ЧЕБЫШЕВСКИЙ СБОРНИК
Том 25. Выпуск 2.

УДК 517.946 DOI 10.22405/2226-8383-2024-25-2-169-180

Теория рассеяния для нагруженного уравнения Кортевега—де Фриза отрицательного порядка

Г. У. Уразбоев, И. И. Балтаева, О. Б. Исмоилов

Уразбоев Гайрат Уразалиевич — доктор физико-математических наук, Ургенчский государственный университет; Институт математики им. В.И. Романовского Академии наук Республики Узбекистан (Хорезмский филиал) (г. Ургенч, Узбекистан).
email: gayrat71@mail.ru
Балтаева Ирода Исмаиловна — кандидат физико-математических наук, Ургенчский государственный университет (г. Ургенч, Узбекистан).
email: iroda-b@mail.ru
Исмоилов Охунджон Бахром оглы — Институт математики им. В.И. Романовского Академии наук Республики Узбекистан (Хорезмский филиал) (г. Ургенч, Узбекистан).
e-mail: bakhromboyevich.oxunjon@gmail.com

Аннотация

В данной работе мы рассматриваем нагруженное уравнение Кортевега–де Фриза отрицательного порядка. Определена эволюция спектральных данных оператора Штурма–Лиувилля с потенциалом, связанным с решением нагруженного уравнения Кортевега–де Фриза отрицательного порядка. Полученные результаты позволяют применить метод обратной задачи для решения нагруженного уравнения Кортевега–де Фриза отрицательного порядка в классе быстро убывающих функций. Приведен пример иллюстрирующий полученные результаты с графиками.

Ключевые слова: Оператор Штурма–Лиувилля, нагруженное уравнение, нагруженное уравнение Кортевега–де Фриза отрицательного порядка, солитонное решение, обратные задачи рассеяния.

Библиография: 23 названий.

Для цитирования:
Г. У. Уразбоев, И. И. Балтаева, О. Б. Исмоилов. Теория рассеяния для нагруженного уравнения Кортевега—де Фриза отрицательного порядка // Чебышевский сборник, 2024, т. 25, вып. 2, с. 169–180.

CHEBYSHEVSKII SBORNIK
Vol. 25. No. 2.

UDC 517.946 DOI 10.22405/2226-8383-2024-25-2-169-180

Scattering theory for the loaded negative order Korteweg–de Vries equation

G. U. Urazboev, I. I. Baltaeva, O. B. Ismoilov

Urazboev Gayrat Urazaliyevich — doctor of physical and mathematical sciences, Urgench State University; V. I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan (Khorezm Branch) (Urgench, Uzbekistan).
email: gayrat71@mail.ru
Baltaeva Iroda Ismailovna — candidate of physical and mathematical sciences, Urgench State University (Urgench, Uzbekistan).
email: iroda-b@mail.ru
Ismailov Oxunjon Bakhrom ugli — V. I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan (Khorezm Branch) (Urgench, Uzbekistan).
e-mail: bakhromboyevich.oxunjon@gmail.com

Abstract

In this paper, we consider the loaded negative order Korteweg–de Vries equation. The evolution of the spectral data of the Sturm–Liouville operator with a potential associated with the solution of the loaded negative order Korteweg–de Vries equation is determined. The obtained results make it possible to apply the inverse problem method to solve the loaded negative order Korteweg–de Vries equation in the class of rapidly decreasing functions. An example of the given problem is given with graphs of the solution.

Keywords: Sturm–Liouville operator, loaded equation, loaded negative order Korteweg–de Vries equation, soliton solution, inverse scattering problems.

Bibliography: 23 titles.

For citation:
Urazboev, G. U., Baltaeva, I. I., Ismoilov, O. B. 2024, “Scattering theory for the loaded negative order Korteweg–de Vries equation”, *Chebyshevskii sbornik*, vol. 25, no. 2, pp. 169–180.

