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Топологические основы динамики нарушения трехмерной симметрии для абелевых и неабелевых моделей Хиггса

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Альмухур Эман — Частный университет прикладных наук (г. Амман, Иордания).*e-mail: e_almuhur@asu.edu.jo***Адайлах Абир** — Частный университет прикладных наук (г. Амман, Иордания).*e-mail: e_almuhur@asu.edu.jo***Саса Тала** — Частный университет прикладных наук (г. Амман, Иордания).*e-mail: e_almuhur@asu.edu.jo***Аль-Нана Абир** — Университет принца Саттама бин Абдулазиза (г. Альхардж, Саудовская Аравия)..*e-mail: e_almuhur@asu.edu.jo***Аннотация**

Мы рассматриваем роль, которую геометрические и топологические концепции сыграли в недавнем развитии теоретической физики, особенно в области теории суперструн и неабелевых калибровочных теорий. Мы также демонстрируем важность этих концепций для лучшего понимания динамических законов физики. В данной работе мы представляем численное исследование динамики трехмерного нарушения симметрии для неабелевых и абелевых моделей Хиггса. Нетривиальная топология многообразия конфигураций вакуумного поля является источником топологических возбуждений в абелевой модели Хиггса и в других теоретико-полевых моделях, которые будут обсуждаться далее. В трехмерных многокомпонентных решетчатых абелево-хиггсовских (ЛАН) моделях, минимально связанных с некомпактным абелевым калибровочным полем, мы изучаем топологические фазовые изменения, происходящие в этих моделях.

Ключевые слова: абелевы модели Хиггса, топологические деконфинитивные переходы, гомотопия.

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Topological foundations of three-dimensional symmetry breaking dynamics for both Abelian and non-abelian Higgs models

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We examine the role that geometrical and topological concepts have played in the recent development of theoretical physics, particularly in the areas of superstring theory and non-Abelian gauge theories. We also demonstrate the importance of these concepts for a better comprehension of the physics' dynamical laws. In this paper, we present a numerical study of the three-dimensional symmetry breaking dynamics for both non-abelian and abelian Higgs models. The non-trivial topology of the manifold of vacuum field configurations is the source of the topological excitations in the abelian Higgs model and in the other field theoretic models that will be discussed. In three-dimensional multicomponent lattice Abelian-Higgs (LAH) models minimally connected to a noncompact Abelian gauge field, we study the topological phase changes that occur in these models.

Keywords: Abelian Higgs models, Topological Deconfinement Transitions, Homotopy.*Bibliography:* 16 titles.**For citation:**

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1. Introduction

The first ten years of the twentieth century saw the development of two of theoretical physics' most revolutionary theories: general relativity and quantum field theory. The seemingly complete description of our universe in a simple Euclidean geometrical framework provided by the classical field theories was drastically altered: Riemannian geometry took the role of Euclidean geometry, and the classical field theories had to be quantized.

These ideas led to the development of the modern theories of elementary particles and gravitation, which physicists believe better capture nature than any earlier theory. Although both theories have an exceptionally high number of accurate predictions, they also have an exceptionally high number of errors. The standard model of elementary particles is described within the framework

of quantum field theory. To construct a quantum field theory, we first need to quantize a classical field theory. Because of divergences in the quantized theory computations, we need to establish renormalizability and use a regularization procedure before computing the physical properties of the theory.

It is clear that these actions are not independent of one another and that each one needs to be done properly. Furthermore, the compatibility of general relativity with the conventional model of elementary particles remains somewhat unclear. This is a difficult task to address, as both theories are expressed in very different mathematical languages. Considerable progress has been made in addressing these issues since the 1970s. Interestingly, several of these contributions involve topological structures, highlighting topology's current importance in theoretical physics.

2. Geometrization of theoretical physics: from geometric quantum theories to Cartan's theory of gravitation

This expository article aims to analyze some of the most significant mathematical developments and the conceptual significance of the geometrization of theoretical physics, from the work of Cartan and Weyl to the more recent non-Abelian gauge theories. Consider, for example, the quantization of a gauge field theory. To quantize such a theory, one selects a particular gauge so as to exclude extra degrees of freedom. The gauge invariance's symmetry property is lost in the process. Since the words that occur in the renormalized theory must be limited by gauge invariance, this has disastrous consequences for the renormalizability proof. Algebraic geometry concepts are used by BRST quantization to tackle this problem. In general, the BRST formalism provides an elegant foundation to work with constrained systems, like those seen in general relativity or string theories. We can request the symmetries that the quantum field theory inherits from the classical theory once the theory has been quantized. Our reflections begin with the question of how to characterize space's properties at the quantum level of physics, including group structures, symmetries, algebraic and topological invariants, and symmetry breaking. Generally speaking, we will attempt to draw attention to a few noteworthy features of the mathematical advancements spurred by efforts to resolve one of the main issues in theoretical physics during the 20th century: the integration of general relativity and quantum field theory into a single, comprehensive theoretical explanation of the physical universe. How to ascertain the topological (global) structure of the universe and the physical parameters of its early formation is another point, which is probably closely related to the previous one. Lastly, we aim to summarize a few theoretical observations that brought up the current advances in theoretical physics related to the aforementioned queries.

