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Полная классификация двух классов двумерных PI-алгебр над любым основным полем

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Аннотация

В статье дана полная классификация двух классов двумерных PI-алгебр над любым основным полем. Выбор этих двух классов обусловлен полиномиальными тождествами классов: тождество одного из них дается с помощью бинарной операции алгебры, а другого - с помощью скобочной операции в тождестве. Приведен список представителей классов изоморфизма. Мы сравниваем наш список со списком, полученным ранее, где подобная классификация была дана при определенных ограничениях на основное поле.

Ключевые слова: алгебра с правильно определёнными кубами, смещенная ассоциативная алгебра, изоморфизм, классификация.

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Complete classification of two classes of two-dimensional PI-algebras over any basic field

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Abstract

In the paper we give complete classification of two classes of two-dimensional PI-algebras over any basic field. The choice of these two classes is predicted by the polynomial identities of the classes: the identity of one of them is given by using the binary operation of the algebra another one involves the bracket operation in the identity. The list of the representatives of isomorphism classes are provided. We compare our list with that obtained earlier, where such a classification was given under certain constraints on the basic field.

Keywords: well-defined cube algebra, mixed associative algebra, isomorphism, classification.

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1. Introduction

The classification problem of a given class of algebras is a challenging and significant task in algebra. It is well-known that complete solution to the classification problem of finite-dimensional algebras remains open. Currently, two approaches have been identified for solving this problem. One approach is the structural (basis-free, invariant) method. Examples of this approach include the classification of finite-dimensional simple and semi-simple associative algebras by Wedderburn, as well as the classification of simple and semi-simple Lie algebras by Cartan. However, it becomes increasingly difficult to apply this approach when dealing with more general types of algebras. Another method to solving the problem is a coordinate base approach. Many researchers have employed this approach to classify various classes of algebras, primarily finite-dimensional ones. For example, Leibniz algebras see [4], [6], [7], [13], Lie algebras see [9], [15], [16], [17], [18], Jordan algebras see [10], and associative algebras refer to [12], [14]. The previous attempts have been made to classify algebras of fixed dimensions under a very strict condition on basic field (most of the results obtained were over algebraic closed fields). In [19], the method of classification of 2-dimensional algebras was based on the basis-free approach. One drawback of the basis-free approach is its limited applicability to the classification of specific classes of algebras. In this regard, the coordinate-based classification approach has an advantage. For the coordinate-based approach, one can refer to [1], [8], [11] for the classification of all 2-dimensional algebras over fields with certain constraints, and [2], [3] for its different applications. Recently, we encountered (see [5]) a result on complete classification of all 2-dimensional algebras over any basic field. We make use the result of [5] to classify two classes of two-dimensional algebras over any basic field.

The outline of the paper is as follows. Section 2 gives preliminaries on classification of two-dimensional algebras over any basic field. This section contains as well as the idea that was used in the classification of all two-dimensional Well-defined cube (WDC) and Mixed-associative (MAA) algebras. In Section 3 we provide computations to classify WDC over an arbitrary basic field $\mathbb{F}$. The results are given as Theorems 4,5 and 6 for $\mathbb{F}$ of $Char(\mathbb{F}) \neq 2,3$, $Char(\mathbb{F}) = 2$ and $Char(\mathbb{F}) = 3$, respectively. Then in Section 4 we give a complete classification of MAA over any basic field $\mathbb{F}$. The results here are given as Theorems 7,8 an 9 over the field $\mathbb{F}$ of $Char(\mathbb{F}) \neq 2,3$, $Char(\mathbb{F}) = 2$ and $Char(\mathbb{F}) = 3$, respectively.

2. Preliminaries

2.1. Algebras

Definition 1. *A vector space* $\mathbb{V}$ *over a field* $\mathbb{F}$ *with a function* $\cdot : \mathbb{V} \otimes \mathbb{V} \to \mathbb{V}$ $((\mathrm{x},\mathrm{y}) \mapsto \mathrm{x}\cdot\mathrm{y})$ *such that*

$$(\alpha \mathrm{x} + \beta \mathrm{y}) \cdot \mathrm{z} = \alpha(\mathrm{x} \cdot \mathrm{z}) + \beta(\mathrm{y} \cdot \mathrm{z}), \quad \mathrm{z} \cdot (\alpha \mathrm{x} + \beta \mathrm{y}) = \alpha(\mathrm{z} \cdot \mathrm{x}) + \beta(\mathrm{z} \cdot \mathrm{y})$$

whenever $\mathrm{x},\mathrm{y},\mathrm{z} \in \mathbb{V}$ *and* $\alpha, \beta \in \mathbb{F}$, *is said to be an algebra* $\mathbb{A} = (\mathbb{V}, \cdot)$.

Definition 2. *Two algebras* $\mathbb{A} = (\mathbb{V}, \cdot_{\mathbb{A}})$ *and* $\mathbb{B} = (\mathbb{V}, \cdot_{\mathbb{B}})$ *are said to be isomorphic if there is an invertible linear map* $f : \mathbb{A} \to \mathbb{B}$ *such that*

$$f(\mathrm{x} \cdot_{\mathbb{A}} \mathrm{y}) = f(\mathrm{x}) \cdot_{\mathbb{B}} f(\mathrm{y})$$

for all $\mathrm{x},\mathrm{y} \in \mathbb{A}$.

