ЧЕБЫШЕВСКИЙ СБОРНИК
Том 25. Выпуск 2.

УДК 519.1 DOI 10.22405/2226-8383-2024-25-2-208-221

Хроматичность полных расщепленных графов

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Аннотация

Соединение нулевого графа O_m и полного графа K_n, $O_m + K_n = S(m, n)$, называется полным разделенным графом. В этой статье мы характеризуем хроматическую уникальность, определяем хроматический номер списка и характеризуем уникальную раскрашиваемость списка для полного графа разделения $G = S(m, n)$. Мы докажем, что G хроматически уникален тогда и только тогда, когда $1\ lem\ le2$, $ch(G) = n + 1$, G является уникальным раскрашиваемым графом с 3-списком тогда и только тогда, когда $m \geqslant 4$, $n \geqslant 4$ и $m + n \geqslant 10$, $m(G) \leqslant 4$ на каждые $1 \leqslant m \leqslant 5$ и $n \geqslant 6$. Также доказано некоторое свойство графа $G = S(m, n)$, когда он представляет собой k-листовой раскрашиваемый граф.

Ключевые слова: хроматически уникальный, список-хроматическое число, уникально списочный раскрашиваемый граф, полный расщепляемый граф.

Библиография: 40 названий.

Для цитирования:
Ле Ван Хонг. Хроматичность полных расщепленных графов // Чебышевский сборник, 2024, т. 25, вып. 2, с. 208–221.

CHEBYSHEVSKII SBORNIK
Vol. 25. No. 2.

UDC 519.1 DOI 10.22405/2226-8383-2024-25-2-208-221

The chromaticity of complete split graphs

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Abstract

The join of null graph O_m and complete graph K_n, $O_m + K_n = S(m, n)$, is called a complete split graph. In this paper, we characterize chromatically unique, determine list-chromatic number and characterize unique list colorability of the complete split graph $G = S(m, n)$. We shall prove that G is chromatically unique if and only if $1 \leqslant m \leqslant 2$, $ch(G) = n + 1$, G is uniquely 3-list colorable graph if and only if $m \geqslant 4$, $n \geqslant 4$ and $m + n \geqslant 10$, $m(G) \leqslant 4$ for every $1 \leqslant m \leqslant 5$ and $n \geqslant 6$. Some the property of the graph $G = S(m, n)$ when it is k-list colorable graph also proved.

Keywords: chromatically unique, list- chromatic number, uniquely list colorable graph, complete split graph.

Bibliography: 40 titles.

For citation:
Le Xuan Hung, 2024. "The chromaticity of complete split graphs", *Chebyshevskii sbornik*, vol. 25, no. 2, pp. 208–221.

1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$, $E(G)$ (or V, E in short) and $\overline{G}$ will denote its vertex-set, its edge-set and its complementary graph, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denoted by $N_W(S)$. If $S = \{v\}$, then $N(S)$ and $N_W(S)$ are denoted shortly by $N(v)$ and $N_W(v)$, respectively. For a vertex $v \in V(G)$, the degree of v (resp., the degree of v with respect to W), denoted by $\deg(v)$ (resp., $\deg_W(v)$), is $|N_G(v)|$ (resp., $|N_W(v)|$). The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. The null graphs and complete graphs of order n are denoted by O_n and K_n, respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [2].

An *acyclic* graph, one not containing any cycles, is called *forest.* A connected forest is called a *tree*, a tree of order n is denoted by T_n.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Their *union* $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their *join* defined is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2.

A graph $G = (V, E)$ is called a *split graph* if there exists a partition $V = I \cup K$ such that $G[I]$ and $G[K]$ are null and complete graphs, respectively. We will denote such a graph by $S(I \cup K, E)$. The join of null graph O_m and complete graph K_n, $O_m + K_n = S(m, n)$, is called a *complete split graph.*The notion of split graphs was introduced in 1977 by Földes and Hammer [14]. A role that split graphs play in graph theory is clarified in [14] and in [7], [9], [27], [30], [34], [35], [36]. These graphs have been paid attention also because they have connection with packing and knapsack problems [11], with the matroid theory [15], with Boolean functions [31], with the analysis of parallel processes in computer programming [18] and with the task allocation in distributed systems [19]. Many generalizations of split graphs have been made. The newest one is the notion of bisplit graphs introduced by Brandstädt et al. [6].

Let $G = (V, E)$ be a graph and λ is a positive integer.