Remarkably, one finds difficulties in formulating quantized gauge theories when gauge fields couple to the two fermion chiral components differently, the so-called chiral anomalies. The difficulties in regularizing such chiral gauge theories without violating chiral symmetry are connected to this puzzle. Anomalies cannot exist in physical theories in terms of local symmetries. This is crucial because it constrains the couplings and particle composition of the standard model, whose electroweak sector is a chiral gauge theory. Because exact chiral symmetry could not be realized on the lattice, anomaly discussions were limited to perturbation theory until recently [1]. One may have been concerned about problems with anomaly cancelations that extended beyond perturbation theory.

This issue also precluded a numerical examination of relevant quantum field theories. Recently, it was discovered that new lattice regularization schemes, known as domain wall, overlap, and perfect action fermions, or more broadly, Ginsparg-Wilson fermions, are consistent with a generalized form of chiral symmetry.

In a fragment [2] that was published in 1833, C. F. Gauß describes a deep topological discovery that he made while looking into a physical issue. He considers the work W_n done by moving a

magnetic monopole with a magnetic charge of g along a closed channel C_1 , in the magnetic field B created by a current I flowing over a closed loop C_2 . The Biot-Savart law provides W_n as follows:

$$W_n = \oint B(x_1) dx_1 = \frac{4\pi g}{c} I \cdot lk\{C_1, C_2\} \dots (1)$$

$$\text{where } lk\{C_1, C_2\} = \frac{1}{4\pi} \oint \oint \frac{(x_2 - x_1)}{|x_2 - x_1|^3} dx_1 dx_2$$

Gauß came to the realization that W_n is independent of both the closed path C_1 and the geometric details of the current carrying loop C_2 .

The value of the Linking Number, $lk\{C_1, C_2\}$, remains constant when these curves are continuously deformed. This quantity can be described by an invariant of topology. The loop C_1 's (signed) total number of crossings of each specific (oriented) surface in $\mathbb{R}^3$, whose edge is the loop C_2 [3] and [4], is represented by an integer.

Since Leibniz proposed "another analysis, purely geometric or linear which also defines the position (situs), as algebra defines magnitude" in 1679, Gauß bemoans the lack of progress made in topology ("Geometria Situs"). Leibniz also thought of applying this new branch of mathematics to the field of physics. However, he was unable to win over physicist Christiaan Huygens to his point of view regarding topology. Topological arguments entered the field of physics with the development of the Helmholtz laws of vortex motion (1858) and the Kelvin circulation theorem (1869).

P. Tait's taxonomy of knots and links resulted from these attempts, despite the fact that it was the first of many subsequent topological attempts to explain the fundamentals of fundamental physics having to fail [5]. These days, topological methods are widely applied in physics to examine system properties. Topological foundations underlie the Aharonov-Bohm effect, Berry's phase, stability of defects in condensed matter systems, quantum liquids, and cosmology.

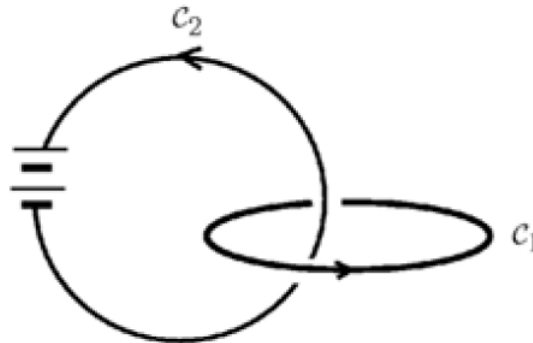


Рис.1

By their very nature, topological approaches are insensitive to details of the systems under study. Because of this, their application frequently reveals unexpected connections between events that seem remarkably different at first glance.

This common ground in the theoretical description holds true for both more visible topological phenomena, such as vortices, which are present in physics on almost all sizes, and less tangible concepts. For instance, the topological charge in gauge theories is closely associated with the topological invariant inviscid fluids known as "Helicity," which was discovered in 1969 [6].

