 = K^n$ and $W_1 = K^m$, and then φ_0 and φ_1 are identified respectively with $n \times n$ and $n \times m$ matrices M_0 and M_1, so the above representation is determined by the quadruple (m, n, M_1, M_0).

Lemma 1. *Consider a representation determined by (m, n, M_1, M_0).*

(i) *If the representation is irreducible, then $m \leqslant n$ and* $\operatorname{rank}(M_1) = m$.

(ii) *If $m \leqslant n$,* $\operatorname{rank}(M_1) = m$ *and M_0 is similar to $J_{n,\alpha}$ for some $\alpha \neq 0$, then the representation is irreducible.*

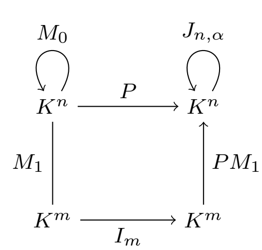
Proof. (i) Suppose that the following representation is irreducible:



Denote by φ_0 and φ_1 mappings given by M_0 and M_1 respectively.

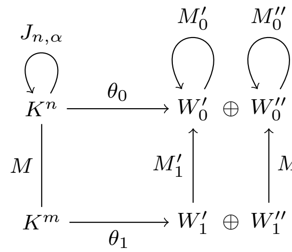
It suffices to prove that $\varphi_1 : K^m \to K^n$ is injective (which clearly implies both $m \leqslant n$ and $\operatorname{rank}(M_1) = m$). Suppose not; then $\ker\varphi_1$ is non-trivial. Find $W \leqslant K^m$ such that $K^m = \ker\varphi_1 \oplus W$. Then the above representation is (equal to) the sum of non-zero representations $(\ker\varphi_1, 0, 0, 0)$ and $(W, K^n, \varphi_{1\restriction W}, \varphi_0)$. This contradicts the irreducibility of the representation.

(ii) Suppose that $m \leqslant n$, $\operatorname{rank}(M_1) = m$ and M_0 is similar to $J_{n,\alpha}$. If $P \in GL_n(K)$ is such that $M_0 = P^{-1} J_{n,\alpha} P$, note that we have the following isomorphism of the representations given by (m, n, M_1, M_0) and $(m, n, PM_1, J_{n,\alpha})$:

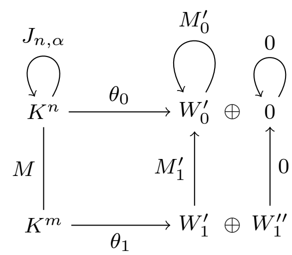


(I_m is the identity $(m \times m)$-matrix.) Clearly, $\operatorname{rank}(PM_1) = m$, so it suffices to prove irreducibility of the representation given by $(m, n, M, J_{n,\alpha})$ where $m \leqslant n$ and $\operatorname{rank}(M) = m$.

Suppose that we have the following reduction:



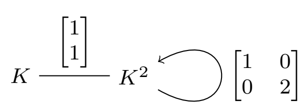
where θ_0 and θ_1 are isomorphisms. By Fact 5(i), one of W_0' and W_0'' must be zero, as otherwise the above reduction in particular would give a reduction of a Jordan block representation of the loop graph (which is irreducible by Fact 5(i)). Without loss of generality we may assume W_0'', so the reduction becomes:



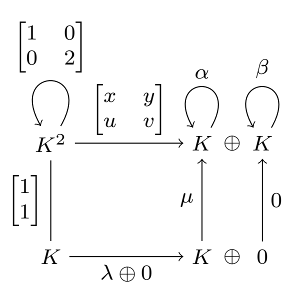
Now, $m \leqslant n$ and $\operatorname{rank}(M) = m$ yield that the homomorphism given by M is injective, so the one given by $M_1' \oplus 0$ on the right-hand side is also injective. This means that $W_1'' = 0$, so the above reduction is in fact trivial. Therefore, our representation is irreducible. ☐

Although, by the previous lemma, M_0 being a Jordan block matrix is a sufficient condition for irreducibility, it is not a necessary condition as the following easy example shows.