A λ-*coloring* of G is a bijection $f : V(G) \to \{1, 2, \ldots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer λ such that G has a λ-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. We say that a graph G is n-*chromatic* if $n = \chi(G)$.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, two λ-colorings f and g are considered different if and only if $f(v_k) \neq g(v_k)$ for some $k = 1, 2, \ldots, n$. Let $P(G, \lambda)$ (or simply $P(G)$ if there is no danger of confusion) denote the number of distinct λ-colorings of G. It is well-known that for any graph G, $P(G, \lambda)$ is a polynomial in λ, called the *chromatic polynomial* of G. The notion of chromatic polynomials was first introduced by Birkhoff [4] in 1912 as a quantitative approach to tackle the four-color problem. Two graphs G and H are called *chromatically equivalent* or in short χ-*equivalent*, and we write in notation $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is called *chromatically unique* or in short χ-*unique* if $G' \cong G$ (i.e., G' is isomorphic to G) for any graph G' such that $G' \sim G$. For examples, all cycles are χ-unique [25]. The notion of χ-unique graphs was first introduced and studied by Chao and Whitehead [10] in 1978. The readers can see the surveys [22], [25], [26] and [36] for more informations about χ-unique graphs.

Let $(L_v)_{v \in V}$ be a family of sets. We call a coloring f of G with $f(v) \in L_v$ for all $v \in V$ is a *list coloring from the lists* L_v. We will refer to such a coloring as an L-coloring. The graph G is called λ-*list-colorable*, or λ-*choosable*, if for every family $(L_v)_{v \in V}$ with $|L_v| = \lambda$ for all v, there is a coloring of G from the lists L_v. The smallest positive integer λ such that G has a λ-choosable is called the *list-chromatic number*, or *choice number* of G and is denoted by $ch(G)$.

Let G be a graph with n vertices and suppose that for each vertex v in G, there exists a list of k colors L_v, such that there exists a unique L-coloring for G, then G is called a *uniquely k-list colorable graph* or a UkLC graph for short. The idea of uniquely colorable graph was introduced independently by Dinitz and Martin [13] and by Mahmoodian and Mahdian [29] (Mahmoodian and Mahdian have obtained some results on the uniquely k-list colorable complete multipartite graphs). There have been many interesting and insightful research results on these issues for different graph classes (see [16], [20], [21], [23], [24], [29]). However, these are still issues that have not been resolved thoroughly, so much more attention is needed.

In this paper, we shall characterize chromatically unique, determine list-chromatic number and characterize unique list colorability of the complete split graph $G = S(m, n)$. Namely, we shall prove that G is chromatically unique if and only if $1 \leqslant m \leqslant 2$ (Section 2), $ch(G) = n + 1$ (Section 3), G is uniquely 3-list colorable graph if and only if $m \geqslant 4$, $n \geqslant 4$ and $m + n \geqslant 10$, $m(G) \leqslant 4$ for every $1 \leqslant m \leqslant 5$ and $n \geqslant 6$ (Section 5), some the property of the graph G when it is k-list colorable graph also proved (Section 4).

2. Chromatically unique

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a k-independent partition in G if each A_i is a non-empty independent set of G. Let $\alpha(G, k)$ denote the number of k-independent partitions in G. Hence, $P(G, \lambda) = \sum\limits_{1 \leqslant k \leqslant n} \alpha(G, k)(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda - 1) \ldots (\lambda - k + 1)$.

The polynomial $\sigma(G, x) = \sum\limits_{1 \leqslant k \leqslant n} \alpha(G, k)x^k$ is called the σ-polynomial of G. For convenience, simply denote $\sigma(G, x)$ by $\sigma(G)$ and $G \cong H$ by $G = H$. The following lemmas will be used to prove our main results.

Lemma 7 ([25]). *If K_n is a complete graph on n vertices then $\chi(K_n) = n$ and G is χ-unique.*

Lemma 8 ([32]). *Let G and H be two χ-equivalent graphs. Then*
(i) $|V(G)| = |V(H)|$;
(ii) $|E(G)| = |E(H)|$;
(iii) $\chi(G) = \chi(H)$;
(iv) G is connected if and only if H is connected;
(v) G is 2-connected if and only if H is 2-connected.

Lemma 9 ([32]). *(i) All trees of the same order are χ-equivalent;*
(ii) A tree T_n is χ-unique if and only if $1 \leqslant n \leqslant 3$.

