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О компактности сильно звездных идеалов топологических пространств

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Аннотация

В этой статье мы вводим понятие сильно звездной I-компактности и изучаем некоторые ее топологические особенности. Мы представляем некоторые свойства конечных пересечений как для I-компактных пространств, так и для сильно звездных I-компактных пространств. Наконец, мы устанавливаем связь между счетно I_{fin}-компактным пространством и сильно звездным I_{fin}-компактным пространством. Для того чтобы выявить разницу между различными версиями компактности, мы приводим несколько контрпримеров. Также в статье поставлены некоторые открытые проблемы.

Ключевые слова: Звездный идеал, звездный I-компакт, I_{fin}-компактное пространство.

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On strongly star ideal compactness of topological spaces

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Abstract

In this article we introduce the concept of strongly star I-compactness and study some of its topological features. We represent some finite intersection like properties for both I-compact spaces and strongly star I-compact spaces. Lastly we establish a relation between the countably I_{fin}-compact space and the strongly star I_{fin}-compact space. In order to identify the difference between the different versions of compactness we represent some counter examples. And some open problems are also posed in this article.

Keywords: Star Ideal, Star I-compact, I_{fin}-compact space.

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1. Introduction

The the year 1933, the concept of ideals in topological spaces were considered by Kuratowski [15] and has been studied extensively by Vaidyanathaswamy [21] in the year 1946. An ideal I in a topological space (X, τ) is a non empty family of subsets of X which satisfies the following properties :

(i) $X \notin I$,
(i) $A, B \in I \Rightarrow A \cup B \in I$,
(ii) $A \in I$ and $B \subseteq A \Rightarrow B \in I$.

If I is an ideal in a topological space (X, τ), then the ordered triplet (X, τ, I) is called an ideal topological space (in short ideal space). If $I \cap \tau = \{\emptyset\}$, then I is called a condensed ideal or a boundary ideal [10]. Some simple ideals on a space (X, τ) are $\{\emptyset\}$, $P(X)$ (power set of X) and I_{fin}, collection of all finite subset of X The concept of compactness modulo an ideal (also called I-compactness) was first established by Newcomb [17] in the year 1967. Recently this generalization of compactness has attracted a lot of mathematicians in this field [12, 20].

On the other hand, Douwen et. al. [9] generalized compactness with the help of the star operator. In a topological space (X,τ), if $M \subseteq X$ and $\mathcal{U}$ is a collection of subsets of X then star of M with respect to $\mathcal{U}$ is denoted by $St(M,\mathcal{U})$ and is defined as $St(M,\mathcal{U}) = \{U \in \mathcal{U} : U \cap M \neq \emptyset\}$. If $M = \{x\}$, we write $St(x,\mathcal{U})$ instead of $St(\{x\},\mathcal{U})$. In recent days, this operator is being used in the study of selection principles [1, 3, 14], covering properties [2, 4, 5, 8, 6, 7, 19]. In this paper we will use the concept of ideal and star operator simultaneously to generalize the study of compactness of an ideal space.

2. Preliminaries

If $A \subseteq X$, $\overline{A}$ will denote closure of A. For general symbols and notation of topology, we follow [11].

A subset A of a space (X,τ) is said to be a g-closed [16] set if $\overline{A} \subseteq U$, whenever $A \subseteq U \in \tau$. Every closed set is a g-closed set but converse may not be true.

Proposition 1. *[17] If $f : (X,\tau,I) \to (Y,\sigma)$ is a function, then $f(I) = \{f(I_1) : I_1 \in I\}$ is an ideal of Y.*

Proposition 2. *[17] If I is an ideal of subsets of X and $Y \subseteq X$, then $I_Y = \{Y \cap I_1 : I_1 \in I\}$ is an ideal of subsets of Y.*

Although Newcomb [17] introduced the concept of I-compactness, Rancin [18], Hamlett and Jankovic [13] studied the concept extensively.