Замечание 1. *Consider the following representation:*



If it is reducible, we would have $\lambda, \mu, \alpha, \beta \in K^\times$ *and* $\begin{bmatrix} x & y \\ u & v \end{bmatrix} \in GL_2(K)$ *such that the following diagram commutes:*



By commutativity of the "square" part we have $\begin{bmatrix} x & y \\ u & v \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \mu \\ 0 \end{bmatrix} \lambda$, *where from we conclude* $u + v = 0$, *i.e.* $v = -u$. *Now, by commutativity of the "loop" part we have* $\begin{bmatrix} x & y \\ u & -u \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} =$

$= \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} x & y \\ u & -u \end{bmatrix}$, *i.e.* $\begin{bmatrix} x & 2y \\ u & -2u \end{bmatrix} = \begin{bmatrix} \alpha x & \alpha y \\ \beta u & -\beta u \end{bmatrix}$. *From the bottom row we obtain* $u = \beta u = 2u$, *so* $u = 0$, *and hence* $v = 0$. *This is a contradiction as* $\begin{bmatrix} x & y \\ u & v \end{bmatrix}$ *is regular.*

As a first step in our investigation, we consider the special case of irreducible representations given by Lemma 1(ii). So we aim to classify irreducible representations given by (m, n, M_1, M_0), where $m \leqslant n$, $\operatorname{rank}(M_1) = m$ (this is necessary by Lemma 1(i)), and M_0 is similar to $J_{n,\alpha}$ for some $\alpha \neq 0$. If $M_0 = P^{-1} J_{n,\alpha} P$, then the representation given by (m, n, M_1, M_0) is clearly isomorphic to the one given by $(m, n, PM_1, J_{n,\alpha})$, so we may suppose that $M_0 = J_{n,\alpha}$. For the representation given my $(m, n, M_1, J_{n,\alpha})$ we say that it is *of type* (m, n, α).

3. Irreducible representations of type $(1, n, \alpha)$

Throughout this section, denote by M_i the $(n \times 1)$-matrix:

$$M_i := \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}^T$$

with 1 in the i-th row.

Lemma 2. *Suppose that we have the following irreducible representation of type* $(1, n, \alpha)$*:*

$$K \xrightarrow{M} K^n \circlearrowleft J_{n,\alpha}$$

where $M = \begin{bmatrix} m_1 & m_2 & \dots & m_n \end{bmatrix}^T$. *Then if* $i \leqslant n$ *is such that* $m_i \neq 0$ *and* $m_j = 0$ *for all* $i < j \leqslant n$, *the above representation is isomorphic to:*

$$K \xrightarrow{M_i} K^n \circlearrowleft J_{n,\alpha}$$

Proof. We need to find $\lambda \in K^\times$ and $A \in GL_n(K)$ such that the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{M} & K^n \circlearrowleft J_{n,\alpha} \\ \lambda \downarrow & & \downarrow A \\ K & \xrightarrow[M_i]{} & K^n \circlearrowleft J_{n,\alpha} \end{array}$$

i.e. such that $AM = M_i \lambda$ and $AJ_{n,\alpha} = J_{n,\alpha} A$. By Fact 5, A is of the following form:

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 \end{bmatrix}$$

where $a_1 \neq 0$, so we must show that the following equation (in variables $a_1, \dots, a_n, \lambda$) has a solution:

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$$

This reduces to the system:

$$\begin{array}{rcrcccrcrcl} a_1m_1 & + & a_2m_2 & + & \dots & + & a_{i-1}m_{i-1} & + & a_im_i & = & 0 \\ & & a_1m_2 & + & \dots & + & a_{i-2}m_{i-1} & + & a_{i-1}m_i & = & 0 \\ & & & & & & & & & \vdots & \\ & & & & & & a_1m_{i-1} & + & a_2m_i & = & 0 \\ & & & & & & & & a_1m_i & = & \lambda \end{array}$$

so we see that we may take e.g. $\lambda = m_i \in K^\times$, $a_1 = 1$ and recursively find $a_2 = -a_1m_{i-1}/m_i, \dots,$ $a_i = -(a_1m_1 + \dots + a_{i-1}m_{i-1})/m_i$; we may also put $a_{i+1} = \dots = a_n = 0$. □

Lemma 3. *If $1 \leqslant i < j \leqslant n$, then the representations:*

$$K \xrightarrow{M_i} K^n \circlearrowleft J_{n,\alpha} \qquad \textit{and} \qquad K \xrightarrow{M_j} K^n \circlearrowleft J_{n,\alpha}$$

are non-isomorphic.

