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Новые оценки задачи Борсука в пространствах ℓ_p[1]

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Аннотация

В 2013 году Андрей Бондаренко сконструировал двумерное множество на единичной сфере $S^{64} \subset \mathbb{R}^{65}$, состоящее из 416 точек, которое нельзя разрезать на 83 части меньшего диаметра. В данной статье мы показываем, что эта конструкция работает не только в евклидовом пространстве, но и во всех ℓ_p-пространствах.

Ключевые слова: гипотеза Борсука, двумерные множества, сильно регулярные графы, пространства ℓ_p, комбинаторная геометрия.

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New bounds on Borsuk's problem in ℓ_p-spaces

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Abstract

In 2013, Andriy Bondarenko constructed a two-distance set on the unit sphere $S^{64} \subset \mathbb{R}^{65}$, consisting of 416 points that cannot be partitioned into 83 parts of smaller diameter. In this paper, we show that this construction works not only for the Euclidean space but for all ℓ_p-spaces.

Keywords: Borsuk's conjecture, Two-distance sets, Strongly regular graphs, ℓ_p spaces, Combinatorial geometry.

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1. Introduction

For the Euclidean space $\mathbb{R}^n$, we define $b(\mathbb{R}^n)$ to be Borsuk's number, which is the minimum number such that any bounded non-singleton set in $\mathbb{R}^n$ can be partitioned into $n+1$ parts of smaller diameter. In 1933, Karol Borsuk [2] proposed his conjecture, which states that $b(\mathbb{R}^n) = n+1$. Borsuk proved in the same paper that this conjecture holds for $n = 2$. Later, it was proved true for $n = 3$ by Eggleston [8] , and with a simpler proof provided by Grünbaum [3].
The conjecture remained open and widely believed to be true for a long time. However, in 1993, Kahn and Kalai [4] disproved it for $n = 1325$ and for all $n \geq 2015$. Subsequently, many mathematicians improved the lowest dimension where the conjecture is false, see [5] [6] [7], culminating in 2013 when Bondarenko [1], found a counterexample using two-distance set for dimension $n = 65$. In 2014, T. Jenrich A. E. Brouwer [9] modified it for $n = 64$, which is the best result known for the low dimensions of the conjecure so far. conjecture remains open for $4 \leq n \leq 63$.
For an n-dimensional normed Minkowski space M^n, its Borsuk's number is denoted by $b(M^n)$. This conjecture was first studied in Minkowski spaces by Grünbaum [10], who proved that $b(M^2) = 4$, when the unit ball of M^2 is a parallelogram; otherwise, $b(M^2) \leq 3$. Later, the conjecture was studied by Gohberg and Boltyanskii [11] in higher dimensions, and they conjectured that $b(M^n) \leq 2^n$.
In this article, we focus on ℓ_p-spaces equipped with the ℓ_p-norm. where, Yu and Zong [12] proved that $b(M^n) \geq 2^n$ holds true for all ℓ_p^n-spaces for $n = 3$, and Wang and Xue [13] proved it for $n = 4$. so far The last result we know for higher dimensions of $b(\ell_p^n)$ was provided by Raigorodskii and Sagdeev [14]. Here in this paper, we generalize Bondarenko's technique to show the following:

2. Main Body of the Paper

Theorem. For $p \geq 1$ There is a point set $\boldsymbol{y_i^*} = y_{i,j} \in \ell_p^{65}$ where $i, j \in V$ the set of vertices of the strongly regular graph Γ, such that $\left\|y_i^* - y_j^*\right\|_p^p =$ a or b, for $i \neq j$, where a, b are integers, which cannot be partitioned into less than 84 parts of smaller diameter.

Hence,

$$b\left(\ell_p^{65}\right) \geq b_2\left(\ell_p^{65}\right) \geq 84.$$

Corollary. For integers $n \geq 1$ and $k \geq 0$, and $p \geq 1$, we have

$$b_2(\ell_p^{66n+k}) \geq 84n + k + 1.$$

2.1. strongly regular graph

A strongly regular graph Γ with parameters (v, k, λ, μ) is an undirected regular graph on v vertices of valency k, such that each pair of adjacent vertices has λ common neighbors, and each pair of nonadjacent vertices has μ common neighbors.