1. Introduction

In 1967 American scientists Gardner, Green, Kruskal, and Miura [1] proposed the inverse scattering problem method for the Sturm-Liouville equation as a method for solving the Cauchy problem for the Korteveg-de Vries (KdV) equation.

$$u_t - 6uu_x + u_{xxx} = 0.$$

Shortly thereafter in 1968, Lax [2] generalized their ideas substantially. Namely, he gave the compatibility condition for linear problems a convenient operator form, presenting the compatibility condition in the form of a commutativity condition for linear differential operators and auxiliary linear problems:

$$L_t = [L, A]$$

where $[L, A] = LA - AL$ commutator of operators L and A, L - Sturm-Lioville operator

$$Ly \equiv -y'' + u(x,t)y,$$

a A - some skew-symmetric operators act in a Hilbert space. This method is called the method of the inverse scattering transform (IST) since it essentially uses the solution of the problem of reconstructing the potential of the Sturm-Liouville operator on the entire axis from scattering data (the inverse problem of scattering theory). The use of the Lax abstract form of the compatibility conditions turned out to be very useful and convenient for many questions related to nonlinear equations integrable by the inverse problem method. Thus, the universality of the method of the inverse scattering problem was shown by considering other operators instead of the operator for which the solution of the inverse scattering problem is known. Soliton equations with self-consistent sources have an important part of physical applications, for example, the KdV equation with a self-consistent source describes the interaction of long and short capillary-gravity waves [3, 4, 5, 6, 7, 8].

Most of the studies about the study of integrable equations with a self-consistent source are related to nonlinear evolution equations of positive order. The works [9, 10] are devoted to the study of the KdV equation of negative order. In particular, J. M. Verosky [9], when studying symmetries and negative powers of a recursive operator, obtained the following KdV equation of negative order:

$$\begin{cases} u_t = v_x, \\ v_{xxx} + 4uv_x + 2u_x v = 0. \end{cases}$$

S. Y. Lou [10] presented additional symmetries based on the invertible recursive operator of the KdV system, and, in particular, derived the KdV equation of negative order in the following form:

$$u_t = 2vv_x, \quad v_{xx} + uv = 0, \iff \quad \left(\frac{v_{xx}}{v}\right)_t + 2vv_x = 0.$$

The study of integrable hierarchies of negative order plays an important role in the theory of pointed solitons.The works [11, 12, 13] studied the hierarchy, the Hamiltonian structure, an infinite set of conservation laws, N-soliton, and quasi-periodic wave solutions for the negative-order KdV equation. The problem of soliton solutions for the negative order KdV equation in the class of rapidly decreasing functions was considered in [14].

In connection with intensive research on problems of optimal control of the agro-economical system, long-term forecasting, and regulating the level of ground waters and soil moisture, it has become necessary to investigate a new class of equations called “loaded equations”. Knezer [15] and Lichtenstein [16] investigated such equations for the first time. Then the term “loaded equation” was used and introduced Nakhushev in [17], where the most general definition of a loaded equation is given, various loaded equations are classified in detail, and numerous applications are described in [18, 19].

Recently works [20, 21, 22, 23] studied integration of the loaded nonlinear equations where has many applications in arterial mechanics via the (G'/G) - expansion method and Inverse scattering problem method.

This paper aims to study the integration of the loaded negative order Korteweg-de Vries equation in the "rapidly decreasing"class via the inverse scattering problem.

2. Statement of problem

We consider the following loaded negative order Korteweg-de Vries equation

$$\begin{cases} u_t = 2vv_x + \gamma(t)u(0,t)u_x, \\ v_{xx} = uv, \end{cases} \tag{1}$$

where $\gamma(t)$ is a given continuous function. Under initial conditions

$$u|_{t=0} = u_0(x), \quad x \in R, \tag{2}$$

The initial function $u_0(x)$ has the following properties:

1. $\int_{-\infty}^{\infty} (1+|x|)\,|u_0(x)|dx < \infty$.
2. The operator $L(0)y \equiv -y'' + u_0(x)y = \lambda y, x \in R^1$ has exactly N number of negative eigenvalues $\lambda_1(0),\ \lambda_2(0), ..., \lambda_N(0)$.