Let $\mathbb{A}$ be an n-dimensional algebra over $\mathbb{F}$ and $\mathbf{e} = (e_1, e_2, ..., e_n)$ be its basis. Then the bilinear map $\cdot$ is represented by a $n \times n^2$ matrix (called the matrix of structure constant, shortly MSC)

$$A = \begin{pmatrix} a_{11}^1 & a_{12}^1 & ... & a_{1n}^1 & a_{21}^1 & a_{22}^1 & ... & a_{2n}^1 & ... & a_{n1}^1 & a_{n2}^1 & ... & a_{nn}^1 \\ a_{11}^2 & a_{12}^2 & ... & a_{1n}^2 & a_{21}^2 & a_{22}^2 & ... & a_{2n}^2 & ... & a_{n1}^2 & a_{n2}^2 & ... & a_{nn}^2 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{11}^n & a_{12}^n & ... & a_{1n}^n & a_{21}^n & a_{22}^n & ... & a_{2n}^n & ... & a_{n1}^n & a_{n2}^n & ... & a_{nn}^n \end{pmatrix}$$

as follows

$$\mathrm{e}_i \cdot \mathrm{e}_j = \sum_{k=1}^{n} a_{ij}^k \mathrm{e}_k, \text{ where } i,j = 1,2,...,n.$$

Therefore, the product on $\mathbb{A}$ with respect to the basis $\mathbf{e}$ is written as follows

$$\mathrm{x} \cdot \mathrm{y} = \mathbf{e}A(x \otimes y) \tag{1}$$

for any $\mathrm{x} = \mathbf{e}x, \mathrm{y} = \mathbf{e}y$, where $x = (x_1, x_2, ..., x_n)^T$, and $y = (y_1, y_2, ..., y_n)^T$ are column coordinate vectors of x and y, respectively, $x \otimes y$ is the tensor(Kronecker) product of the vectors x and y. Now and onward for the product "x · y" on $\mathbb{A}$ we use the juxtaposition "xy". Further we assume that the basis $\mathbf{e}$ is fixed and we do not make a difference between the algebra $\mathbb{A}$ and its MSC A.

Recently in [5] the following result on the classification of two-dimensional algebras over any basic field was obtained.

THEOREM 1. *Any non-trivial 2-dimensional algebra over a field $\mathbb{F}$ ($Char(\mathbb{F}) \neq 2,3$) is isomorphic to only one of the following listed, by their matrices of structure constants, such algebras:*

- $A_1(\mathrm{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & 1+\alpha_2 & \alpha_4 \\ \beta_1 & -\alpha_1 & 1-\alpha_1 & -\alpha_2 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{F}^4$,
- $A_2(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$ *and* $\alpha_4 \neq 0$,
- $A_3(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ *and* $a \neq 0$,
- $A_4(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$, *where* $\mathrm{c} = (\beta_1, \beta_2) \in \mathbb{F}^2$,
- $A_5(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1 - 1 & 1-\alpha_1 & 0 \end{pmatrix}$,
  where $\mathrm{c} = \alpha_1 \in \mathbb{F}$,
- $A_6(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$ *and* $\alpha_4 \neq 0$,
- $A_7(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$,
- $A_8(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathrm{c} = \beta_1 \in \mathbb{F}$,
- $A_9 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$,
- $A_{10}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & -1 \end{pmatrix}$,
  where $\mathrm{c} = \beta_1 \in \mathbb{F}$, *the polynomial* $(\beta_1 t^3 - 3t - 1)(\beta_1 t^2 + \beta_1 t + 1)(\beta_1^2 t^3 + 6\beta_1 t^2 + 3\beta_1 t + \beta_1 - 2)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $\beta_1'(t) = \frac{(\beta_1^2 t^3 + 6\beta_1 t^2 + 3\beta_1 t + \beta_1 - 2)^2}{(\beta_1 t^2 + \beta_1 t + 1)^3}$,
- $A_{11}(\mathrm{c}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, *where the polynomial* $\beta_1 - t^3$ *has no root in* $\mathbb{F}$, $a, \mathrm{c} = \beta_1 \in \mathbb{F}$ *and* $a, \beta_1 \neq 0$,
- $A_{12}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2\beta_1 & 0 & 0 & -1 \end{pmatrix}$, *where* $a, \mathrm{c} = \beta_1 \in \mathbb{F}$ *and* $a \neq 0$,
- $A_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

THEOREM 2. *Any non-trivial* 2*-dimensional algebra over a field* $\mathbb{F}$ ($Char(\mathbb{F}) = 2$) *is isomorphic to only one of the following listed by their matrices of structure constants, such algebras:*

- $A_{1,2}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & 1+\alpha_2 & \alpha_4 \\ \beta_1 & \alpha_1 & 1+\alpha_1 & \alpha_2 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{F}^4$,

- $A_{2,2}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & \beta_2 & 1+\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$ *and* $\alpha_4 \neq 0$,

- $A_{2,2}(\alpha_1, 0, 1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 1 & 1+\alpha_1 & 0 \end{pmatrix}$, *where* $\alpha_1 \in \mathbb{F}$,

- $A_{3,2}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1+\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1+\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ *and* $a \neq 0$,

- $A_{4,2}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1+\alpha_1 & 1 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 + (1+\beta_2)a + a^2 & \beta_2 & 1+\alpha_1 & 1 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{F}^3$,

- $A_{5,2}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & 1+\alpha_1 & \alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$ *and* $\alpha_4 \neq 0$,

- $A_{5,2}(1, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$,

- $A_{6,2}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1+\alpha_1 & \alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & 1+\alpha_1 & \alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$,

- $A_{7,2}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 & 1+\alpha_1 & \alpha_1 & 1 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 + a\alpha_1 + a + a^2 & 1+\alpha_1 & \alpha_1 & 1 \end{pmatrix}$, *where* $\mathrm{c} =$ $= (\alpha_1, \beta_1) \in \mathbb{F}^2$ *and* $a \in \mathbb{F}$,

- $A_{8,2}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & 1 \end{pmatrix}$, *where the polynomial* $(\beta_1 t^3 + t + 1)(\beta_1 t^2 + \beta_1 t + 1)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $\beta_1'(t) = \frac{(\beta_1^2 t^3 + \beta_1 t + \beta_1)^2}{(\beta_1 t^2 + \beta_1 t + 1)^3}$,

- $A_{9,2}(\mathrm{c}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, *where* $a, \mathrm{c} = \beta_1 \in \mathbb{F}$ *and* $a \neq 0$, *the polynomial* $\beta_1 + t^3$ *has no root in* $\mathbb{F}$,