Proof. We have to show that for no $\lambda \in K^\times$ and A of the form:

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 \end{bmatrix}$$

where $a_1 \neq 0$, $AM_i = M_j\lambda$, i.e.

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i \quad = \quad \begin{bmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{bmatrix} \leftarrow j$$

holds. This is obvious as it implies $a_1 \cdot 1 = 0$ as $i < j$. □

As a direct corollary of the previous two lemmas we obtain:

Theorem 1. *Up to isomorphism, all non-isomorphic irreducible representations of type $(1, n, \alpha)$ are given by (for $i \leqslant n$):*

$$K \xrightarrow{M_i} K^n \circlearrowleft J_{n,\alpha}$$

In particular, there are exactly n non-isomorphic representations of type $(1, n, \alpha)$.

4. Irreducible representations of type $(n-1, n, \alpha)$

Throughout this section, denote by M_i the $(n \times (n-1))$-matrix:

$$M_i := \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow & i-1 \\ \leftarrow & i \\ \leftarrow & i+1 \\ \\ \\ \end{matrix}$$

so the matrix I_{n-1} with a zero-row added as the i-th row. Moreover, we also fix the following notation. For an $(n \times (n-1))$-matrix M, $\operatorname{rank}_i(M)$ denotes the rank of the matrix obtained by deleting the i-th row from M.

Lemma 4. *Suppose that we have the following irreducible representation of type* $(n-1, n, \alpha)$*:*

$$K^{n-1} \xrightarrow{M} K^n \circlearrowleft J_{n,\alpha}$$

Then if $i \leqslant n$ is such that $\operatorname{rank}_i(M) = n-1$ and $\operatorname{rank}_j(M) < n-1$ for all $j < i$, the above representation is isomorphic to:

$$K^{n-1} \xrightarrow{M_i} K^n \circlearrowleft J_{n,\alpha}$$

Proof. Suppose that $\operatorname{rank}_i(M) = n-1$ and $\operatorname{rank}_j(M) < n-1$ for all $j < i$. Since $\operatorname{rank}(M_i) = n-1$, by elementary transformations of columns only, we may transform M to the matrix of the following form:

$$M' := MQ = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ m_1 & \dots & m_{i-1} & m_i & \dots & m_{n-1} \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow & i-1 \\ \leftarrow & i \\ \leftarrow & i+1 \\ \\ \\ \end{matrix}$$

where $Q \in GL_{n-1}(K)$ is the product of all elementary matrices used in the transformation. Since elementary transformations of columns don't change the row-rank, $\operatorname{rank}_j(MQ) < n-1$ for all $j < i$ too. From here we directly see that it must be $m_1 = \dots = m_{i-1} = 0$, so:

$$M' = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & m_i & \dots & m_{n-1} \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow & i-1 \\ \leftarrow & i \\ \leftarrow & i+1 \\ \\ \\ \end{matrix}$$

Thus we have an isomorphism of representations:

$$\begin{array}{ccc} K^{n-1} & \xrightarrow{M} & K^n \circlearrowleft J_{n,\alpha} \\ {\scriptstyle Q^{-1}}\downarrow & & \downarrow{\scriptstyle I_n} \\ K^{n-1} & \xrightarrow[M']{} & K^n \circlearrowleft J_{n,\alpha} \end{array}$$

Now, it suffices to find an isomorphism of representations of the following form:

$$\begin{array}{ccc} K^{n-1} & \xrightarrow{M'} & K^n \circlearrowleft J_{n,\alpha} \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle B} \\ K^{n-1} & \xrightarrow[M_i]{} & K^n \circlearrowleft J_{n,\alpha} \end{array}$$

So we need $A \in GL_{n-1}(K)$ and $B \in GL_n(K)$ such that $M_iA = BM'$ and $BJ_{n,\alpha} = J_{n,\alpha}B$. Since, $BJ_{n,\alpha} = J_{n,\alpha}B$, by Fact 5(ii), B must be found in the following form:

$$B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \\ 0 & b_1 & \dots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_1 \end{bmatrix}, \quad b_1 \neq 0,$$

so we have to check that for such B, $BM' = M_iA$ has a solution (for A and B). Note that the i-th row of M_iA (for any A) is zero, so let us first look at the i-th row of BM'. We have:

$$(BM')_i = \begin{bmatrix} 0 & \dots & 0 & b_1 & b_2 & \dots & b_{n-i+1} \end{bmatrix} \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & m_i & \dots & m_{n-1} \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

and the i-th row equals:

$$\begin{bmatrix} 0 & \dots & 0 & b_1m_i + b_2 & b_1m_{i+1} + b_3 & \dots & b_1m_{n-1} + b_{n-i+1} \end{bmatrix}.$$