Definition 1. *[17] A subset A of an ideal space (X,τ,I) is said to be compact modulo I or I-compact subset, if for every τ-open cover $\{U_\alpha : \alpha \in \Lambda\}$ of A there exists a finite subset $\{U_{\alpha_i} : i = 1,2,3,\dots k\}$ such that $X \setminus \bigcup_{i=1}^{k} U_{\alpha_i} \in I$. If X itself is a I-compact subset, then (X,τ,I) is called an I-compact space.*

Definition 2. *[9] A topological space (X,τ) is called a strongly star compact space if for every open cover $\mathcal{U}$ of X, there exists a finite subset $F \subseteq X$ such that $St(F,\mathcal{U}) = X$.*

3. I-compactness

Since $\emptyset \in I$ for every ideal I of X, every compact space is an I-compact space.

Remark 1. *There exists an ideal space (X,τ,I) which is I compact but not compact.*

Let $X = N, A = \{1,3,5,...\}, \tau = \{\emptyset, X\} \cup \{X/P : p \in P(A)\}$ and $I = P(A)$.

Clearly (X,τ,I) is an ideal space. Let $\mathcal{U}$ be an arbitrary non-trivial open cover of X and $\mathcal{U}' = \{u_1,u_2,...,u_k\}$ is a finite subset of $\mathcal{U}$. Then there exist $p_n \in P(A)$ such that $U_n = X/P_n \ \forall n \in \{1,2,3,...,k\}$ and $\mathcal{U}' = \bigcup_{n=1}^{k} u_n = \bigcup_{n=1}^{k}(X/P_n) = X/(\bigcap_{n=1}^{k} P_n) = X/(\cap_{n=1}^{k} P_n)$ Therefore, $X/\cup\mathcal{U}' = \cap_{n=1}^{k} P_n \in P(A) = I$.

Hence (X,τ,I) is an I-compact space.

Now consider the countable open cover $\mathcal{V} = \{V_n : n \in N\}$, where $V_n = X/\{2n-1, 2n+1, 2n+3, ...\}$.

If possible let $\mathcal{V}' = \{V_{n_1}, V_{n_2}, V_{n_3}..., V_{n_k}\}$ is a finite sub cover of X. Then there exists $n_{max} = \{n_1, n_2, n_3, ...n_k\} \in N$.

Therefore, $\cup\mathcal{V} = V_{n_{max}} = X/\{2_{n_{max}} - 1, 2_{n_{max}} + 1, 2_{n_{max}} + 3, ...\} \neq X$

So, $\mathcal{V}$ can not have finite sub cover. Hence (X,τ,I) can not a compact space.

Definition 3. *In an ideal space (X,τ,I), a family $\{H_\alpha : \alpha \in \Lambda\}$ is said to have idealized finite intersection property if for every finite subset $\Lambda_0 \in \Lambda, \bigcap_{\alpha\in\Lambda_0} H_\alpha \notin I$*

THEOREM 1. *For an ideal space (X,τ,I), following statements are equivalent:*

1. *Every family $\mathcal{H}$ of closed sets having idealized finite intersection property have $\bigcap\mathcal{H}\neq\emptyset$*