Lemma 10 ([8]). *Let G and H be two disjoint graphs. Then*

$$\sigma(G + H, x) = \sigma(G, x)\sigma(H, x).$$

Lemma 11 ([28]). *Let G and H be two graphs. Then $P(G, \lambda) = P(H, \lambda)$ if and only if $\sigma(G, x) = \sigma(H, x)$.*

Lemma 12 ([36]). *Let $G = S(I \cup K, E)$ be a split graph with $|K| = n$ and $k = \max\{\deg(u) \mid u \in I\}$. Then*
(i) G is n-chromatic if and only if $k < n$;
(ii) G is $(n + 1)$-chromatic if and only if $k = n$.

Lemma 13 ([21]). *The graph $G = K_2^m + K_n$ is χ-unique.*

Now we characterize χ-unique complete split graphs.

THEOREM 1. *$G = S(m,n)$ is chromatically unique if and only if $1 \leqslant m \leqslant 2$.*

PROOF. Let $V(G) = I \cup K$ is a partition of $V(G)$ such that $G[I] = O_m$, $G[K] = K_n$, $G = G[I] + G[K]$, $I = \{u_1, u_2, \ldots, u_m\}$ and $K = \{v_1, v_2, \ldots, v_n\}$.

Firrst we prove the necessity. Suppose that $G = S(m,n)$ is χ-unique. For suppose on the contrary that $m \geqslant 3$. If $n = 1$ then $G = T_{m+1}$, where T_{m+1} is a tree of order $m+1$. By (ii) of Lemma 9, G is not χ-unique because $m+1 \geqslant 4$, a contradiction. So $n \geqslant 2$. Set $G' = (I' \cup K', E')$ with

$$I' = \{u_1, u_2, \ldots, u_m\}, K' = \{v_1, v_2, \ldots, v_n\}$$

and $E' = E_1 \cup E_2 \cup E_3$ with

$$E_1 = \{v_1u_1, u_2u_3, \ldots, u_{m-1}u_m\},$$

$$E_2 = \{u_iv_j \mid i = 1, 2, \ldots, m, j = 2, \ldots, n\},$$

$$E_3 = \{v_iv_j \mid i \neq j; i, j = 1, 2, \ldots, n\}.$$

It is not difficult to see that

$$G = G[\{v_2, v_3, \ldots, v_n\}] + G[\{v_1, u_1, u_2, \ldots, u_m\}] = K_{n-1} + T_{m+1},$$

$$G' = G'[\{v_2, v_3, \ldots, v_n\}] + G'[\{v_1, u_1, u_2, \ldots, u_m\}] = K_{n-1} + T'_{m+1},$$

where T_{m+1} and T'_{m+1} are trees of order $m+1$. By (i) of Lemma 9, $P(T_{m+1}, \lambda) = P(T'_{m+1}, \lambda)$. By Lemma 10 and Lemma 11, it follows that $P(G, \lambda) = P(G', \lambda)$, ie., $G \sim G'$. It is clear that

$$|\{u \in V(G) \mid \deg_G(u) = \Delta(G) = m+n-1\}| = |\{v_1, v_2, \ldots, v_n\}| = n,$$

$$|\{u \in V(G') \mid \deg_{G'}(u) = \Delta(G') = m+n-1\}| = |\{v_2, v_3, \ldots, v_n\}| = n-1.$$

So, $G \ncong G'$ and G is not χ-unique, a contradiction. Thus, $1 \leqslant m \leqslant 2$.

Now we prove the sufficiency. Suppose that $1 \leqslant m \leqslant 2$. If $m = 1$ then G is a complete graph K_{n+1}. By Lemma 7, G is χ-unique. If $m = 2$ then $G = K_2^1 + K_n$. By Lemma 13, G is χ-unique. □

3. List-chromatic number

We need the following Lemmas 14, 15 to prove our results.

LEMMA 14 ([12]). *For a graph G, $ch(G) \geqslant \chi(G)$.*

We determine list-chromatic number for complete graphs.

LEMMA 15. *If K_n is a complete graph on n vertices then $ch(K_n) = n$.*

PROOF. By Lemma 7 and Lemma 14, $ch(K_n) \geqslant n$. Set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and L_{v_i} is a list of colors of V_i such that $|L_{v_i}| = n$ for every $i = 1, 2, \ldots, n$. Let f be a coloring of K_n such that

$$f(v_1) \in L_{v_1}, f(v_2) \in L_{v_2} \setminus \{f(v_1)\}, \ldots, f(v_n) \in L_{v_n} \setminus \{f(v_1), f(v_2), \ldots, f(v_{n-1}\}.$$

Then f is n-choosable for K_n, i.e., $ch(K_n) \leqslant n$. Thus, $ch(K_n) = n$. □

Now we determine list-chromatic number for complete split graphs.