2.1.1. The Euclidean representation:

A graph $\Gamma = (V, E)$ is denoted by $\mathrm{srg}(v, k, \lambda, \mu)$, where The (0,1)-adjacency matrix of a $\mathrm{srg}(v, k, \lambda, \mu)$, has an eigenvalue k with multiplicity 1 , and two more eigenvalues:

$$\begin{aligned} r &= \frac{1}{2}\left((\lambda - \mu) + \sqrt{(\lambda - \mu)^2 + 4(\mathrm{k} - \mu)}\right) \\ s &= \frac{1}{2}\left((\lambda - \mu) - \sqrt{(\lambda - \mu)^2 + 4(\mathrm{k} - \mu)}\right) \end{aligned} \tag{1}$$

With multiplicities:

$$f = \frac{1}{2}\left((\lambda-\mu) - \frac{(v-1)(\lambda-\mu)+2k}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right)$$
$$g = \frac{1}{2}\left((\lambda-\mu) + \frac{(v-1)(\lambda-\mu)+2k}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right) \quad (2)$$

Respectively.

The given properties of the eigenvalues imply that $\dim P(V) = f$, the properties of the graph and its vectors configurations that is made with eigenvalues of the adjacency matrix can be found in [15].

We use these notations: I is the identity matrix of size v, $y = \mathrm{A} - \mathrm{sI}$, y_i where $i \in V$ are the columns of y, and $y_{i,j}$ where $i, j \in V$, are the entries of y, and for any $W \subseteq V$, the corresponding point set P(W) is $\{y_i : \mathrm{i} \in \mathrm{W}\}$.

For $i, j \in V$

$$y_{i,j} = \begin{cases} -s & \text{if } i = j \\ 1 & \text{if } (i,j) \in E \\ 0 & \text{if } i \neq j \end{cases}$$

For $i \in V$: y_i consists of one s at position i, 1 at k positions, and 0 at $(v-k-1)$ positions, elsewhere. its Euclidean norm is the square root of $s^2 + k$

For different $i, j \in V$

$$\|y_i - y_j\|^2 = \begin{cases} 2\left(k-\lambda-1+(-\mathrm{s}-1)^2\right) \text{ if } (i,j) \in E \\ 2\left(\mathrm{k}-\mu+s^2\right) \text{ otherwise} \end{cases}$$

The distance for the non-adjacent case exceeds the distance for the adjacent case by:

$$2(\lambda-\mu-2s) = 2\sqrt[2]{(\lambda-\mu)^2+4(k-\mu)}$$

If the graph is not complete, this excess is positive, and we can conclude: For any two different $i, j \in V$, the distance between y_i and y_j is smaller than the diameter of the complete vector set if and only if i and j are neighbors. Thus, for each $W \subseteq V$, the diameter of the corresponding point set P(W) is smaller than that of P(V) if and only if W is a clique. Furthermore, P(V) can be divided into x parts of smaller diameter if and only if V can be divided into x cliques.

2.1.2. the representation in ℓ_p-spaces:

The vectors $\boldsymbol{y_i} = (y_{i_1}, y_{i_2}, y_{i_3}, \dots, y_{i_n}) \in \mathbb{R}^n$ will have ℓ_p- norms as follows:

$$\|y_i\|_p = \left(\sum_{j=1}^{n} |y_{i,j}|^p\right)^{\frac{1}{p}}, p \geq 1, i \in \mathrm{V}.$$

We map every $\boldsymbol{y_i}$ to $\boldsymbol{y_i^*} = y_{i,j} \in \ell_p^n$, $j = 1, \dots, n$

For $i \in V$: y_i^* consists of one s at position i , 1 at k positions, and 0 at $\mathrm{v}-\mathrm{k}-10$ positions. its norm is:

$$\|\boldsymbol{y_i^*}\|_p^p = |-s|^p + k|1|^p + (v-k-1)|0|^p = |s|^p + k$$

For different $i, j \in V$, we have the distance as:

$$\|\boldsymbol{y_i^*} - \boldsymbol{y_j^*}\|_p = \begin{cases} 2(k-\lambda-1+|-s-1|^p) & \text{if } (i,j) \in E \\ 2(k-\mu+|-s|^p) & \text{otherwise} \end{cases}$$

which is positive in case of a non-empty or complete graph, the diameter of this graph is the distance when i, j are non-adjacent, which is obviously the maximum distance among them.