- $A_{10,2}(\mathrm{c}) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 + a + a^2 & 1 & 1 & 1 \end{pmatrix}$, *where* $a, \mathrm{c} = \beta_1 \in \mathbb{F}$,

- $A_{11,2}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1 + a^2) & 0 & 0 & 1 \end{pmatrix}$, *where* $a, b \in \mathbb{F}$ *and* $b \neq 0$,

- $A_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

THEOREM 3. *Any non-trivial* 2*-dimensional algebra over a field* $\mathbb{F}$ $(Char(\mathbb{F}) = 3)$ *is isomorphic to only one of the following, listed by their matrices of structure constants, such algebras:*

- $A_{1,3}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 + 1 & \alpha_4 \\ \beta_1 & 2\alpha_1 & 1+2\alpha_1 & +2\alpha_2 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{F}^4$,

- $A_{2,3}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & \beta_2 & 1+2\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$, *and* $\alpha_4 \neq 0$,

- $A_{3,3}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1+2\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ *and* $a \neq 0$,

- $A_{4,3}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & 2 \end{pmatrix}$, *where* $\mathrm{c} = (\beta_1, \beta_2) \in \mathbb{F}^2$,

- $A_{5,3}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1+2 & 1+2\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = \alpha_1 \in \mathbb{F}$,

- $A_{6,3}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & 1+2\alpha_1 & 2\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$ *and* $\alpha_4 \neq 0$,

- $A_{7,3}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1+2\alpha_1 & 2\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & 1+2\alpha_1 & 2\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$,

- $A_{8,3}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & 2 \end{pmatrix}$, *where* $\mathrm{c} = \beta_1 \in \mathbb{F}$,

- $A_{9,3}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & 2 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & 2 \end{pmatrix}$, *where the polynomial* $(\beta_1 - t^3)(\beta_1 t^2 + \beta_1 t + 1)(\beta_1^2 t^3 + \beta_1 + 1)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $\beta_1'(t) = \frac{(\beta_1^2 t^3 + \beta_1 + 1)^2}{(\beta_1 t^2 + \beta_1 t + 1)^3}$,

- $A_{10,3}(\mathrm{c}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, *where the polynomial* $\beta_1 + 2t^3$ *has no root,* $a, \mathrm{c} = \beta_1 \in \mathbb{F}$ *and* $a, \beta_1 \neq 0$,

- $A_{11,3}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 2 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2\beta_1 & 0 & 0 & 2 \end{pmatrix}$, *where* $a, \mathrm{c} = \beta_1 \in \mathbb{F}$, $a \neq 0$,

- $A_{12,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \end{pmatrix}$,

- $A_{13,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

The next sections are devoted to the classification of all two-dimensional WDC and MA-algebras over any basic field relying on the theorems above.

2.2. Well-defined cube algebras

Definition 3. *An algebra* $(\mathbb{A}, \cdot)$ *is said to be a well-defined-cube (WDC) algebra if*

$$\mathrm{x}^2\mathrm{x} = \mathrm{x}\mathrm{x}^2, \quad \forall \mathrm{x} \in \mathbb{A}. \tag{2}$$

Consider two-dimensional case, i.e., $\mathbb{A}$ is represented by a 2×4 matrix

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

as a matrix of structure constants (MSC) of $\mathbb{A}$ on the basis $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2)$, i.e.,

$$\mathrm{xy} = \mathbf{e}A(x \otimes y)$$

for any $\mathrm{x} = \mathbf{e}x, \mathrm{y} = \mathbf{e}y$, where $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2)$ is a basis of $\mathbb{A}$, $x = (x_1, x_2)^T$, and $y = (y_1, y_2)^T$ as above with $n = 2$.

Let write

$$\begin{aligned} \mathrm{x}^2 &= \mathbf{e}Ax^{\otimes 2} \\ \mathrm{x}^2\mathrm{x} &= \mathbf{e}A((Ax^{\otimes 2}) \otimes x) \\ \mathrm{x}\mathrm{x}^2 &= \mathbf{e}A(x \otimes (Ax^{\otimes 2})) \end{aligned} \tag{3}$$

Then, the WDC algebra axiom

$$\mathrm{x}^2\mathrm{x} = \mathrm{x}\mathrm{x}^2$$

is written as

$$\mathbf{e}A((Ax^{\otimes 2}) \otimes x) = \mathbf{e}A(x \otimes (Ax^{\otimes 2}))$$

and in terms of the elements of A it can be rewritten as follows

$$\begin{aligned} \beta_1(\alpha_2 - \alpha_3) &= 0 \\ (\alpha_1 - \beta_2 - \beta_3)(\alpha_2 - \alpha_3) &= 0 \\ (\alpha_2 + \alpha_3 - \beta_4)(\alpha_2 - \alpha_3) &= 0 \\ \alpha_4(\alpha_2 - \alpha_3) &= 0 \\ \beta_1(\beta_2 - \beta_3) &= 0 \\ (\alpha_1 - \beta_2 - \beta_3)(\beta_2 - \beta_3) &= 0 \\ (\alpha_2 + \alpha_3 - \beta_4)(\beta_2 - \beta_3) &= 0 \\ \alpha_4(\beta_2 - \beta_3) &= 0 \end{aligned} \tag{4}$$

So, we use this system of equations to classify two-dimensional well-defined-cube algebras.

2.3. Mixed-associative algebras

Definition 4. *An algebra* $(\mathbb{A}, \cdot)$ *is said to be a mixed-associative algebra (MAA) if*

$$[\mathrm{x}, \mathrm{y}]\mathrm{z} = \mathrm{x}[\mathrm{y}, \mathrm{z}], \quad \forall \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathbb{A}, \tag{5}$$

where $[\mathrm{u}, \mathrm{v}] = \mathrm{uv} - \mathrm{vu}$, *for* $\mathrm{u}, \mathrm{v} \in \mathbb{A}$.