Theorem 2. *List-chromatic number of $G = S(m, n)$ is*

$$ch(G) = n + 1.$$

Proof. By (ii) of Lemma 12 and Lemma 14, $ch(G) \geqslant n+1$. Let $V(G) = I \cup K$ is a partition of $V(G)$ such that $G[I] = O_m$, $G[K] = K_n$, $I = \{u_1, u_2, \ldots, u_m\}$ and $K = \{v_1, v_2, \ldots, v_n\}$. Let

$$L_{u_1}, L_{u_2}, \ldots, L_{u_m}, L_{v_1}, L_{v_2}, \ldots, L_{v_n}$$

be the lists of colors of

$$u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n,$$

respectively, such that

$$|L_{u_1}| = |L_{u_2}| = \ldots = |L_{u_m}| = |L_{v_1}| = |L_{v_2}| = \ldots = |L_{v_n}| = n + 1.$$

By Lemma 15, there exists $(n+1)$-choosable g of $G[K \cup \{u_1\}] = K_{n+1}$ with the lists of colors $L_{v_1}, L_{v_2}, \ldots, L_{v_n}, L_{u_1}$. Let f be the coloring of G such that
$f(v_i) = g(v_i)$ for every $i = 1, 2, \ldots, n$,
$f(u_1) = g(u_1)$,
$f(u_i) \in L_{u_i} \setminus g(N(u_i))$ for every $i = 2, 3, \ldots, m$.
Then f is $(n+1)$-choosable for G, i.e., $ch(G) \leqslant n+1$. Thus $ch(G) = n+1$. □

4. Property of $S(m, n)$ when it is k-list colorable

If a graph G is not uniquely k-list colorable, we also say that G has property $M(k)$. So G has the property $M(k)$ if and only if for any collection of lists assigned to its vertices, each of size k, either there is no list coloring for G or there exist at least two list colorings. The least integer k such that G has the property $M(k)$ is called the *m-number* of G, denoted by $m(G)$. This conception was originally introduced by Mahmoodian and Mahdian in [29].

Lemma 16 ([29]). *Each UkLC graph is also a $U(k-1)LC$ graph.*

Lemma 17 ([29]). *The graph G is UkLC if and only if $k < m(G)$.*

Lemma 18 ([29]). *A connected graph G has the property $M(2)$ if and only if every block of G is either a cycle, a complete graph, or a complete bipartite graph.*

Lemma 19 ([29]). *For every graph G we have $m(G) \leqslant |E(\overline{G})| + 2$.*

Lemma 20 ([29]). *Every UkLC graph has at least $3k - 2$ vertices.*

For example, one can easily see that the graph $S(2, 2)$ has the property M(3) and it is U2LC, so $m(S(2, 2)) = 3$.

Proposition 1. *Let $G = S(m, n)$ be a UkLC graph with $k \geqslant 2$. Then*
(i) $m \geqslant 2$;
(ii) $k < \frac{m^2-m+4}{2}$;
(iii) $k \leqslant \lfloor \frac{m+n+2}{3} \rfloor$.

Proof. (i) If $m = 1$ then G is a complete graph K_{n+1}. Lemma 18, G has the property $M(2)$, a contradiction.

(ii) It is not difficult to see that $|E(\overline{G})| = \frac{m(m-1)}{2}$. By Lemma 19, we have

$$m(G) \leqslant |E(\overline{G})| + 2 = \frac{m^2 - m + 4}{2}.$$

By Lemma 17, we have $k < \frac{m^2-m+4}{2}$.

(iii) Assertion (iii) follows immediately from Lemma 20. $\square$

Let $G=S(m,n)$ be a UkLC graph with $V(G)=I\cup K, G[I]=O_m, G[K]=K_n, m\geqslant 2, n\geqslant 1, k\geqslant 3$. Set

$$I=\{u_1,u_2,\ldots,u_m\}, K=\{v_1,v_2,\ldots,v_n\}.$$

Suppose that, for the given k-list assignment L:

$L_{u_i}=\{a_{i,1},a_{i,2},\ldots,a_{i,k}\}$ for every $i=1,\ldots,m$,

$L_{v_i}=\{b_{i,1},b_{i,2},\ldots,b_{i,k}\}$ for every $i=1,\ldots,n$,

there is a unique k-list color f:

$f(u_i)=a_{i,1}$ for every $i=1,\ldots,m$,

$f(v_i)=b_{i,1}$ for every $i=1,\ldots,n$.