2. *(X,τ,I) is I-compact.*

PROOF.
(1) $\Rightarrow$ (2)
Let condition (1) holds and $\mathcal{U}=\{U_\alpha:\alpha\in\Lambda\}$ is an open cover of the ideal space (X,τ,I) Therefore, $\mathcal{H}=\{H_\alpha=X/U_\alpha:\alpha\in\Lambda\}$ is a family of closed sets and
$\bigcap\mathcal{H}=\bigcap_{\alpha\in\Lambda}H_\alpha=\bigcap_{\alpha\in\Lambda}(X/U_\alpha)=X/\bigcup_{\alpha\in\Lambda}U_\alpha=X/X=\emptyset$
Therefore by condition (1), the family $\mathcal{H}$ of closed sets must not have IFI property. Thus there exists a finite subset $\Lambda_0\subseteq\Lambda$
Such that. $\bigcap_{\alpha\in\Lambda_o}H_\alpha\in I$
$\Rightarrow\bigcap_{\alpha\in\Lambda_o}(X/U_\alpha)\in I\Rightarrow X/\bigcup_{\alpha\in\Lambda_o}(U_\alpha)\in I$
So,$\{U_\alpha:\alpha\in\Lambda_0\}$ is a finite subset of $\mathcal{U}$ such that $X/\bigcup_{\alpha\in\Lambda_o}(U_\alpha)\in I$. Hence (X,τ,I) is an I-compact space.
(2) $\Rightarrow$ (1)
Let (X,τ,I) is an I-compact space and $\mathcal{H}=\{H_\alpha:\alpha\in\Lambda\}$ is a family of closed sets having finite intersection property. If possible let $\cap\mathcal{H}=\emptyset$.
Then $\mathcal{U}=\{U_\alpha=X/H_\alpha:\alpha\in\Lambda\}$ is a family of open sets such that
$\mathcal{U}=\bigcup_{\alpha\in\Lambda}(X/H_\alpha)=X/\bigcap_{\alpha\in\Lambda}H_\alpha=X/\emptyset=X$.
$\mathcal{U}$ is an open cover of X. But (X,τ,I) is I compact.
Therefore, there exists a finite subset $\Lambda_0\subseteq\Lambda$ Such that $X/\bigcup_{\alpha\in\Lambda_0}U_\alpha\in I$
So, $\{H_\alpha:\alpha\in\Lambda_0\}$ is a finite subset of $\mathcal{H}$ and $\bigcap_{\alpha\in\Lambda_0}\mathcal{H}_\alpha\in I$ which is a contradiction to the fact that H has IFI property. Hence, $\bigcap\mathcal{H}\neq\emptyset$ □

4. Strongly star I-compact space

DEFINITION 4. *In an ideal space (X,τ,I), a subset $B\subseteq X$ is said to be strongly star I-compact subset if for every τ-open cover of B, there exists a finite subset $M\subseteq B$ Such that $X/St(M,\mathcal{U})\in I$. If X it self is strongly star I-compact, then we say that (X,τ,I) is a strongly star I-compact space.*
Since $\emptyset\in I$, every strongly star compact space is a strongly star I-compact space. But the converse may not true.

REMARK 2. *There exists a strongly star I-compact space which is not strongly star compact. Let $X=N$, $\beta=\{\emptyset\}\cup\{\{n\}:n\in 2N\}\cup\{\{1,2n-1\}:n\in N\},I=\mathcal{P}(2N)$ and τ be the topology generated by β.*
Let $\mathcal{U}$ be an arbitrary open cover of X. Then for $F=\{1\}$, $N/2N\subseteq St(F,\mathcal{U})\Rightarrow X/St(F,\mathcal{U})\subseteq$ $\subseteq 2N\Rightarrow X/St(F,\mathcal{U})\in I$.
Therefore, (X,τ,I) is a strongly star I-compact space Now, consider the basic open cover β, then for every finite subset $F\subseteq X$,
$St(F,\beta)\subseteq F\cup(N/2N)\neq X$
Therefore, (X,τ,I) can not be a strongly star compact space.

PROPOSITION 3. *Every I-compact space is a strongly star I-compact space.*
$\Rightarrow$ Let (X,τ,I) is an I-compact space and $\mathcal{U}$ be an arbitrary open cover of X. Then there exists a finite subset $\mathcal{U}' =\{U_1,U_2,U_3,...,U_k\}\subseteq\mathcal{U}$ such that $X/\cup_{i=1}^{k}U_k\in I$
If we take $x_i\in U_i,\forall i=1,2,3,...,k$ then $F=\{x_1,x_2,x_3,...,x_k\}\subseteq X$ is finite and $\cup_{i=1}^{k}U_k\subseteq St(F,\mathcal{U})$
$\Rightarrow X/St(F,\mathcal{U})\subseteq X/\cup_{i=1}^{k}U_k\in I$

Therefore, $X/St(F,\mathcal{U}) \in I$

Hence (X, τ, I) *is strongly star* I*-compact space. But the converse of the above proposition may not be true.*