Let's assume that, the functions $u(x,t)$ is sufficiently smooth, and $u(x,t)$, $v(x,t)$ tends to its limits rapidly enough when $x \to \pm\infty$ and satisfying following conditions:

$$\int_{-\infty}^{\infty} \left((1+|x|)\,|u| + \left| \frac{\partial u}{\partial x} \right| \right) dx < \infty.,$$

$$v^2(x,t) \to 1,\ \ v_x(x,t) \to 0,\ \ v_{xx}(x,t) \to 0,\ \text{ in } \ |x| \to \infty. \tag{3}$$

3. Scattering problem

In this section, the dependence of the function $u(x,t)$ on t will be omitted. Consider the Sturm-Liouville equations on the axis

$$Lg \equiv -g'' + u(x)g = k^2 g, \quad -\infty < x < \infty, \tag{4}$$

with potential function $u(x)$ satisfying the condition of "rapidly decreasing"

$$\int_{-\infty}^{\infty} (1+|x|)\,|u(x)|\,dx < \infty. \tag{5}$$

This section contains information on the direct and inverse scattering problems for problem (4)-(5) which is necessary for our further exposition. Condition (5) provides that equation (4) possesses the Jost solutions $f(x,k)$ and $g(x,k)$ with the following asymptotic formulas

$$\lim_{x\to-\infty} g(x,k)e^{ikx} = 1, \quad \lim_{x\to\infty} f(x,k)e^{-ikx} = 1, \quad Imk = 0. \tag{6}$$

When k are real, the pairs $\{f(x,k), f(x,-k)\}$ and $\{g(x,k), g(x,-k)\}$ are of pairs of linearly independent solutions for equation (4).

Therefore,

$$g(x,k) = -b(-k)f(x,k) + a(k)f(x,-k). \tag{7}$$

The Jost solutions $f(x,k)$ and $g(x,k)$ admits an analytic continuation into the upper half-plane $Imk > 0$ via variable k.

The coefficients $a(k)$ and $b(k)$ have following properties:

$$a(k) = -\frac{1}{2ik} W\{f,g\}, \tag{8}$$

where $W\{f,g\} = fg' - f'g$, and for real k

$$|a(k)|^2 = 1 + |b(k)|^2. \tag{9}$$

The function $a(k)$ admits an analytic continuation into the upper half-plane $Imk > 0$ and has a finite number of simple zeroes $k_n = i\chi_n, \quad n = 1, 2, ..., N$, meanwhile, $\lambda_n = -\chi_n^2$ is an eigenvalue of L_0.

For $Imz > 0$ the function $a(z)$ recovers from its zero $i\chi_n,\ n = 1, 2, ..., N$ and $r^+(k) = \frac{b(-k)}{a(k)}-$ given function in $Imk = 0$,

$$a(z) = \prod_{j=1}^{N} \frac{z - i\chi_j}{z + i\chi_j} exp \left\{ -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{ln\left(1 - |r^+(k)|^2\right)}{k - z} dk \right\}.$$

From (7), (8) and properties the function's $a(k)$

$$g(x, i\chi_j) = B_j f(x, i\chi_j),\ j = 1, 2, ..., N. \tag{10}$$

Solutions $f(x, k), g(x, k)$ have the following representations

$$f(x, k) = e^{ikx} + \int_{x}^{\infty} A^+(x, k) e^{ikz} dz,$$

$$g(x, k) = e^{-ikx} + \int_{-\infty}^{x} A^-(x, k) e^{-ikz} dz, \tag{11}$$

where the kernels $A^+(x, z), A^-(x, z)$ are real functions and connected with the potential function $u(x)$ by the equalities

$$u(x) = -2\frac{d}{dx}A^+(x, x), \quad u(x) = 2\frac{d}{dx}A^-(x, x). \tag{12}$$

The kernel $A^+(x, y)$ in representation (11) is a solution of the Gelfand-Levitan-Marchenko integral equation

$$\Omega^+(x + y) + A^+(x, y) + \int_{x}^{\infty} A^+(x, z)\Omega^+(z + y) dz = 0, \quad (y > x), \tag{13}$$

where

$$\Omega^+(x) = -\sum_{j=1}^{N} \frac{iB_j}{\frac{da(z)}{dz}|_{z=i\chi_j}} exp(-\chi_j x) - \frac{1}{2\pi} \int_{-\infty}^{\infty} r^+(k) e^{ikx} dx,$$

a $a(z)-$ analytic continuation of the function $a(k), (Imk = 0)$ into the upper half plane.