Theorem 3. *(i) $b_{i,1}\neq b_{j,1}$, where $1\leqslant i,j\leqslant n$ and $i\neq j$;*

(ii) $a_{i,1}\neq b_{j,1}$, where $1\leqslant i\leqslant m, 1\leqslant j\leqslant n$;

(iii) $a_{i,1}\notin\{a_{j,2},a_{j,3},\ldots,a_{j,k}\}$, where $i,j=1,2,\ldots,m$.

Proof. (i) Since $G[K]=K_n$, it is not difficult to see that $b_{i,1}=f(v_i)\neq f(v_j)=b_{j,1}$, where $1\leqslant i,j\leqslant n$ and $i\neq j$.

(ii) Since $G[K\cup\{u_i\}]=K_{n+1}$ for every $i=1,\ldots,m$, it is not difficult to see that

$$a_{i,1}=f(u_i)\neq f(v_j)=b_{j,1},$$

where $1\leqslant i\leqslant m, 1\leqslant j\leqslant n$.

(iii) If $i=j$, then it is obvious that the conclusion is true. If $i\neq j$, then we suppose that there exists i_0,j_0 such that $i_0,j_0=1,\ldots,m; i_0\neq j_0$ and $a_{i_0,1}\in\{a_{j_0,2},a_{j_0,3},\ldots,a_{j_0,k}\}$. It is clear that $a_{i_0,1}\neq a_{j_0,1}$. Let f' be the coloring of G such that

(a) $f'(u_{j_0})=a_{i_0,1}$;

(b) $f'(u_i)=a_{i,1}$ for every $i\in\{1,\ldots,m\}, i\neq j_0$;

(c) $f'(v_i)=b_{i,1}$ for every $i=1,\ldots,n$.

Then f' is a k-list coloring for G and $f'\neq f$, a contradiction. $\square$

Set $\overline{f(v)}=L_v\setminus\{f(v)\}$ for every $v\in V(G)=I\cup K$.

Theorem 4. *(i) $2\leqslant|f(I)|$;*

(ii) $|f(I)|\leqslant m-2$, where $m\geqslant 4$;

(iii) $\cup_{v\in I}\overline{f(v)}\subseteq f(K)$;

(iv) $\cup_{v\in V(G)}\overline{f(v)}\subseteq f(V(G))$;

(v) There exists $i\in\{1,\ldots,n\}$ such that $\overline{f(v_i)}\subseteq f(I)$.

Proof. (i) For suppose on the contrary that $|f(I)|=1$, then $a_{1,1}=a_{2,1}=\ldots=a_{m,1}=a$. Set $H=G-I$, it is not difficult to see that H is a complete graph K_n. We assign the following lists L'_v for the vertices v of H:

If $a\in L_v$ then $L'_v=L_v\setminus\{a\}$,

If $a\notin L_v$ then $L'_v=L_v\setminus\{b\}$, where $b\in L_v$ and $b\neq f(v)$.

It is clear that $|L'_v|=k-1\geqslant 2$ for every $v\in V(H)$. By Lemma 18, H has the property $M(2)$. So by Lemma 16, H has the property $M(k-1)$. It follows that with lists L'_v, there exist at least two list colorings for the vertices v of H. So it is not difficult to see that with lists L_v, there exist at least two list colorings for the vertices v of G, a contradiction.

(ii) For suppose on the contrary that $|f(I)|\geqslant m-1$. We consider separately two cases.

Case 1: $|f(I)|=m-1$.

Without loss of generality, we may assume that $a_{1,1} = a_{2,1}$ and $a_{i,1} \neq a_{j,1}$ for every $i, j \in \{2, \ldots, m\}, i \neq j$. Set graph $G' = (V', E')$, with

$$V' = I \cup K, E' = (E(G) \cup \{u_i u_j | i, j = 1, 2, \ldots, m; i \neq j\}) \setminus \{u_1 u_2\}.$$

It is clear that G' is complete split graph $S(2, m+n-2)$ with $V(G') = I' \cup K'$, where

$$I' = \{u_1, u_2\}, K' = \{u_3, u_4, \ldots, u_m, v_1, v_2, \ldots, v_n\}$$

Since $a_{1,1} = a_{2,1}$, it is not difficult we have got a contradiction.

Case 2: $|f(I)| = m$.

In this case, $a_{i,1} \neq a_{j,1}$ for every $i, j \in \{1, 2, \ldots, m\}, i \neq j$. Set graph $G'' = (V'', E'')$, with

$$V'' = I \cup K, E'' = E(G) \cup \{u_i u_j | i, j = 1, 2, \ldots, m; i \neq j\}.$$

It is clear that G'' is a complete graph K_{m+n}. By Lemma 18, G'' has the property $M(2)$, so with lists L_v, there exist at least two list colorings for the vertices v of G''. Since $V(G) = V(G'')$, it is not difficult to see that with lists L_v, there exist at least two list colorings for the vertices v of G, a contradiction.