Remark 3. *There exists a strongly star* I*-compact space which is not strongly star compact. Let* $X = N$, $\beta = \{\emptyset\} \cup \{2N\} \cup \{\{1, 2n-1\} : n \in N\}$, $I = \mathbf{P}(2N)$ *and* τ *be the topology generated by* β*. Clearly, for every open cover* $\mathcal{U}$ *of* X*, if we take* $F = \{1\}$*, then* $N/2N \subseteq St(F,\mathcal{U})$

$\Rightarrow X/St(F,\mathcal{U}) \subseteq 2N$

$\Rightarrow X/St(F,\mathcal{U}) \in I$ *Therefore* (X, τ, I) *is a strongly star* I*-compact.*

Now consider the countable open cover $\mathcal{U} = \{U_n : n \in N\}$ *Where* $U_n = 2N \cup \{1, 3, 5, ..., 2n-1\}$

Suppose that $\{U_{n_1}, U_{n_2}, U_{n_3}, ..., U_{n_k}\}$ *is a finite subset of* X*. Then there exists a* n_{max} *such that* $n_{max} = max\{n_1, n_2, ..., n_k\}$*.*

Therefore, $\cup_{i=1}^{k} U_{n_i} = U_{n_{max}} = 2N \cup \{1, 3, 5, ..., 2n_{max} - 1\}$

$\Rightarrow X/\cup_{i=1}^{k} U_{n_i} = \{2n_{max} + 1, 2n_{max} + 3, 2n_{max} + 5, ...\}$

$\Rightarrow X/\cup_{i=1}^{k} U_{n_i} \notin I$ $\mathcal{U}$ *can not have a finite subset* $\mathcal{U}'$ *such that* $X/\cup\mathcal{U}' \in I$*.*

Therefore (X, τ, I) *is not* I*-compact.*

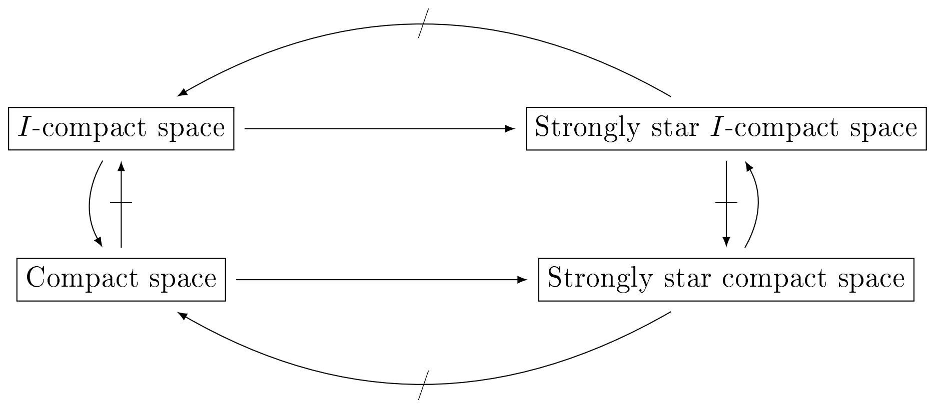

Рис. 1: Relation among several variations of compactness.

Theorem 2. *g-closed subset of a strongly star* I*-compact space is strongly star* I*-compact subset.*

Proof. Let B be a g-closed subset of a strongly star I-compact space (X, τ, I), and let $\mathcal{U}$ be a τ-open cover of B. i.e., $B \subseteq \cup\mathcal{U}$. But B is a g-closed. Therefore, $\overline{B} \subseteq \cup\mathcal{U}$.

So, $X/(\cup\mathcal{U}) \subseteq X/\overline{B}$.

Now, $\mathcal{V} = \mathcal{U} \cup (X/\overline{B})$ become an open cover of X. But X is a strongly star I-compact space.

Therefore, there exists a finite subset $F \subseteq X$ Such that $X/St(F, \mathcal{V}) \in I$.

$\Rightarrow X/(St(F,\mathcal{U}) \cup (X/\overline{B})) \in I$ or $X/St(F,\mathcal{U}) \in I$.

$\Rightarrow (X/St(F,\mathcal{U})) \cap \overline{B})) \in I$ or $B/St(F,\mathcal{U}) \subseteq X/St(F,\mathcal{U}) \in I$.