Then, the potential function $u(x)$ is determined from the equality (12).

The set $\{r^+(k), B_1, B_2, ..., B_N, \chi_1, \chi_2, ..., \chi_n\}$ is called scattering data for the problem (4)-(5). The direct problem consists of determining scattering data by the potential function $u(x)$, and the inverse problem consists in recovering the potential function $u(x)$ of the equation (4) by the scattering data.

It's easy to check that, the functions

$$h_n(x) = \frac{\frac{d}{dk}\left(g(x, k) - B_n f(x, k)\right)|_{k=i\chi_n}}{\dot{a}(i\chi_n)} \tag{14}$$

are solutions of the equations $L_0 y = -\chi_n^2 y$. By equality (14), we obtain the following asymptotic expressions

$$h_n \sim e^{\chi_n x} \ \text{ in } \ x \to \infty, \tag{15}$$

$$h_n \sim -B_n e^{-\chi_n x} \ \text{ in } \ x \to -\infty. \tag{16}$$

Using (15) and (16) we obtained

$$W\left\{h_n(x), f(x, i\chi_n)\right\} = -2\chi_n,$$

$$W\left\{h_n(x), g(x, i\chi_n)\right\} = -2B_n\chi_n, \quad n = 1, 2, ..., N. \tag{17}$$

4. Evolution of scattering data

In this section, we will consider the system of equations:

$$u_t = 2vv_x + G, \quad v_{xx} = uv, \tag{18}$$

where $G(x,t)-$ sufficiently smooth function for any non-negative t satisfying the conditions

$$G(x,t) = o(1) \text{ in } x \to \pm\infty.$$

Equation (18) is considered with initial condition (2). Similar to the work [7] we can bring the following main lemma:

Lemma 6. *If the potential of the operator* $L(t) = -\frac{d^2}{dx^2} + u(x,t)$ *are solutions of problem* (18)-(2) *in the class of functions satisfying conditions (3), then the scattering data of the operator* $L(t)$ *changes over* t *as follows:*

$$\frac{\partial r^+}{\partial t} = -\frac{i}{k}r^+ - \frac{1}{2ika^2(k)}\int_{-\infty}^{\infty} Gg^2dx, \; Imk = 0,$$

$$\frac{dB_n}{dt} = -\frac{B_n}{\chi_n} - \frac{1}{2\chi_n}\int_{-\infty}^{\infty} Gg(x, i\chi_n, t)h_n(x,t)dx,$$

$$\frac{d\chi_n}{dt} = -\frac{1}{2\chi_n}\int_{-\infty}^{\infty} G\Phi_n^2(x,t)dx. \;\; n = 1, 2, 3, ..., N.$$

where $\Phi_n(x,t)$ *is the normalized eigenfunction of the operator* $L(t)$ *corresponding to the eigenvalue* $\lambda_n = -\chi_n^2(t)$.

We apply the result of Lemma 1 for

$$G(x,t) \equiv \gamma(t)u(0,t)u_x. \tag{19}$$

Let's find the evolution of the eigenvalues of the operator $L(t)$:

$$\frac{\partial r^+}{\partial t} = -\frac{i}{k}r^+ - \frac{\gamma(t)u(0,t)}{2ika^2}\int_{-\infty}^{\infty} u_x g^2 dx,$$

$$\int_{-\infty}^{\infty} u_x g^2 dx = ug^2 \mid_{-\infty}^{\infty} -2\int_{-\infty}^{\infty} ugg'dx = -2\int_{-\infty}^{\infty}(k^2g + g'')g'dx =$$

$$= -\lim_{R\to\infty}\int_{-R}^{R}[k^2(g^2)' + ((g')^2)']dx = 4k^2a(k)b(-k).$$

Consequently, for $Imk = 0$ we get

$$\frac{\partial r^+}{\partial t} = \left(-\frac{i}{k} + 2ik\gamma(t)u(0,t)\right)r^+. \tag{20}$$