Some gauge theory flaws are closely related to defects in nematic liquid crystals. Field theoretic investigations did not commonly employ topological methods until the development of non-abelian gauge theories [7], which are replete with non-perturbative phenomena. In 1931, Dirac's research on magnetic monopoles made topology relevant to field theory investigations in physics.

3. Lagrangian Abelian Higgs model

In theoretical physics, a Nielsen-Olesen vortex [8] is a point-like object isolated in two spatial dimensions, or an equivalent feature found in a classical field theory solution. This solution holds if the configuration space of scalar fields contains non-contractible circles.

lies. A circle that encircles the vortex at infinity in the configuration space may be "wrapped" once on another circle. This complex topological property is found in a structure known as the Nielsen-Olesen vortex, which bears the names of Holger Bech Nielsen and Poul Olesen (1973). Formally, the solution is equivalent to the superconductor solution's quantum vortex. Finding analytical solutions to the classical field theory equations of motion that apply to particular branches of physics is a topic of great interest.

Among these field theories is the Abelian Higgs model, which is important for cosmology, condensed matter physics, and particle physics [8, 9]. The full Euler-Lagrange equations of motion for this model have not yet been successfully solved. This results from the connected and highly nonlinear nature of these second order partial differential equations. Because the relationships between these second order partial differential equations are very nonlinear.

The abelian Higgs model is utilized as a gauge theory. It also includes the Higgs field, a self-interacting scalar field that has a weak relationship to the electromagnetic field. Conceptually speaking, it is useful to consider this field theory in two plus one dimensions of space-time before expanding it to three plus one dimensions for practical uses.

The abelian Higgs model Lagrangian is as follows:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D_\mu\phi) - V(\phi).....(2)$$

The self-interacting scalar matter field ϕ which is complex (charged) is present in the Higgs entropy model

$$V(\phi) = \frac{1}{4}\lambda(|\phi|^2 - a^2)^2.....(3)$$

Along the circle $|\phi| = a$ in the complex plane, the Higgs potential is smallest. The parameter that controls the strength of the Higgs field's self-interaction is taken to be positive λ for stability reasons.

In the minimal coupling of the Higgs field to the radiation field A_μ , the covariant derivative takes the place of the partial derivative ∂_μ .

$$D_\mu = \partial_\mu + ieA_\mu.....(4)$$

Gauge fields is related to field strengths by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{1}{ie}[D_\mu, D_\nu].....(5)$$

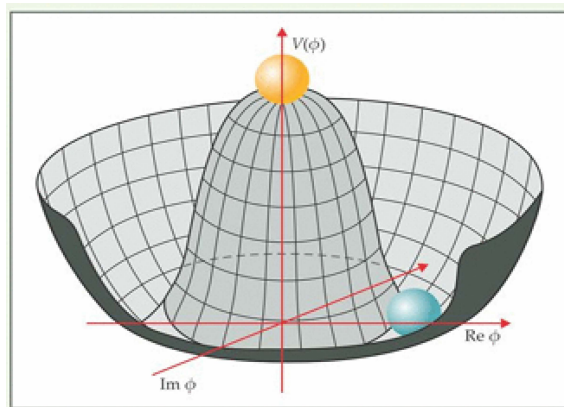


Рис.2

4. Equations for Maxwell's homogeneous and inhomogeneous systems

The theoretical foundations of classical electromagnetism, classical optics, and electric circuits are the Lorentz force law and a system of related partial differential equations known as Maxwell's equations [10], also referred to as Maxwell-Heaviside equations. The equations provide a mathematical description for electrical, optical, and radio technologies such as electric motors, power generation, wireless communication, lenses, radar, etc. They describe how electric and magnetic fields are created by charges, currents, and field variations. Even in cases when the gauge fields are not dynamical equations of motion, the field strength can be expressed in terms of them thanks to the homogeneous Maxwell equations. The homogeneous equations follow from the Jacobi identity of the covariant derivative.