$\Rightarrow (X/St(F,\mathcal{U})) \cap B)) \in I$ or $B/St(F,\mathcal{U}) \in I$.

$\Rightarrow B/St(F,\mathcal{U}) \in I$ or $B/St(F,\mathcal{U}) \in I$.

$\Rightarrow B/St(F,\mathcal{U}) \in I$ Therefore, B is a strongly star I-compact subset.

□

Corollary 1. *Every closed subset of a strongly star* I*-compact space is a strongly star* I*-compact subset.*

Theorem 3. *If* A *and* B *be two strongly star* I*-compact subset in an ideal space* (X, τ, I) *then* $A \cup B$ *is also an strongly star* I*-compact subset.*

Proof. Let $\mathcal{U} = \{\mathcal{U}_\alpha : \alpha \in \Lambda\}$ be an τ-open cover of $A \cup B$, where A and B are strongly star I-compact subset in the ideal space (X, τ, I).

Therefore, $\mathcal{U}$ is an τ-open cover of A as well as for B. Thus there exists a finite sets

$M \subseteq A$ and $N \subseteq B$ such that $A/St(M,\mathcal{U}) = I_1 \in I$ and $B/St(N,\mathcal{U}) = I_2 \in I$.

Therefore, $A = St(M,\mathcal{U}) \cup I_1$ and $B = St(N,\mathcal{U}) \cup I_2$.

$\Rightarrow A \cup B = St(M,\mathcal{U}) \cup St(N,\mathcal{U}) \cup (I_1 \cup I_2)$

$\Rightarrow A \cup B = St(M \cup N,\mathcal{U}) \cup (I_1 \cup I_2)$

$\Rightarrow A \cup B/St(M \cup N,\mathcal{U}) = (I_1 \cup I_2) \in I$

Since M and N are finite, Hence $M \cup N \subseteq A \cup B$ is also finite.

Hence $A \cup B$ is a strongly star I-compact subset of X.

□

Corollary 2. *Finite union of strongly star I-compact subset is a strongly star I-compact subset.*

Theorem 4. *In a strongly star I-compact space (X, τ, I) if $B \subseteq X$ is a clopen subset of X, then (B, τ_B, I_B) is a strongly star I_B-compact space.*

Proof. ⇒Let (X, τ, I) is strongly star I-compact space and $B \subseteq X$ is cl-open. Let $\mathcal{U}$ be a τ_B open cover of B. But B is open, therefore, $\mathcal{U} \subseteq \tau$. Also B is closed. Hence $\mathcal{V} = \mathcal{U} \cup \{X/B\}$ is a τ open cover of X. But (X, τ, I) is strongly star I-compact. Therefore, there exists a finite subset $M \subseteq X$ such that $X/St(M, \mathcal{V}) \in I \Rightarrow B/St(M, \mathcal{V}) \in I_B$. We take $P = M \cap B \subseteq B$ Which is a finite subset of B.

Now $St(M, \mathcal{V}) = St(P,\mathcal{U}) \cup (X/B)$ or $St(M, \mathcal{V}) = St(P,\mathcal{U})$ In both cases $B/St(P,\mathcal{U}) =$ $= St(M, \mathcal{V}) \in I_B$ Hence, (B, τ_B, I_B) is strongly star I_B-compact space. □

Theorem 5. *$f : (X, \tau, I) \to (Y, \sigma)$ be function from a strongly star I-compact space to a topological space (Y, σ) If f is continuous then $f(x)$ is strongly star $f(I)$ compact subset of Y.*

Proof.

Let $f : (X, \tau, I) \to (Y, \sigma)$ is a continuous function and (X, τ, I) is strongly star I-compact.

Let, $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be σ open cover of $f(X)$.