Let $\mathbb{A}$ be a two-dimensional MA-algebra and

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

be its MSC on a basis $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2)$. Then

$$\mathrm{xy} = \mathbf{e}A(x \otimes y)$$

for any $\mathrm{x} = \mathbf{e}x, \mathrm{y} = \mathbf{e}y$, $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$. Hence, one has

$$\begin{aligned} [\mathrm{x}, \mathrm{y}] &= \mathrm{xy} - \mathrm{yx} &&= \mathbf{e}A((x \otimes y) - (y \otimes x)) \\ [\mathrm{y}, \mathrm{z}] &= \mathrm{yz} - \mathrm{zy} &&= \mathbf{e}A((y \otimes z) - (z \otimes y)) \\ [\mathrm{x}, \mathrm{y}]\mathrm{z} &= (\mathrm{xy} - \mathrm{yx})\mathrm{z} &&= \mathbf{e}A(A((x \otimes y) - (y \otimes x)) \otimes z) \\ \mathrm{x}[\mathrm{y}, \mathrm{z}] &= \mathrm{x}(\mathrm{yz} - \mathrm{zy}) &&= \mathbf{e}A(x \otimes A((y \otimes z) - (z \otimes y))) \end{aligned} \tag{6}$$

Therefore, the MAA axiom

$$[\mathrm{x}, \mathrm{y}]\mathrm{z} = \mathrm{x}[\mathrm{y}, \mathrm{z}]$$

in terms of the elements of A turns to

$$\begin{array}{rcl} (2\alpha_1+\beta_2-\beta_3)\alpha_2-\alpha_3(2\alpha_1-\beta_2+\beta_3) & = & 0 \\ \alpha_2^2-\alpha_2\alpha_3+\alpha_4(\beta_2-\beta_3) & = & 0 \\ (\alpha_1+\beta_2-\beta_3)\alpha_2-\alpha_3\alpha_1 & = & 0 \\ \alpha_2\alpha_3-\alpha_3^2+\alpha_4(\beta_2-\beta_3) & = & 0 \\ \alpha_1\alpha_2-\alpha_3(\alpha_1-\beta_2+\beta_3) & = & 0 \\ \alpha_2^2-\alpha_3^2+2\alpha_4(\beta_2-\beta_3) & = & 0 \\ (\alpha_2-\alpha_3+2\beta_4)\beta_2+\beta_3(\alpha_2-\alpha_3-2\beta_4) & = & 0 \\ \beta_2\beta_3-\beta_3^2+\beta_1(\alpha_2-\alpha_3) & = & 0 \\ \beta_2\beta_4+\beta_3(\alpha_2-\alpha_3-\beta_4) & = & 0 \\ \beta_2^2-\beta_2\beta_3+\beta_1(\alpha_2-\alpha_3) & = & 0 \\ (\alpha_2-\alpha_3+\beta_4)\beta_2-\beta_3\beta_4 & = & 0 \\ \beta_2^2-\beta_3^2+2\beta_1(\alpha_2-\alpha_3) & = & 0 \end{array} \quad (7)$$

So, this is a system of equations which we make use to classify two-dimensional MA-algebras.

Further, for the sake of simplification, W_b^a and M_b^a stand for two-dimensional WDC-algebra and MA-algebra (MAA), respectively, where a is the number of the class from [5] and b is the enumeration in this paper.

3. Classification of two-dimensional WDC-algrbras

3.1. The characteristic is not 2 and 3

Theorem 4. *Any non-trivial 2-dimensional well-defined-cube algebra over a field $\mathbb{F}$, (Char($\mathbb{F} \neq \neq 2,3$)) is isomorphic to only one of the following listed by their matrices of structure constants algebras:*

1. $W_1^1 := \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$,
2. $W_2^2(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c}=(\alpha_1,\alpha_4)\in\mathbb{F}^2$,
3. $W_3^3(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c}=(\alpha_1,\alpha_4)\in\mathbb{F}^2$,
4. $W_3^4(\alpha_1) := \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 2\alpha_1-1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\alpha_1\in\mathbb{F}$,
5. $W_4^5(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & -1 \end{pmatrix}$, *where* $\beta_1\in\mathbb{F}$,
6. $W_5^6 := \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$,
7. $W_7^7 := \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$,
8. $W_{10}^8(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix}$, *where* $\beta_1\in\mathbb{F}$,
9. $W_{11}^9(\beta_1) := \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}$, *where* $\beta_1\in\mathbb{F}$,

10. $W_{12}^{10}(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix}$, *where* $\beta_1 \in \mathbb{F}$,

11. $W_{13}^{11} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Proof.

We check each algebra from Theorem 1 one by one to be a WDC algebra.

$A_1(\mathrm{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & 1+\alpha_2 & \alpha_4 \\ \beta_1 & -\alpha_1 & 1-\alpha_1 & -\alpha_2 \end{pmatrix}$ is WDC if $\alpha_1 = 1/3,\ \alpha_2 = -1/3,\ \alpha_4,\ \beta_1 = 0$.

$A_2(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$ is WDC if $\beta_2 = 1-\alpha_1,\ \mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$.

$A_3(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$ is WDC algebra in the following two cases:

- $\beta_2 = 1-\alpha_1$
- $\beta_2 = 1-2\alpha_1$ and $\alpha_4 = 0$.

$A_4(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$ is WDC algebra if and only if $\beta_2 = 1$.

$A_5(\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0 \end{pmatrix}$, is WDC algebra if $\alpha_1 = \frac{2}{3}$.

$A_7(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, where $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ and $a \neq 0$, gives a WDC algebra only for $\alpha_1 = \frac{1}{3}$ and $\alpha_4 = 0$.