The (inhomogeneous) Maxwell equations are derived using the principle of least action using variation of δ with respect to the gauge fields as follows:

$$\delta S = \delta \iint d^4x \mathcal{L} = 0 \dots (6).$$

$$\text{Typically, } -F_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} \dots (7)$$

$$\text{Also, } -K^\nu = \frac{\delta \mathcal{L}}{\delta A_\nu} \dots (8)$$

$$\text{where } K^\nu = ie(\phi^* \partial_\nu \phi) - 2e^2 \phi^* \phi A_\nu$$

Gauge theories have repeated variables. This redundancy is brought to light by the existence of local symmetry transformations, or "gauge transformations"

$$U(x) = e^{ie\alpha(x)} \dots (9)$$

(9) modifies the matter field's phase and the gauge field's value in a location- and time-dependent manner as the subsequent

$$\phi \rightarrow \phi^U = U(x)\phi(x) \dots (10)$$

$$A_\mu \rightarrow A_\mu^U = A_\mu + U(x) \frac{1}{ie} \partial_\mu U(x) \dots (11)$$

Now that the covariant derivative D_μ defines $D_\mu \phi(x) \rightarrow U(x) D_\mu \phi(x)$, it transforms like $\phi(x)$.

Together with the invariance of $F_{\mu\nu}$, this transformation condition guarantees the invariance of $\mathcal{L}$ and the equations of motion.

It is possible to study classical statistical systems approximately with the help of quantum field theory, or QFT. As the correlation length in lattice units increases, so does the approximation's accuracy of classical statistical systems. The accuracy of the approximation increases with the correlation length in lattice units. Changing the statistical model's temperature equates to deforming the QFT by a specific operator, often one that takes into account all symmetries.

The easiest way to define a renormalization group theory transformation is using the one-dimensional Ising model. The key point is that the renormalization theory can be applied to Ising models in higher dimensions easily, unlike the previously discussed transfer matrix method or other procedures, which makes it instructive to recover the same conclusion from the perspective of renormalization group theory. Using the transfer matrix method, we have already shown that the one-dimensional Ising model does not present a phase transition.

5. Three-dimensional noncompact Lattice Abraham-Higgs (LAH) models: Theoretical deconfinement of transitions

Many emergent collective phenomena in condensed matter physics are described by effective three-dimensional (3D) scalar Abelian gauge models, where scalar fields are coupled with an Abelian gauge field [11, 12]. To ascertain the possible universality classes of the continuous transitions occurring in generic scalar gauge systems, several lattice scalar gauge models utilizing both compact and non-compact gauge variables have been considered. They provide examples of topological

transitions brought about by extended charged excitations with no local order parameter, nonlocal topological gauge modes, and long-range scalar fluctuations.

The basic variables in the LAM model are complex vectors with $(u_x, \overline{u_x}) = 1$ components ($A_{x\mu} \in \mathbb{R}$ variables), where $\mu = 1, 2, 3$.

The non-trivial topology of the manifold of vacuum field configurations is the source of the topological excitations in the abelian Higgs model and the other field theoretic models that will be discussed later. We follow the same steps as for ground state configurations and consider static fields [13], adding the possibility of partially disappearing energy densities. One can only generate finite energy if $|x| \rightarrow \infty$ asymptotically, which is evident from the energy density expression [14].

Depending on (12), the scalar field phase in (13) asymptotically defines the gauge field

$$D_{\phi(x)} = (\nabla - ieA(x))\phi(x) \rightarrow 0 \dots (12)$$

$$A(x) = \frac{1}{ie} \nabla \ln(\phi(x)) = \frac{1}{e} \nabla \theta(x) \dots (13)$$

By design, the vector potential is asymptotically a pure gauge [15], and $A(x)$ is independent of the strength of the magnetic field.

It can be concluded from the structure of the asymptotic gauge field (13) that the magnetic flux of field configurations with finite energy is quantized. Applying Stokes' theorem to a surface Υ enclosed by the asymptotic curve C yields the following result:

$$\phi_k^n = \iint k d^2x = \oint A ds = \frac{1}{e} \oint \nabla \theta(x) ds = 2n \frac{\pi}{e} \dots (14)$$

ϕ_k^n is a conserved quantity that does not change with time because it is an integer multiple of the basic unit of magnetic flux. Rather than an underlying symmetry, topological factors are what give rise to the appearance of this preserved quantity [16]. Moreover, ϕ_k^n is regarded as a topological invariant since it cannot be changed by a continuous deformation of the asymptotic curve. The topological significance of this finding is highlighted by assuming that the asymptotic curve C is a circle. Consequently, the scalar field $\phi(x)$ gives a mapping from the asymptotic circle C to the Higgs potential's circle of zeros $V(a) = 0$.

6. Conclusion

The fundamental group characterizes the topological properties of the space where the loops are defined by demonstrating the behavior of loops under continuous deformations. Using this approach, one may find a limited class of non-trivial topological properties. It has already been observed that loops in dimensions greater than two are unable to detect a point defect, hence the idea of homotopy groups needs to be extended to higher dimensions. A circle cannot encircle a pointlike defect in $\mathbb{R}^3$, but a 2-sphere can.

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