(iii) For suppose on the contrary that $\cup_{v \in I} \overline{f(v)} \nsubseteq f(K)$. Then there exists i_0, j_0 such that $a_{i_0,j_0} \notin f(K)$ with $1 \leqslant i_0 \leqslant m, 2 \leqslant j_0 \leqslant k$. Let f' be the coloring of G such that

(a) $f'(u_{i_0}) = a_{i_0,j_0}$;

(b) $f'(u_i) = a_{i,1}$ for every $i \in \{1, \ldots, m\}, i \neq i_0$;

(c) $f'(v_i) = b_{i,1}$ for every $i = 1, \ldots, n$.

Then f' is a k-list coloring for G and $f' \neq f$, a contradiction.

(iv) For suppose on the contrary that $\cup_{v \in I \cup K} \overline{f(v)} \nsubseteq f(V(G)$. We consider separately two cases.

Case 1: There exists i_0, j_0 such that $a_{i_0,j_0} \notin f(V(G))$ with $1 \leqslant i_0 \leqslant m, 2 \leqslant j_0 \leqslant k$.

Let f' be the coloring of G such that

(a) $f'(u_{i_0}) = a_{i_0,j_0}$;

(b) $f'(u_i) = a_{i,1}$ for every $i \in \{1, \ldots, m\}, i \neq i_0$;

(c) $f'(v_i) = b_{i,1}$ for every $i = 1, \ldots, n$.

Then f' is a k-list coloring for G and $f' \neq f$, a contradiction.

Case 2: There exists i_0, j_0 such that $b_{i_0,j_0} \notin f(V(G))$ with $1 \leqslant i_0 \leqslant n, 2 \leqslant j_0 \leqslant k$.

Let f'' be the coloring of G such that

(a) $f''(u_i) = a_{i,1}$ for every $i \in \{1, \ldots, m\}$;

(b) $f''(v_{i_0}) = b_{i_0,j_0}$;

(c) $f''(v_i) = b_{i,1}$ for every $i \in \{1, \ldots, n\}, i \neq i_0$.

Then f'' is a k-list coloring for G and $f'' \neq f$, a contradiction.

(v) For suppose on the contrary that $\overline{f(v_i)} \nsubseteq f(I)$ for every $i \in \{1, \ldots, n\}$, then $|\overline{f(v_i)} \setminus f(I)| \geqslant 1$ for every $i \in \{1, \ldots, n\}$. So $|L_{v_i} \setminus f(I)| \geqslant 2$ for every $i \in \{1, \ldots, n\}$. Set graph

$$H = G - I = G[K] = K_n.$$

Let $L'_{v_i} \subseteq L_{v_i} \setminus f(I)$ such that $|L'_{v_i}| = 2$ for every $i \in \{1, \ldots, n\}$. By Lemma 18, H has the property $M(2)$, it follows that with lists L'_{v_i}, there exist at least two list colorings for the vertices v_i for every $i \in \{1, \ldots, n\}$. So it is not difficult to see that with lists L_v, there exist at least two list colorings for the vertices v of G, a contradiction. □

5. Uniquely 3-list colorable complete split graphs

We need the following Lemmas 21–29 to prove our results.

Lemma 21. *(i) $m(S(1,n)) = 2$ for every $n \geqslant 1$;*
(ii) $m(S(r,1)) = 2$ for every $r \geqslant 1$;
(iii) $m(S(2,n)) = 3$ for every $n \geqslant 2$.

Proof. (i) It is clear that $S(1,n)$ is a complete graph for every $n \geqslant 1$, by Lemma 18, $m(S(1,n)) = 2$ for every $n \geqslant 1$.

(ii) It is clear that $S(r,1)$ is a complete bipartite graph for every $r \geqslant 1$, by Lemma 18, $m(S(r,1)) = 2$ for every $r \geqslant 1$.

(iii) By Lemma 18, $G = S(2,n)$ is U2LC for every $n \geqslant 2$.

It is not difficult to see that $|E(\overline{G})| = 1$. By Lemma 19, $m(S(2,n)) \leqslant 3$ for every $n \geqslant 2$.