$A_{10}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & -1 \end{pmatrix}$, where $\beta_1 \in \mathbb{F}$, the polynomial $(\beta_1 t^3 - 3t - 1)(\beta_1 t^2 + \beta_1 t + 1)(\beta_1^2 t^3 + 6\beta_1 t^2 + 3\beta_1 t + \beta_1 - 2)$ has no root in $\mathbb{F}$, $a \in \mathbb{F}$ and $\beta_1'(t) = \frac{(\beta_1^2 t^3 + 6\beta_1 t^2 + 3\beta_1 t + \beta_1 - 2)^2}{(\beta_1 t^2 + \beta_1 t + 1)^3}$, is WDC.

$A_{11}(\beta_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, where the polynomial $\beta_1 - t^3$ has no root in $\mathbb{F}$, $a,\ \beta_1 \in \mathbb{F}$ and $a, \beta_1 \neq 0$, is WDC.

$A_{12}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2\beta_1 & 0 & 0 & -1 \end{pmatrix}$, where $a,\ \beta_1 \in \mathbb{F}$ and $a \neq 0$, is a WDC.

$A_{13} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is a WDC algebra.

Note that the last four cases all the equations of (4) become identities. □

3.2. Comparison

In this section we compare the list of Theorem 4 with that of Theorem 4.7 (I_5) from [2].

	WDC algebra from [2]	Isomorphism	WDC algebra from this paper
Char $(\mathbb{F}) \neq 2,3$	$A_1\left(\frac{1}{3}, -\frac{1}{3}, 0, 0\right)$	$=$	W_1^1
	$A_2(\alpha_1, \beta_1, 1-\alpha_1)$	$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\beta_1} \end{pmatrix}$	$W_2^2(\alpha_1, \alpha_4),\ \alpha_4 = \beta_1^2$
	$A_3(\beta_1, 1)$	$=$	$W_4^5(\beta_1)$
	$A_4(\alpha_1, 1-\alpha_1),\ \alpha_1 \neq \frac{2}{3}$	$\cong$	$W_3^3(\mathrm{c}),\ \mathrm{c} = (\alpha_1, 0) \in \mathbb{F}^2, \alpha_1 \neq \frac{2}{3}$
	$A_4(\alpha_1, 2\alpha_1 - 1)$	$\cong$	$W_3^4(\alpha_1)$
	$A_5\left(\frac{2}{3}\right)$	$\cong$	W_5^6
	$A_8\left(\frac{1}{3}\right)$	$\cong$	W_7^7
	A_{10}	$\cong$	$W_{12}^{10}(0)$
	A_{11}	$\cong$	$W_{12}^{10}(1)$
	A_{12}	$\cong$	W_{13}^{11}

3.3. The characteristic is two

The following result holds true.

THEOREM 5. *Any non-trivial 2-dimensional well-defined-cube algebra over a field* $\mathbb{F}$, *(*Char($\mathbb{F}$ = = 2)*) is isomorphic to only one of the following listed by their matrices of structure constants algebras:*

1. $W_{12,2}^1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$,

2. $W_{11,2}^2(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1 + a^2) & 0 & 0 & 1 \end{pmatrix}$, *where* $a, b, \beta_1 \in \mathbb{F}$ *and* $b \neq 0$,

3. $W_{10,2}^3(\beta_1) := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & 1 \end{pmatrix}$, *where* $\beta_1 \in \mathbb{F}$,

4. $W_{9,2}^4(\beta_1) := \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}$, *where* $\beta_1 \in \mathbb{F}$,

5. $W_{8,2}^5(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix}$, *where* $\beta_1 \in \mathbb{F}$,

6. $W_{6,2}^6 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

7. $W_{4,2}^7(\alpha_1, \beta_1) := \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 & \alpha_1 + 1 & \alpha_1 + 1 & 1 \end{pmatrix} \cong \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 + (\alpha_1 + 2)a + a^2 & \alpha_1 + 1 & \alpha_1 + 1 & 1 \end{pmatrix}$
where $a, \alpha_1, \beta_1 \in \mathbb{F}$,

8. $W_{3,2}^8(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \alpha_1 + 1 & \alpha_1 + 1 & 0 \end{pmatrix}, \cong \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \alpha_1 + 1 & \alpha_1 + 1 & 0 \end{pmatrix}$,
where $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2, a \in \mathbb{F}, a \neq 0$,

9. $W_{3,2}^{9}(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1 & \alpha_1+1 & 0 \end{pmatrix}$, *where* $\alpha_1 \in \mathbb{F}$,

10. $W_{2,2}^{9}(\alpha_1) := \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & \alpha_1+1 & \alpha_1+1 & 0 \end{pmatrix}$, *where* $\alpha_1 \in \mathbb{F}$,

11. $W_{2,2}^{10}(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & \alpha_1+1 & \alpha_1+1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2, \alpha_4 \neq 0$,

12. $W_{2,2}^{11} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$,

13. $W_{1,2}^{12} := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

PROOF.
Let check each algebra from Theorem 2 one by one to be a WDC algebra.

$A_{12,2} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is a WDC algebra.

$A_{11,2} := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1+a^2) & 0 & 0 & 1 \end{pmatrix}$, where $a, b \in \mathbb{F}$ and $b \neq 0$ is a WDC algebra.

$A_{10,2} := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & 1 \end{pmatrix}$, where $\beta_1 \in \mathbb{F}$ is a WSC algebra.

$A_{9,2} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}$, where $\beta_1 \in \mathbb{F}$ is a WDC algebra.

$A_{8,2} := \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix}$, where $\beta_1 \in \mathbb{F}$ is a WDC algebra.

$A_{6,2}$ is WDC if $\alpha_1 = 1$ and $\alpha_4 = 0$, i.e.,

$A_{6,2} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is WDC algebra.

$A_{4,2} := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1+a+a^2 & 0 & 0 & 1 \end{pmatrix}$, where $a, \beta_1 \in \mathbb{F}$. $A_{3,2}$ is WDC if $\beta_2 = \alpha_1 + 1$, i.e., $\begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \alpha_1+1 & \alpha_1+1 & 0 \end{pmatrix}$, where $\alpha_1, \alpha_4 \in \mathbb{F}$ is MSC of a WDC algebra.