2.2. Proof of the main theorem:

Consider the point set $\{\boldsymbol{y}_i^* : y_i \in A - sI\}$ of the strongly regular graph Γ - $(416, 100, 36, 20)$, by (2) we get $f = 65$, and since $\lambda > \mu$, and by (1) $s = -4$, we have that:

$$2(k - \mu + |-s|^p) > 2\left(k - \lambda - 1 + |-s-1|^p\right)$$

then the diameter of this set is the distance between $\boldsymbol{y}_i^*, \boldsymbol{y}_j^*$ where i,j are non-adjacent.
Then this point set can't be partitioned into less than v/m, where m is the clique number of this graph Γ, in [16] [1] you can see the construction of the strongly regular graph $(416, 100, 36, 20)$, this graph is member of so called Suzuki tower, also called $G_2(4)$graph.

1-its local graph is the Hall-Janko graph $(100, 36, 14, 12)$ with maximal clique size 4.

2-The local graph of the Hall-Janko graph $(36, 14, 4, 6)$ is the graph $U_3(3) = G_2(2)$ (with maximal clique 3 since its local graph is bipartite)

3-The first subconstituent of U3(3) $= G_2(2)$ is the point-line non-incidence graph Δ of the Fano plane on $v = 14$ (has no triangles). See (10.68, 10.32, 10.14) in [16]

hence, the maximal clique in this graph is of size 5 , then the point set $\{\boldsymbol{y}_i^* : y_i \in A - sI\}$ can't be partitioned into less than 84 parts of smaller diameter.

□

2.3. Proof of the Corollary

For $k = 0$ and $n \in \mathbb{N}$, we put $m = 66n$, we construct vectors $y \in \mathbb{R}^m$

$$y = (y_1, ..., y_n | a_1, ..., a_n)$$

where $y_k \in \mathbb{R}^{65}$ and $a_k \in \mathbb{R}$ for $k = 1, \ldots, n$, now we take this set of vectors that lie in $\mathbb{R}^m$: S $= s_{ij}$. where

$$s_{ik} = (0, \ldots, 0, y_i^*, 0, \ldots, 0 | 0, \ldots, 0, 1, 0, \ldots, 0)$$

where $y_i^* \in \ell_p^{66n}$, now these s_{ij} vectors have only two nonzero coordinates y_k (one coordinate in the subvectors) and a_k (one coordinate in the scalars), now we can measure the distance between higher-order vectors as follows:

$$d_p(s_{ik}, s_{jk}) = \left(\sum_{k=1}^{n} \left\|y_{ik}^* - y_{jk}^*\right\|_p^p\right)^{1/p} \tag{1}$$

where, $\left\|y_{ik}^* - y_{jk}^*\right\|_p^p$ is the $\mathcal{L}_p$ distance between the vectors y_i^* and y_j^*, then in our case:

$$\begin{aligned}\|s_{ik} - s_{jk}\|_p^p &= \left(\left\|y_i^* - y_j^*\right\|_p^p\right)^{1/p} \\ &= \begin{cases} \left(2\left(k - \lambda - 1 + (-\mathrm{s} - 1)^2\right)\right)^{1/p} \text{ if } (i,j) \in E \\ \left(2\left(\mathrm{k} - \mu + s^2\right)\right)^{1/p} \text{ otherwise} \end{cases}\end{aligned}$$

and that shows that the set S is a two-distance set of $416n$ vectors. By our main theorem, this set cannot be partitioned into fewer than $84n$ pieces of smaller diameter. by adding one more vector that is at distance R from each vector in S, we get a set of $416n + 1$ vectors that span at most $66n$, so they lie in $\mathbb{R}^{66n}$. Then, $b_2(\ell_p^{66n}) \geq 84n + 1$. Again, adding one more vector that has the same R

distance from each of these $416n+1$ vectors, we get a set of $416n+2$ vectors that lie in $\mathbb{R}^{66n+1}$, giving $b_2(\ell_p^{66n+1}) \geq 84n+2$. Inductively applying this procedure, we get the desired result:

$$b_2(\ell_p^{66n+k}) \geq 84n+k+1$$

□

3. Conclusion

In this paper, we extended Bondarenko's construction to ℓ_p-spaces and established new lower bounds for Borsuk's problem in these settings. By utilizing strongly regular graphs, we proved that in dimension 65, the Borsuk number satisfies $b(\ell_p^{65}) \geq 84$ for all $p \geq 1$, improving upon known results. Furthermore, for a general formula for the higher dimensions, we proved that $b_2(\ell_p^{66n+k}) \geq 84n+k+1$ for all $p \geq 1$, meaning that Borsuk's conjecture has a negative answer for all dimensions $n \geq 65$ in ℓ_p-spaces.

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