Thus, $m(S(2,n)) = 3$ for every $n \geqslant 2$. □

Lemma 22 ([16]). *$m(S(3,n)) = 3$ for every $n \geqslant 2$;*

Lemma 23 ([16]). *For every $r \geqslant 2$, $m(S(r,3)) = 3$.*

Lemma 24 ([17]). *Graphs $S(5,4)$ and $S(4,4)$ have property $M(3)$.*

Lemma 25 ([33]). *The graph $S(4,5)$ has property $M(3)$.*

Lemma 26. *$G = S(4,n)$ has the property $M(4)$ for every $n \geqslant 2$;*

Proof. Let $G = S(4,n)$ is a complete split graph with $V(G) = I \cup K, G[I] = O_4, G[K] = K_n, n \geqslant 2$. Set

$$I = \{u_1, u_2, u_3, u_4\}, K = \{v_1, v_2, \ldots, v_n\}.$$

For suppose on the contrary that graph $G = S(4,n)$ is U4LC. So there exists a list of 4 colors L_v for each vertex $v \in V(G)$, such that there exists a unique L-coloring f for G. By (i) and (ii) of Theorem 4, $|f(I)| = 2$.

Let $f(I) = \{a,b\}$. Set graph $H = G - I$, it is not difficult to see that H is a complete graph K_n. We assign the following lists L'_v for the vertices v of H:

(a) If $a, b \in L_v$ then $L'_v = L_v \setminus \{a,b\}$,

(b) If $a \in L_v, b \notin L_v$ then $L'_v = L_v \setminus \{a,c\}$, where $c \in L_v$ and $c \neq f(v)$,

(c) If $a \notin L_v, b \in L_v$ then $L'_v = L_v \setminus \{b,c\}$, where $c \in L_v$ and $c \neq f(v)$,

(d) If $a, b \notin L_v$ then $L'_v = L_v \setminus \{c,d\}$, where $c, d \in L_v, c \neq d$ and $c, d \neq f(v)$.

It is clear that $|L'_v| = 2$ for every $v \in V(H)$. By Lemma 18, H has the property $M(2)$. It follows that with lists L'_v, there exist at least two list colorings for the vertices v of H. So it is not difficult to see that with lists L_v, there exist at least two list colorings for the vertices v of G, a contradiction. □

Lemma 27 ([39]). *(i) For every $n \geqslant 2$, $S(5,n)$ has the property $M(4)$;*
(ii) If $n \geqslant 5$ then $m(S(5,n)) = 4$.

Lemma 28 ([38]). *For every $m \geqslant 1, k \geqslant 2$, $S(m, 2k-3)$ has the property $M(k)$.*

Lemma 29 ([38]). *For every $n \geqslant 1, k \geqslant 2$, $S(2k-3, n)$ has the property $M(k)$.*

Now we prove our results.

Theorem 5. *The graph $G = S(m,n)$ is uniquely 3-list colorable graph if and only if $m \geqslant 4$, $n \geqslant 4$ and $m + n \geqslant 10$.*

Proof. Firrst we prove the necessity. Suppose that $G = S(m,n)$ is U3LC. If $m < 4$ or $n < 4$ then by Lemma 28 and Lemma 29, it is not difficult to see that G has the property $M(3)$, a contradiction. Therefore, $m \geqslant 4$ and $n \geqslant 4$. It follows that $m + n \geqslant 8$. If $m + n = 8$ then $m = 4$ and $n = 4$, by Lemma 24, G has property $M(3)$, a contradiction. If $m + n = 9$ then $(m,n) \in \{(4,5),(5,4)\}$, by Lemma 24 and Lemma 25, G has property $M(3)$, a contradiction. Thus, $m + n \geqslant 10$.

Now we prove the sufficiency. Suppose that $m \geqslant 4$, $n \geqslant 4$ and $m + n \geqslant 10$. Let $V(G) = I \cup K, G[I] = O_m, G[K] = K_n$, $I = \{u_1,u_2,\dots,u_m\}, K = \{v_1,v_2,\dots,v_n\}$. We prove G is U3LC by induction on $m+n$. If $m+n = 10$, then we consider separately three cases.

(i) $m = 4$ *and* $n = 6$.

We assign the following lists for the vertices of G:

$L_{u_1} = \{1,3,4\}, L_{u_2} = \{1,7,8\}, L_{u_3} = \{2,5,6\}, L_{u_4} = \{2,7,8\}$;

$L_{v_1} = \{1,2,3\}, L_{v_2} = \{1,2,4\}, L_{v_3} = \{1,2,5\}, L_{v_4} = \{1,2,6\}, L_{v_5} = \{1,2,7\}, L_{v_6} = \{1,2,8\}$.