Therefore, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is an open cover of X. But X is strongly star I-compact. Therefore there exists a finite set $F \subseteq X$, such that $X/St(F, \mathcal{V} \in I. \Rightarrow f(X/St(F, \mathcal{V})) \in f(I)$

$\Rightarrow f(X)/f(St(F, \mathcal{V})) \subseteq f(X/St(F, \mathcal{V})) \in f(I)$

$\Rightarrow f(X)/f(\cup\{f^{-1}(U_\alpha) \in \mathcal{V} : F \cap f^{-1}(U_\alpha) \neq \emptyset\}) \in f(I)$

$\Rightarrow f(X)/ \cup \{U_\alpha \in \mathcal{U} : f(F) \cap U_\alpha \neq \emptyset\} \in f(I)$

$\Rightarrow f(X/St(F,\mathcal{U})) \in f(I)$

Here, $|f(F)| \leq |F|$, therefore $f(F)$ is finite. Hence, $f(X)$ is a strongly star $f(I)$-compact space.

□

5. Modified and idealized finite intersection property:

In an ideal space (X, τ, I), a family $\mathcal{H}$ of subsets of X is said to have MIFIP if for every finite subset $P \subseteq X$, if $\cap\{H \in \mathcal{H} : P \cap (X/H) \neq \emptyset\} \notin I$

Theorem 6. *In an ideal space (X, τ, I), following statements are equivalent:*

1. *Every family of closed sets having MIFIP has non empty intersection.*
2. *(X, τ, I) is strongly star I-compact space.*

PROOF. (1) $\Rightarrow$ (2)

Let condition 1 holds and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is an open cover of X. Then $\mathcal{H} = \{X/U_\alpha : \alpha \in \Lambda\}$ is a family of closed sets such that $\cap\mathcal{H} = X/\bigcup_{\alpha\in\Lambda} U_\alpha = X/X = \emptyset$.

Since, intersection is empty. The family $\mathcal{H}$ Must not have MIFIP. Therefore there exists a finite set $P \subseteq X$ such that

$\bigcap\{H \in \mathcal{H} : P \cap (X/H) \neq \emptyset\} \in I$

$\Rightarrow \bigcap\{X/U_\alpha : \alpha \in \Lambda$ and $P \cap U_\alpha \neq \emptyset\} \in I$

$\Rightarrow X/\bigcup\{U_\alpha : \alpha \in \Lambda$ and $P \cap U_\alpha \neq \emptyset\} \in I$

$\Rightarrow X/St(P,\mathcal{U}) \in I$.

Therefore (X, τ, I) is a strongly star I-compact space.

(2) $\Rightarrow$ (1) Let (X, τ, I) is a strongly star I-compact space and $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$ be a family of closed sets having MIFIP. If possible assume that cap $\mathcal{H} = \emptyset$.

Then for the family $\mathcal{U} = \{U_\alpha = X/H_\alpha : \alpha \in \Lambda\}$ is a family of open sets and $\cup\mathcal{U} = X/(\cap_{\alpha\in\Lambda} H_\alpha) = X/\emptyset = X$.

Therefore, $\mathcal{U}$ is an open cover of X. But X is strongly star I-compact space. So, there exists a finite subset $P \subseteq X$ such that $X/St(P,\mathcal{U} \in I$.

$\Rightarrow X/\bigcup\{U \in \mathcal{U} : P \cap U \neq \emptyset\} \in I$.

$\Rightarrow \bigcap\{X/U : U \in \mathcal{U}$ and $P \cap U \neq \emptyset\} \in I$.

$\Rightarrow \bigcap\{H \in \mathcal{H} : P \cap (X/H) \neq \emptyset\} \in I$, which is a contradiction to the fact that the family $\mathcal{H}$ has MIFIP. Therefore, $\cap\mathcal{H} \neq \emptyset$ □

6. Countably I-compact

DEFINITION 5. *An ideal space (X, τ, I) is called an countably I-compact space if for every countable open cover $\mathcal{U}$, there exists a finite subset $\{U_1, U_2, U_3, ..., U_k\} \subseteq \mathcal{U}$ such that $X/(\cup_{i=1}^{k} U_i) \in I$*

THEOREM 7. *Every closed subspace of a countably I-compact space is countably I-compact.*

PROOF. Let (A, τ_A) be a closed subspace of a countably I-compact space (X, τ, I).

Let, $\mathcal{U}_A = \{U_{A_n} : n \in N\}$ be a τ_A-open cover of A.

Therefore, for every $U_A \in \mathcal{U}_A$ there exist $U \in \tau$ such that $U_A = U \cap A$.