$A_{2,2}$ is WDC if $\beta_2 = \alpha_1 + 1$, i.e.,

$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & \alpha_1+1 & \alpha_1+1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$ is MSC of a WDC algebra.

$A_{2,2}(\alpha_1, 0, 1) := \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 1 & 1+\alpha_1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$ is WDC if $\alpha_1 = 0$, i.e., $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ is MSC of a WDC algebra.

$A_{1,2}(\mathrm{c}) := \begin{pmatrix} \alpha_1 & \alpha_2 & 1+\alpha_2 & \alpha_4 \\ \beta_1 & \alpha_1 & 1+\alpha_1 & \alpha_2 \end{pmatrix}$, where $\mathrm{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{F}^4$ is WDC if $\alpha_1 = 1,\ \alpha_2 = 1$, $\alpha_4 = 0,\ \beta_1 = 0$, i.e., $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ is a MSC of a WDC algebra.

□

3.4. Comparison

This section is a comparison of the result of Theorem 5 with that of Theorem 4.9 (I_5) from [2].

	WDC algebra from [2]	Isomorphism	WDC algebra from this paper
$\mathrm{Char}(\mathbb{F}) = 2$	$A_{1,2}(1,1,0,0)$	$\cong$	$W_{1,2}^{12}$
	$A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$	$\cong$	$W_{2,2}^{10}(\alpha_1,\beta_1)$
	$A_{3,2}(\alpha_1,1+\alpha_1)$	$\cong$	$W_{4,2}^{7}(\alpha_1,0)$
	$A_{4,2}(\alpha_1,1+\alpha_1),\ \alpha_1\neq\frac{2}{3}$	$\cong$	$W_{3,2}^{8}(\alpha_1,0),\ \mathrm{c}=(\alpha_1,0)\in\mathbb{F}^2,\alpha_1\neq\frac{2}{3}$
	$A_{4,2}(\alpha_1,1)$	$\cong$	$W_{3,2}^{9}(\alpha_1)$
	$A_{5,2}(0)$	$\cong$	$W_{2,2}^{11}$
	$A_{8,2}(1)$	$\cong$	$W_{6,2}^{6}$
	$A_{10,2}$	$\cong$	$W_{11,2}^{2}(0)$
	$A_{11,2}$	$\cong$	$W_{10,2}^{3}(0)$
	$A_{12,2}$	$\cong$	$W_{12,2}^{1}$

3.5. The characteristic is three

In this case WDC algebras come out from the following classes of Theorem 3. It is immediate that the algebras $A_{9,3}$ - $A_{13,3}$ are WDC algebras. In these cases all the equations of the system (4) turn into identities. The algebra $A_{7,3}$ is WDC if $\alpha_1 = 1$ and $\alpha_4 = 0$ and the algebra $A_{4,3}$ is WDC if $\beta_2 = 1$. Finally, the algebras $A_{3,3}$ and $A_{2,3}$ are WDC if $\beta_2 = 1 - \alpha_1$. The other classes of algebras in Theorem 3 do not produce WDC algebras. Thus, we get the following theorem.

Theorem 6. *Any non-trivial 2-dimensional well-defined-cube algebra over a field $\mathbb{F}$, ($Char(\mathbb{F}) = 3$) is isomorphic to only one of the following listed by their matrices of structure constants algebras:*

1. $W_{2,3}^1(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1,\alpha_4) \in \mathbb{F}^2$, *and* $\alpha_4 \neq 0$,

2. $W_{3,3}^2(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$,
where $\mathrm{c} = (\alpha_1,\alpha_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$,

3. $W_{3,3}^3(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 2\alpha_1+2 & 2\alpha_1+1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & 2\alpha_1+2 & 2\alpha_1+1 & 0 \end{pmatrix}$,
where $\mathrm{c} = (\alpha_1,\alpha_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$,

4. $W_{4,3}^4(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & 2 \end{pmatrix}$, *where* $\beta_1 \in \mathbb{F}$,

5. $W_{4,3}^5 := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 \end{pmatrix}$,

6. $W_{7,3}^6 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$,

7. $W_{9,3}^7(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & 2 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & 2 \end{pmatrix}$, *where the polynomial*
$(\beta_1 - t^3)(\beta_1 t^2 + \beta_1 t + 1)(\beta_1^2 t^3 + \beta_1 - 2)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $\beta_1'(t) = \frac{(\beta_1^2 t^3+\beta_1-2)^2}{(\beta_1 t^2+\beta_1 t+1)^3}$,

8. $W_{10,3}^8(\beta_1) := \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$,
where the polynomial $\beta_1 - t^3$ *has no root,* $a, \beta_1 \in \mathbb{F}$ *and* $a, \beta_1 \neq 0$,

9. $W_{11,3}^{9}(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 2 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2\beta_1 & 0 & 0 & 2 \end{pmatrix}$, *where* $a, \beta_1 \in \mathbb{F}$, $a \neq 0$,

10. $W_{12,3}^{10} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \end{pmatrix}$,

11. $W_{13,3}^{11} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

3.6. Comparison

Now we compare the list of Theorem 6 with that of Theorem 4.10 (I_5) from [2].