A unique coloring f of G exists from the assigned lists:

$f(u_1) = 1, f(u_2) = 1, f(u_3) = 2, f(u_4) = 2$;

$f(v_1) = 3, f(v_2) = 4, f(v_3) = 5, f(v_4) = 6, f(v_5) = 7, f(v_6) = 8$.

(ii) $m = 5$ *and* $n = 5$.

We assign the following lists for the vertices of G:

$L_{u_1} = \{1,4,5\}, L_{u_2} = \{1,3,6\}, L_{u_3} = \{2,3,7\}, L_{u_4} = \{2,4,5\}, L_{u_5} = \{2,6,7\}$;

$L_{v_1} = \{1,2,3\}, L_{v_2} = \{1,2,4\}, L_{v_3} = \{1,2,5\}, L_{v_4} = \{1,2,6\}, L_{v_5} = \{1,2,7\}$.

A unique coloring f of G exists from the assigned lists:

$f(u_1) = 1, f(u_2) = 1, f(u_3) = 2, f(u_4) = 2, f(u_5) = 2$;

$f(v_1) = 3, f(v_2) = 4, f(v_3) = 5, f(v_4) = 6, f(v_5) = 7$.

(iii) $m = 6$ *and* $n = 4$.

We assign the following lists for the vertices of G:

$L_{u_1} = \{1,3,5\}, L_{u_2} = \{1,4,5\}, L_{u_3} = \{2,3,6\}, L_{u_4} = \{2,3,4\}, L_{u_5} = \{2,4,6\}, L_{u_6} = \{2,5,6\}$;

$L_{v_1} = \{1,2,3\}, L_{v_2} = \{1,2,4\}, L_{v_3} = \{1,2,5\}, L_{v_4} = \{1,2,6\}$.

A unique coloring f of G exists from the assigned lists:

$f(u_1) = 1, f(u_2) = 1, f(u_3) = 1, f(u_4) = 2, f(u_5) = 2, f(u_6) = 2$;

$f(v_1) = 3, f(v_2) = 4, f(v_3) = 5, f(v_4) = 6$.

Now let $m+n > 10$ and assume the assertion for smaller values of $m+n$. We consider separately two cases.

Case 1: $m \geqslant 5$.

Set $G' = G - u_m = S(m-1,n)$. By the induction hypothesis, for each vertex v in G', there exists a list of 3 colors L'_v, such that there exists a unique f' for G'. We assign the following lists for the vertices of G:

$L_{u_m} = L'_{u_{m-1}}$, $L_v = L'_v$ if $v \in V(G')$.

A unique coloring f of G exists from the assigned lists:

$f(u_m) = f'(u_{m-1})$, $f(v) = f'(v)$ if $v \in V(G')$.

Case 2: $n \geqslant 5$.

Set $G' = G - v_n = S(m,n-1)$. By the induction hypothesis, for each vertex v in G', there exists a list of 3 colors L'_v, such that there exists a unique f' for G'. We assign the following lists for the vertices of G:

$L_{v_n} = \{f'(v_{n-1}), f'(v_{n-2}), t\}$ with $t \notin f'(G')$, $L_v = L'_v$ if $v \in V(G')$.

A unique coloring f of G exists from the assigned lists:

$f(v_n) = t$, $f(v) = f'(v)$ if $v \in V(G')$. □

Corollary 1. $m(S(4,n)) = 4$ *for every* $n \geqslant 6$.

Proof. It follows from Theorem 5 and Lemma 26. □

Theorem 6. $m(S(r,n)) \leqslant 4$ *for every* $1 \leqslant r \leqslant 5$ *and* $n \geqslant 6$.

Proof. It follows from Lemma 21 to Lemma 27. □

6. Conclusion

The coloring problem and the list coloring problem are interesting topics in graph theory. Coloring graphs found application in many practical problems, for example, coding theory or security. Clearly, to estimate the chromatic as well as the chromatic uniqueness is very important. So far there have been many research results on this topic for different graph layers. However, the problem has not been generally solved, and further research is needed. This article contributes to enriching the research results on the problem of listing colors.

The main results of the paper have identified the characterized chromatically unique (Theorem 1), list-chromatic number (Theorem 2) and characterized unique list colorability (Theorem 5 and Theorem 6) of complete split graph $G = S(m, n)$. Some the property of the graph $G = S(m, n)$ when it is k-list colorable graph also proved (Theorem 3 and Theorem 4).The desire in the future will achieve deeper results on the issues raised in this article.

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Получено: 11.12.2023
Принято в печать: 28.06.2024