Assume that $\mathcal{U} = \{U_n : U_n \cap A \in \mathcal{U}_A\}$

Clearly, $\mathcal{V} = \mathcal{U} \cup (X/A)$ is an open cover of X. But X is countably I-compact.

Therefore, it has finite subset $\{X/A, V_1, V_2, V_3, ..., V_K\} \subseteq \mathcal{V}$ such that $X/((\cup_{i=1}^{k} V_i) \cup (X/A)) =$
$= I_1 \in I$

$(\cup_{i=1}^{k} V_i) \cup (X/A) \cup I_1 = X$

$A \cap \{(\cup_{i=1}^{k} V_i) \cup (X/A) \cup I_1\} = A$

$\Rightarrow (A \cap (\cup_{i=1}^{k} V_i)) \cup (A \cap (X/A)) \cup (A \cap I_1) = A$

$\Rightarrow \cup_{i=1}^{k} (A \cap V_i)) \cup \emptyset \cup (A \cap I_1) = A$

$\Rightarrow \cup_{i=1}^{k} U_{A_i} \cup (A \cap I_1) = A$

$\Rightarrow A/\cup_{i=1}^{k} U_{A_i} = A \cap I_1 \in I_A$

Therefore $(A, \tau_A)$is an countable compact subspace of (X, τ, I). □

THEOREM 8. *Every countably I_{fin}-compact space is strongly star I_{fin}-compact space.*

PROOF. Suppose that (X, τ, I) is countably I-compact but not strongly star I_{fin}-compact. Let $\mathcal{U}$ be an arbitrary open cover of X then for every finite subset $B \subseteq X$, $X/St(B,\mathcal{U}) \notin I_{fin}$. i.e. $X/St(B,\mathcal{U}) \neq \emptyset$

Consider a point $x_0 \in X$

Set other points as $x_n \in X/St(\{x_0, x_1, ..., x_{n-1}\},\mathcal{U})$ and construct a countable open cover $\mathcal{V} = \{V_n = St(x_{n-1},\mathcal{U}) : n \in N\}$

Let, $A = \{x_{n-1} :\in N\}$. If $y \in \overline{A}$, we have an open set $U \in \mathcal{U}$ (since $\mathcal{U}$ is an open cover)such that $y \in U$. But $y \in \overline{A}$, therefore, $A \cap U \neq \emptyset$.

Let, $x_{k-1} \in A \cap U \Rightarrow U \subseteq St(x_{k-1},\mathcal{U}) \Rightarrow y \in St(x_{k-1},\mathcal{U}) \Rightarrow y \in V_k$ for some $k \in N$.

Therefore, $\mathcal{V}$ is a countable cover of $\overline{A}$. But $\overline{A}$ is a closed subspace of (X, τ, I), $\overline{A}$ is also countably I_{fin}-compact (by Theorem 7). Therefore, there exists finite subset $\{V_{n_1}, V_{n_2}, V_{n_3}, ..., V_{n_p}\} \subseteq \mathcal{V}$ such that

$\overline{A}/(\bigcup_{i=1}^{p} V_{n_i}) = I_1 \in I_{fin}$

$\Rightarrow (\bigcup_{i=1}^{p} V_{n_i}) \cup I_1 = \overline{A}$

$\Rightarrow A \subseteq (\bigcup_{i=1}^{p} V_{n_i}) \cup I_1$.

But the construction of $\mathcal{V}$, each V_{n_i} can contain only one element of A also I, is a finite subset of X which contradicts the fact that A is a countable infinite set.

Hence, (X, τ, I) is an strongly star I_{fin}-compact space.

□

Open Problem Does there exists a strongly star I_{fin}-compact space which is not countably I_{fin}-compact.

7. Conclusion

Strongly star I-compactness has a very unique structure in comparison to other variations of compactness. This covering property can also be expressed in terms of family of closed sets by means of modified and idealized finite intersection property. All countably I_{fin}-compact spaces are strongly star I_{fin}-compact space although their structures are very different. These topological properties can further be used in the study of selection principles and topological games involbing ideals.

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