Put $b_1 = 1, b_2 = -m_i, b_3 = -m_{i+1}, \dots, b_{n-i+1} = -m_{n-1}$, then the obtained row is zero, and further put $b_{n-i+2} = \dots = b_n = 0$. For the obtained matrix B, BM' has the i-th row zero. Now set A to be BM' after deleting the i-th row. It is easy to see that A is an upper triangular matrix with ones on the diagonal, thus it is regular, and that $BM' = M_iA$. This finishes the proof. □

Lemma 5. *If $1 \leqslant i < j \leqslant n$, then the representations:*

$$K^{n-1} \xrightarrow{M_i} K^n \circlearrowleft J_{n,\alpha} \quad \textit{and} \quad K^{n-1} \xrightarrow{M_j} K^n \circlearrowleft J_{n,\alpha}$$

are non-isomorphic.

Proof. We have to show that there are no $A \in GL_{n-1}(K)$ and $B \in GL_n(K)$ of the form:

$$B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \\ 0 & b_1 & \dots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_1 \end{bmatrix}$$

where $b_1 \neq 0$, such that $BM_i = M_jA$. The j-th row of M_jA is zero, while j-th row of BM_i has $b_1 \neq 0$ in the place $(j, j-1)$ (as $j-1 \geqslant i$). Therefore, the two representations are non-isomorphic. $\square$

As a direct corollary of the previous two lemmas we obtain:

Theorem 2. *Up to isomorphism, all irreducible representations of type $(n-1, n, \alpha)$ are given by (for $i \leqslant n$):*

$$K \xrightarrow{M_i} K^n \circlearrowright J_{n,\alpha}$$

In particular, there are exactly n non-isomorphic representations of type $(n-1, n, \alpha)$.

5. Irreducible representations of type $(2, n, \alpha)$

Throughout this section, denote by $M_{m,k}(x_1, \dots, x_{m-1})$ the $(n \times 2)$-matrix:

$$M_{m,k}(x_1, \dots, x_{m-1}) := \left[\begin{array}{ccc} x_1 & 0 & \\ x_2 & 0 & \\ \vdots & \vdots & \\ x_{m-1} & 0 & \\ 1 & 0 & \leftarrow m \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \\ 0 & 1 & \leftarrow k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{array}\right]$$

where $1 \leqslant m < k \leqslant n$ and $x_1, x_2, \dots, x_{m-1} \in K$. For $i < m$ and $x_1, \dots, x_{m-2} \in K$ denote $M_{m,k,i}(x_1, \dots, x_{m-2}) := M_{m,k}(x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{m-2})$.

Lemma 6. *If $(m_1, k_1) \neq (m_2, k_2)$, then the representations:*

$$K^2 \xrightarrow{M_{m_1,k_1}(\vec{x})} K^n \circlearrowright J_{n,\alpha} \quad \text{and} \quad K^2 \xrightarrow{M_{m_2,k_2}(\vec{y})} K^n \circlearrowright J_{n,\alpha}$$

are non-isomorphic for arbitrary $\vec{x}, \vec{y} \in K^{m-1}$.

Proof. We have to show that there are no $A \in GL_2(K)$ and $B \in GL_n(K)$ of the form:

$$B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \\ 0 & b_1 & \dots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_1 \end{bmatrix}$$

where $b_1 \neq 0$, such that $BM_{m_1,k_1} = M_{m_2,k_2}A$. If $k_1 < k_2$, the k_1-th row of BM_{m_1,k_1} has 1 in place $(k_1, 2)$, while the same place in k_1-th row of $M_{m_2,k_2}A$ is 0. If $k_1 = k_2$ and $m_1 < m_2$, then m_2-th row of BM_{m_1,k_1} is zero, while m_2-th row of $M_{m_2,k_2}A$ has 1 in place $(m_2, 1)$. Therefore, the two representations are non-isomorphic. □

Consider an irreducible representation of type $(2, n, \alpha)$:

$$K^2 \xrightarrow{M} K^n \circlearrowleft J_{n,\alpha}$$

Recall that the rank of M is two. Define:

$$k_M := \max\left\{k \leqslant n \colon (\exists m < k)\ \operatorname{rank}\begin{bmatrix} M_m \\ M_k \end{bmatrix} = 2\right\},$$

where M_i denotes the i-th row of M, and then:

$$m_M := \max\left\{m < k_M \colon \operatorname{rank}\begin{bmatrix} M_m \\ M_{k_M} \end{bmatrix} = 2\right\}.$$

From now on we fix the previous representation, i.e. the matrix M, so to simplify the notation, we denote k_M and m_M only by k and m.