	WDC algebra from [2]	Isomorphism	WDC algebra from this paper
Char ($\mathbb{F}$) = 3	$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1)$	$\cong$	$W_{2,3}^{1}$
	$A_{3,3}(\beta_1, 1)$	$\cong$	$W_{4,3}^{4}(\beta_1)$
	$A_{3,3}(0, -1)$	$\cong$	$W_{4,3}^{5}$
	$A_{4,3}(\alpha_1, 1-\alpha_1)$	$\cong$	$W_{3,3}^{2}(\alpha_1, 0)$
	$A_{4,3}(\alpha_1, 2\alpha_1 - 1)$	$\cong$	$W_{3,3}^{3}(\alpha_1, 0)$
	$A_{9,3}$	$\cong$	$W_{11,3}^{9}(1)$
	$A_{10,3}$	$\cong$	$W_{11,3}^{9}(0)$
	$A_{11,3}$	$\cong$	$W_{12,3}^{10}$
	$A_{12,3}$	$\cong$	$W_{13,3}^{11}$

4. Classification of two-dimensional mixed-associative algebras

4.1. The characteristic is not two and three

The following classes from Theorem 1 produce MAA as follows:

$A_2(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, where $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$, $\alpha_4 \neq 0$,

$A_3(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$,

where $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$, $a \in \mathbb{F}, a \neq 0$,

$A_4(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & -1 \end{pmatrix}$, and $A_5(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1 - 1 & 1-\alpha_1 & 0 \end{pmatrix}$,

under the conditions $\beta_2 = 1-\alpha_1$ (in $A_2(\mathrm{c})$), $\beta_2 = 1-\alpha_1$ (in $A_3(\mathrm{c})$), $\beta_2 = 1$ (in $A_4(\mathrm{c})$) and $\alpha_1 = \frac{2}{3}$ (in $A_5(\mathrm{c})$) respectively, i.e.,

$$A_2 = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}, \text{ where } \mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^3,\ \alpha_4 \neq 0,$$

$A_3(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, where $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^3$, $a \in \mathbb{F}, a \neq 0$,

$$A_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & -1 \end{pmatrix}, \text{ where } \beta_1 \in \mathbb{F}$$

and

$$A_5 = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

are mixed-associative algebras.

It is immediate to check that the algebras

- $A_{10}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix}$ where $\beta_1 \in \mathbb{F}$,
- $A_{11}(\beta_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}$ where $\beta_1 \in \mathbb{F}$,
- $A_{12}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix}$ where $\beta_1 \in \mathbb{F}$,
- $A_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

from Theorem 1 also are mixed associative. Thus, we get the following result.

Theorem 7. *Any non-trivial* 2*-dimensional mixed-associative algebra over a field* $\mathbb{F}, (Char(\mathbb{F}) \neq \neq 2, 3)$ *is isomorphic to only one of the following listed by their matrices of structure constants algebras:*

1. $M_2^1(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2,\ \alpha_4 \neq 0$,
2. $M_3^2(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1-\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$,
   where $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2,\ a \in \mathbb{F},\ a \neq 0$,
3. $M_4^3(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & -1 \end{pmatrix}$,
4. $M_5^4 := \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$,
5. $M_{10}^5(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix}$,
6. $M_{11}^6(\beta_1) := \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}$,
7. $M_{12}^7(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix}$,
8. $M_{13}^8 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

4.2. Comparison

Let compare the list of Theorem 7 with that of Theorem 7.4 from [2] as below.

	MMA from [2]	Isomorphism	MMA from this paper
$\mathrm{Char}(\mathbb{F}) \neq 2,3$	$A_2(\alpha_1, \beta_1, 1-\alpha_1)$	$\cong$	$M_2^1(\mathrm{c}), \quad \mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$ $\alpha_4 \neq 0$
	$A_3(\beta_1, 1)$	$\cong$	$M_4^3(\beta_1)$
	$A_4(\alpha_1, 1-\alpha_1),\ \alpha_1 \neq \frac{2}{3}$	$\cong$	$M_3^2(\alpha_1, 0),\ \alpha_1 \neq \frac{2}{3},$
	$A_5\left(\frac{2}{3}\right)$	$\cong$	M_5^4
	A_{10}	$\cong$	$M_{12}^7(0)$
	A_{11}	$\cong$	$M_{12}^7(1)$
	A_{12}	$\cong$	M_{13}^8

4.3. The characteristic is two

The list of MA-algebras appear among the classes of Theorem 2 as follows. For the classes
$A_{12,2} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$,
$A_{11,2} := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1 + a^2) & 0 & 0 & 1 \end{pmatrix}$, where $a, b \in \mathbb{F}$ and $b \neq 0$,
$A_{10,2} := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & 1 \end{pmatrix}$, where $\beta_1 \in \mathbb{F}$,
$A_{9,2} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}$, where $\beta_1 \in \mathbb{F}$,
$A_{8,2} := \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix}$, where $\beta_1 \in \mathbb{F}$,
all the equations of (7) become identities.
$A_{3,2}$ is MAA if $\beta_2 = \alpha_1 + 1$, i.e., $\begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \alpha_1 + 1 & \alpha_1 + 1 & 0 \end{pmatrix}$, where $\alpha_1, \alpha_4 \in \mathbb{F}$ is MAA.
$A_{2,2}$ is MAA if $\beta_2 = \alpha_1 + 1$, i.e., $\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & \alpha_1 + 1 & \alpha_1 + 1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$ is MAA.

Theorem 8. *Any non-trivial 2-dimensional mixed-associative algebra over a field $\mathbb{F}$, ($Char(\mathbb{F} = = 2)$) is isomorphic to only one of the following listed by their matrices of structure constants algebras:*

1. $M_{12,2}^1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$,

2. $M_{11,2}^2(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1 + a^2) & 0 & 0 & 1 \end{pmatrix}$, *where $a, b \in \mathbb{F}$ and $b \neq 0$,*

3. $M_{10,2}^3(\beta_1) := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 + a + a^2 & 1 & 1 & 1 \end{pmatrix}$, *where $a, \beta_1 \in \mathbb{F}$,*

4. $M_{9,2}^4(\beta_1) := \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, *where $a, \beta_1 \in \mathbb{F},\ a \neq 0$, and the polynomial $\beta_1 + t^2$ has no root in $\mathbb{F}$,*