Lemma 7. *There is $\vec{x} \in K^{m-1}$ such that the above representation is isomorphic to the one given by $M_{m,k}(\vec{x})$. Moreover, if $k < 2m$, then the above representation is isomorphic to the one given by $M_{m,k,2m-k}(\vec{x})$ for some $\vec{x} \in K^{m-2}$.*

Proof. By elementary transformations of columns only, and the fact that m-th and k-th rows are linearly independent, we may transform M to the matrix of the following form:

$$M' := MQ = \begin{bmatrix} a_{11} & a_{12} & \\ \vdots & \vdots & \\ a_{m-1,1} & a_{m-1,2} & \\ 1 & 0 & \leftarrow\ m \\ a_{m+1,1} & a_{m+1,2} & \\ \vdots & \vdots & \\ a_{k-1,1} & a_{k-1,2} & \\ 0 & 1 & \leftarrow\ k \\ a_{k+1,1} & a_{k+1,2} & \\ \vdots & \vdots & \\ a_{n,1} & a_{n,2} & \end{bmatrix}$$

where $Q \in GL_2(K)$ is the product of all elementary matrices used in the transformation. Recall that elementary transformations of columns don't change the rank of rows. Hence, for $i > k$, by the choice of k, i-th row is linearly dependent with k-th and with m-th row, so we see that $a_{i,1} = a_{i,2} = 0$.

Similarly, for $m < i < k$, by the choice of m now, i-th row is lineary dependent with k-th row, so we see that $a_{i,1} = 0$. Therefore, our matrix M' equals:

$$M' = \begin{bmatrix} a_{11} & a_{12} & \\ \vdots & \vdots & \\ a_{m-1,1} & a_{m-1,2} & \\ 1 & 0 & \leftarrow\ m \\ 0 & a_{m+1,2} & \\ \vdots & \vdots & \\ 0 & a_{k-1,2} & \\ 0 & 1 & \leftarrow\ k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix}$$

Clearly, representations given by M and M' are isomorphic. It suffices to find an isomorphism of representations given by M' and $M_{m,k}(\vec{x})$ for some $\vec{x} \in K^{m-1}$. We do that by finding $B \in GL_n(K)$ such that $BM' = M_{m,k}(\vec{x})$ and $BJ_{n,\alpha} = J_{n,\alpha}B$. Since, $BJ_{n,\alpha} = J_{n,\alpha}B$, by Fact 5(ii), B must be found in the following form:

$$B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \\ 0 & b_1 & \dots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_1 \end{bmatrix}, \quad b_1 \neq 0. \tag{1}$$

Consider BM':

$$BM' = \begin{bmatrix} \sum_{i=1}^{m-1} b_i a_{i,1} + b_m & \sum_{i=1}^{k-1} b_i a_{i,2} + b_k & \\ \sum_{i=1}^{m-2} b_i a_{i+1,1} + b_{m-1} & \sum_{i=1}^{k-2} b_i a_{i+1,2} + b_{k-1} & \\ \vdots & \vdots & \\ b_1 & \sum_{i=1}^{k-m} b_i a_{i+m-1,2} + b_{k-m+1} & \leftarrow\ m \\ 0 & \sum_{i=1}^{k-m-1} b_i a_{i+m,2} + b_{k-m} & \\ \vdots & \vdots & \\ 0 & b_1 a_{k-1,2} + b_2 & \\ 0 & b_1 & \leftarrow\ k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix}$$

It is clear that if we recursively put $b_1 = 1$ and $b_{k-j} = -\sum_{i=1}^{k-j-1} b_i a_{i+m,2}$ for $j = k-2, k-3, \dots, 0$, we obtain the desired zeroes in the second column, i.e. we obtain $M_{m,k}(\vec{x})$, where $\vec{x}$ can be easily calculated.

For the "moreover" part, suppose that $k < 2m$, and set $i = 2m - k$. We prove that $B \in GL_n(K)$ of the form (1), and $\vec{y} \in K^{m-2}$ can be found such that $BM_{m,k}(\vec{x}) = M_{m,k,i}(\vec{y}) \begin{bmatrix} 1 & -x_i \\ 0 & 1 \end{bmatrix}$; clearly, this finishes the proof. For, put $b_1 = 1$, and consider $BM_{m,k}(\vec{x}) = M_{m,k,i}(\vec{y}) \begin{bmatrix} 1 & -x_i \\ 0 & 1 \end{bmatrix}$:

$$\begin{bmatrix} \sum_{j=1}^{m-1} b_j x_j + b_m & b_k & \\ \sum_{j=1}^{m-2} b_j x_{j+1} + b_{m-1} & b_{k-1} & \\ \vdots & \vdots & \\ \sum_{j=1}^{m-i+1} b_j x_{j+i-1} + b_{m-i+2} & b_{k-i+2} & \\ \sum_{j=1}^{m-i} b_j x_{j+i-1} + b_{m-i+1} & b_{k-i+1} & \leftarrow i \\ \sum_{j=1}^{m-i-1} b_j x_{j+i-1} + b_{m-i} & b_{k-i} & \\ \vdots & \vdots & \\ x_{m-1} + b_2 & b_{k-m+2} & \\ 1 & b_{k-m+1} & \leftarrow m \\ 0 & b_{k-m} & \\ \vdots & \vdots & \\ 0 & b_2 & \\ 0 & 1 & \leftarrow k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix} = \begin{bmatrix} y_1 & -x_i y_1 & \\ y_2 & -x_i y_2 & \\ \vdots & \vdots & \\ y_{i-1} & -x_i y_{i-1} & \\ 0 & 0 & \leftarrow i \\ y_i & -x_i y_i & \\ \vdots & \vdots & \\ y_{m-2} & -x_i y_{m-2} & \\ 1 & -x_i & \leftarrow m \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \\ 0 & 1 & \leftarrow k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix}$$

We first note that we must set $b_2 = \cdots = b_{k-m} = 0$ and $b_{k-m+1} = -x_i$. Now, we calculate that $y_{m-2} = x_{m-1}, \ldots, y_i = x_{i+1}$, and we set $b_{k-m+1} = -x_i y_{m-2}, \ldots, b_{k-i} = -x_i y_i$. Look at the i-th row. On the left hand side, since $m-i = m-2m+k = k-m$ and $m-i+1 = m-2m+k+1 = k-m+1$, we have $x_i + b_{m-i+1} = x_i + b_{k-m+1} = x_i - x_i = 0$ (note that other terms in the sume are zero), so it just remains to set $b_{k-i+1} = 0$. Finally, we can now calculate y_{i-1}, then set $b_{k-i+2} = -x_i y_{i-1}$, calculate b_{k-i+3}, then set $b_{k-i+3} = -x_i y_{i-2}$, etc. □

Lemma 8. *If $k \geqslant 2m$, representation determined by $M_{m,k}(\vec{x})$ and $M_{m,k}(\vec{y})$ are non-isomorphic for distinct $\vec{x}, \vec{y} \in K^{m-1}$. If $k = 2m - i$, representation determined by $M_{m,k,i}(\vec{x})$ and $M_{m,k,i}(\vec{y})$ are non-isomorphic for distinct $\vec{x}, \vec{y} \in K^{m-2}$.*

Proof. We have to show that there are no $A \in GL_2(K)$ and $B \in GL_n(K)$ of the form:

$$B = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \\ 0 & b_1 & \ldots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_1 \end{bmatrix}$$

where $b_1 = 1$, such that $BM_{m,k}(\vec{x}) = M_{m,k}(\vec{y})A$. We have the following equation:

$$\begin{bmatrix} 1 & b_2 & b_3 & \dots & b_{n-1} & b_n \\ 0 & 1 & b_2 & \dots & b_{n-2} & b_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_2 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & 0 & \\ x_2 & 0 & \\ \vdots & \vdots & \\ x_{m-1} & 0 & \\ 1 & 0 & \leftarrow \ m \\ 0 & 0 & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 1 & \leftarrow \ k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix} = \begin{bmatrix} y_1 & 0 & \\ y_2 & 0 & \\ \vdots & \vdots & \\ y_{m-1} & 0 & \\ 1 & 0 & \leftarrow \ m \\ 0 & 0 & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 1 & \leftarrow \ k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