5. $M^5_{8,2}(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & 1 \end{pmatrix}$,
where the polynomial $\beta_1 t^3 + t + 1)(\beta_1 t^2 + \beta_1 t + 1)$ *has no root in* $\mathbb{F}$ *and* $\beta_1'(t) = \frac{(\beta_1^2 t^3 + \beta_1 t + \beta_1)^2}{(\beta_1 t^2 + \beta_1 t + 1)^3}$,

6. $M^6_{4,2} := \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 & 1+a_1 & 1+a_1 & 1 \end{pmatrix} \cong \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 + \alpha_1 a + a^2 & 1+\alpha_1 & 1+\alpha_1 & 1 \end{pmatrix}$, $\alpha_1, \beta_1 \in \mathbb{F}^2$,

7. $M^7_{3,2}(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \alpha_1+1 & \alpha_1+1 & 0 \end{pmatrix} \cong \begin{pmatrix} \alpha_1 & 0 & 0 & a\alpha_4 \\ 0 & \alpha_1+1 & \alpha_1+1 & 0 \end{pmatrix}$, *where* $\alpha_1, a \in \mathbb{F}$ *and* $a \neq 0$,

8. $M^8_{2,2} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$,

9. $M^9_{2,2}(\alpha_1) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & \alpha_1+1 & \alpha_1+1 & 0 \end{pmatrix}$, *where* $\alpha_1, \alpha_4 \in \mathbb{F}$ *and* $\alpha_4 \neq 0$,

4.4. Comparison

Below is a comparison of the list of Theorem 8 with that of Theorem 4.9 (I_8) from [2].

	MMA from [2]	Isomorphism	MMA from this paper
Char ($\mathbb{F}$) = 2	$A_{2,2}(\alpha_1, \beta_1, 1+\alpha_1)$	$\cong$	$M^1_{2,2}(\mathrm{c}),\ \mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$
	$A_{3,2}(\alpha_1, 1+\alpha_1)$	$\cong$	$M^6_{4,2}(\beta_1)$
	$A_{4,2}(\alpha_1, 1+\alpha_1),\ \alpha_1 \neq \frac{2}{3}$	$\cong$	$M^7_{3,2}(\alpha_4 = 0),\ \alpha_1 \neq \frac{2}{3}$,
	$A_{5,2}(0)$	$\cong$	$M^8_{2,2}$
	$A_{10,2}$	$\cong$	$M^2_{11,2}(0)$
	$A_{11,2}$	$\cong$	$M^3_{10,2}(0)$
	$A_{12,2}$	$\cong$	$M^1_{12,2}$

4.5. The characteristic is three

In this case MAA come out from the following classes of Theorem 3. It is immediate that the algebras $A_{9,3}$ - $A_{13,3}$ are MAA algebras. In these cases all the equations of the system (4) turn into identities. The algebra $A_{4,3}$ is MAA if $\beta_2 = 1$. Finally, the algebras $A_{3,3}$ and $A_{2,3}$ are MAA if $\beta_2 = 1 - \alpha_1$. The other classes of algebras of Theorem 3 do not produce MAA algebras. Thus, we get the following theorem.

THEOREM 9. *Any non-trivial* 2*-dimensional mixed-associative algebra over a field* $\mathbb{F}, (Char(\mathbb{F}) = 3)$ *is isomorphic to only one of the following listed by their matrices of structure constants algebras:*

1. $M^1_{2,3}(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & 1+2\alpha_1 & 1+2\alpha_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$, *and* $\alpha_4 \neq 0$,

2. $M^2_{3,3}(\mathrm{c}) := \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1+2\alpha_1 & 1+2\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & 1+2\alpha_1 & 1+2\alpha_1 & 0 \end{pmatrix}$,
where $\mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$,

3. $M^3_{4,3}(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & 2 \end{pmatrix}$, *where* $\beta_1 \in \mathbb{F}$,

4. $M^4_{9,3}(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & 2 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & 2 \end{pmatrix}$, *where the polynomial* $(\beta_1 - t^3)(\beta_1 t^2 + \beta_1 t + 1)(\beta_1^2 t^3 + \beta_1 - 2)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $\beta_1'(t) = \frac{(\beta_1^2 t^3 + \beta_1 - 2)^2}{(\beta_1 t^2 + \beta_1 t + 1)^3}$,

5. $M^5_{10,3}(\beta_1) := \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$,
where the polynomial $\beta_1 - t^3$ *has no root,* $a, \beta_1 \in \mathbb{F}$ *and* $a, \beta_1 \neq 0$,

6. $M^6_{11,3}(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 2 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2\beta_1 & 0 & 0 & 2 \end{pmatrix}$, *where* $a, \beta_1 \in \mathbb{F}$, $a \neq 0$,

7. $M^7_{12,3} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \end{pmatrix}$,

8. $M^8_{13,3} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

4.6. Comparison

Here is a comparison of the list of Theorem 9 with that of Theorem 4.9 (I_8) from [2].

	MMA from [2]	Isomorphism	MMA from this paper
$\mathrm{Char}(\mathbb{F}) = 3$	$A_{2,3}(\alpha_1, \beta_1, 1+2\alpha_1)$	$\cong$	$M^1_{2,3}(\mathrm{c}),\ \mathrm{c} = (\alpha_1, \alpha_4) \in \mathbb{F}^2$
	$A_{3,3}(\beta_1, 1)$	$\cong$	$M^3_{4,3}(\beta_1)$
	$A_{4,3}(\alpha_1, 1+2\alpha_1),\ \alpha_1 \neq \frac{2}{3}$	$\cong$	$M^7_{3,2}(\alpha_4 = 0),\ \alpha_1 \neq \frac{2}{3}$,
	$A_{9,3}$	$\cong$	$M^6_{11,3}(1)$
	$A_{10,3}$	$\cong$	$M^6_{11,3}(0)$
	$A_{11,3}$	$\cong$	$M^7_{12,3}$
	$A_{12,3}$	$\cong$	$M^8_{13,3}$

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