Then we have:

$$\begin{bmatrix} x_1 + b_2x_2 + b_3x_3 + \dots + b_m & b_k & \\ x_2 + b_2x_3 + \dots + b_{m-1} & b_{k-1} & \\ \vdots & \vdots & \\ x_{m-1} + b_2 & b_{k-m+2} & \\ 1 & b_{k-m+1} & \leftarrow \ m \\ 0 & b_{k-m} & \\ 0 & b_{k-m-1} & \\ \vdots & \vdots & \\ 0 & b_2 & \\ 0 & 1 & \leftarrow \ k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix} = \begin{bmatrix} y_1 & y_1b & \\ y_2 & y_2b & \\ \vdots & \vdots & \\ y_{m-1} & y_{m-1}b & \\ 1 & b & \leftarrow \ m \\ 0 & 0 & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 1 & \leftarrow \ k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix}$$

Put $b = b_{k-m+1}$, $b_2 = b_3 = \dots = b_{k-m} = 0$, and we can choose $b_k = y_1b_{k-m+1}, b_{k-1} = = y_2b_{k-m+1}, \dots, b_{k-m+2} = y_{m-1}b_{k-m+1}$. For $k \geqslant 2m$ we have $k - m \geqslant m$, and it yields that $x_i = y_i$ for $1 \leqslant i < m$.

In the case $k = 2m - i$ for $1 \leqslant i < m$ we have following equation:

$$\begin{bmatrix} 1 & b_2 & b_3 & \dots & b_{n-1} & b_n \\ 0 & 1 & b_2 & \dots & b_{n-2} & b_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_2 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & 0 & \\ x_2 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \\ \vdots & \vdots & \\ x_{m-1} & 0 & \\ 1 & 0 & \leftarrow m \\ 0 & 0 & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 1 & \leftarrow k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix} = \begin{bmatrix} y_1 & 0 & \\ y_2 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \leftarrow i \\ \vdots & \vdots & \\ y_{m-1} & 0 & \\ 1 & 0 & \leftarrow m \\ 0 & 0 & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 1 & \leftarrow k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + b_2x_2 + \dots + b_{m-1}x_{m-1} + b_m & b_k & \\ x_2 + b_2x_3 + \dots + b_{m-1} & b_{k-1} & \\ \vdots & \vdots & \\ b_2x_{i+1} + \dots + b_{m-i}x_{m-i} + b_{m-i+1} & b_{k-i} & \\ \vdots & \vdots & \\ x_{m-1} + b_2 & b_{k-m+2} & \\ 1 & b_{k-m+1} & \leftarrow m \\ 0 & b_{k-m} & \\ 0 & b_{k-m-1} & \\ \vdots & \vdots & \\ 0 & b_2 & \\ 0 & 1 & \leftarrow k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix} = \begin{bmatrix} y_1 & y_1b & \\ y_2 & y_2b & \\ \vdots & \vdots & \\ 0 & 0 & \\ \vdots & \vdots & \\ y_{m-1} & y_{m-1}b & \\ 1 & b & \leftarrow m \\ 0 & 0 & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 1 & \leftarrow k \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix}$$

Notice that $b_2 = \dots = b_{k-m} = b_{m-i} = 0$, $b_{k-i} = 0$ $b = b_{k-m+1} = b_{m-i+1}$. Then $\vec{x}_i = \vec{y}_i$ for $\vec{x}, \vec{y} \in K^{m-2}$. This finishes the proof. $\square$

Directly from the previous three lemmas we have:

THEOREM 3. *Up to isomorphism, all irreducible representations of type $(2, n, \alpha)$ are given by the following matrices:*

- $M_{m,k}(\vec{x})$ *where* $1 \leqslant m < k \leqslant n$, $k \geqslant 2m$ *and* $\vec{x} \in K^{m-1}$, *and*
- $M_{m,k,2m-k}(\vec{x})$ *where* $1 \leqslant m < k \leqslant n$, $k < 2m$ *and* $\vec{x} \in K^{m-2}$.

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Получено: 12.01.2025
Принято в печать: 07.04.2025