Let us apply Lemma 1 for

$$\frac{d\chi_n}{dt} = -\frac{\gamma(t)u(0,t)}{2\chi_n}\int_{-\infty}^{\infty} u_x\phi_n^2 dx,$$

according to the following calculation

$$\int_{-\infty}^{\infty} u_x\phi_n^2dx = -2\int_{-\infty}^{\infty} u\phi_n\phi_n'dx = -2\int_{-\infty}^{\infty}(\chi_n^2\phi_n+\phi'')\phi_n'dx =$$

$$= -2\int_{-\infty}^{\infty}(-\chi_n^2(\phi_n^2)' + (\phi_n'^2)')dx = 0.$$

Hence,

$$\frac{d\chi_n}{dt} = 0, \quad n = 1,2,3,...,N. \tag{21}$$

According to the representation (11) and by virtue of asymptotic formulas (16), (17) we have:

$$\int_{-\infty}^{\infty} u_x g_n h_n dx = -\int_{-\infty}^{\infty} u(g_n'h_n + g_nh_n')dx = -\int_{-\infty}^{\infty}(g'(h_n''+\lambda_nh_n)+$$

$$+h_n'(g_n''+\lambda_ng_n))dx = -\int_{-\infty}^{\infty}((g'h_n')' + \lambda_n(g_nh_n)')dx = 4\chi_n^2B_n.$$

With a glance at this calculation, we get

$$\frac{dB_n}{dt} = -\left(\frac{1}{\chi_n} + 2\chi_n\gamma(t)u(0,t)\right)B_n, n = 1,2,3,...,N. \tag{22}$$

By using the obtained equalities (20), (21), and (22), we have to infer the following theorem.

Theorem 1. *If the functions $u(x,t), v(x,t)$ are a solution to the problem (1)-(3), then the scattering data of the operator $L(t) \equiv -\frac{d^2}{dx^2} + u(x,t)$ change in t as follows*

$$\frac{\partial r^+}{\partial t} = \left(-\frac{i}{k} + 2ik\gamma(t)u(0,t)\right)r^+, \quad Imk = 0,$$

$$\frac{dB_n}{dt} = -\left(\frac{1}{\chi_n} + 2\chi_n\gamma(t)u(0,t)\right)B_n,$$

$$\frac{d\chi_n}{dt} = 0, \ n = 1,2,...,N.$$

We note that for the nKdV equation, this result was obtained in [14]. The resulting equality completely determines the evolution of the scattering data, which makes it possible to apply the inverse problem method for solving problems (1)-(3).

4.1. Example

We will solve the following Cauchy‘s problem:

$$u_t = 2vv_x + \gamma(t)u(0,t)u_x, \ \ v_{xx} = uv,$$

$$u|_{t=0} = -\frac{2}{\cosh^2 x},$$

where

$$\gamma(t) = \frac{1}{4}\sqrt{1+t^2}(\sqrt{1+t^2}-2).$$

For finding the general solution to this problem we use the inverse scattering problem method. First of all, we find a solution of the Direct Problem for the following equation:

$$-y'' - \frac{2}{\cosh^2 x}y = k^2 y, -\infty < x < \infty,$$

We find the Jost solutions

$$f(x,k) = \frac{ik - thx}{ik - 1}e^{ikx},$$

$$g(x,k) = \frac{ik + thx}{ik - 1}e^{-ikx}.$$

According to equalities (8) and (9)

$$a(k) = -\frac{1}{2ik}W\left\{f(x,k)g(x,k)\right\} = \frac{k-i}{k+i}, \quad b(k) = 0.$$

Since the function $a(k)$ has only one zero $k = i$ to $\chi_1 = 1,\ N = 1$. In addition,

$$f(x,i) = \frac{1 + thx}{2}e^{-x},\ g(x,i) = \frac{1 - thx}{2}e^{x}.$$

Thus

$$B_1 = \frac{g(x,i)}{f(x,i)} = 1.$$

As a result, we obtain the following Scattering Data

$$N = 1,\ a(k) = \frac{k-i}{k+i},\ r^+(k,0) = 0,\ B_1(0) = 1,\ \chi_1(0) = 1.$$

Using Theorem 1 we find the evolution of Scattering Data depending on t:

$$r^+(k,t) = 0,\ B_1(t) = \exp^{2\delta(t)},\ \chi_1(t) = 1,$$

here

$$\delta(t) = -\frac{t}{2} - \int_0^t \gamma(t)u(0,t)dt.$$

We find a solution to the Inverse scattering problem using this Scattering Data. The solution of considering equation is defined by the following formulae:

$$A^+(x,y;t) = -\frac{2\exp^{-x-y+2\delta(t)}}{1 + \exp^{-2x+2\delta(t)}},$$

As a result, from (12) the general solution $u(x,t)$ and $v(x,t)$ of the considering problem are expressed as follows:

$$u(x,t) = -\frac{2}{\cosh^2(x + t - arcsinh(t))}, v(x,t) = \tanh\left(x + t - arcsinh(t)\right).$$

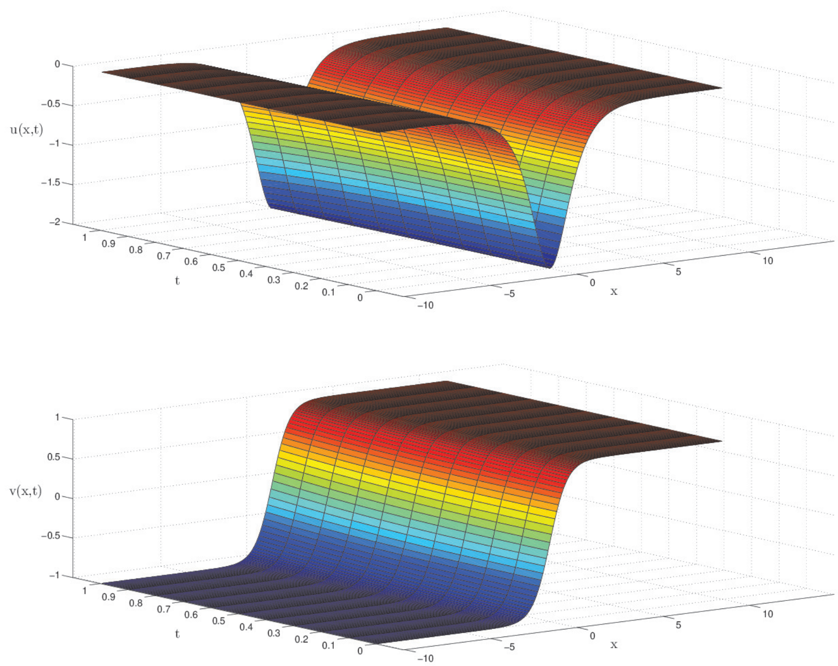

Рис. 1: Soliton solution of the loaded negative order KdV equation

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Gardner C. S., Greene J. M., Kruskal M. D., Miura R. M. Method for Solving Korteweg-Devries Equation // Phys. Rev. Lett. 1967. Vol. 19. P. 1095–1097.

2. Lax P. D. Integrals of Nonlinear Equations of Evolution and Solitary Waves // Communications on Pure and Applied Mathematics. 1968. Vol. 21. P. 467–490.

3. Leon J., Latifi A. Solution of an initial-boundary value problem for coupled nonlinear waves // J. Phys. A. 1990. Vol. 23. P. 1385–1403.

4. Mel'nikov V. K. Exact solutions of the Korteweg-de Vries equation with a self-consistent source // Phys. Lett. 1988. Vol. 128. P. 488–492.

5. Mel'nikov V. K. Integration of the Korteweg-de Vries equation with a source // Inverse Problem. 1990. Vol. 6. P. 233–246.

6. Khasanov A. B., Khasanov T. G. Integration of a Nonlinear Korteweg–de Vries Equation with a Loaded Term and a Source // J. Appl. Ind. Math. 2022. Vol. 16. P. 227–239.

7. Khasanov A. B., Urazboev G.U. Integration of the sine-Gordon equation with a self-consistent source of the integral type in the case of multiple eigenvalues // Russ Math. 2009. Vol. 53. P. 45–55.

8. Urazboev G. U., Hasanov M.M. Integration of the negative order Korteweg-de Vries equation with a self-consistent source in the class of periodic functions // Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki. 2022. Vol. 32. P. 228–239.

9. Verosky J. M. Negative powers of Olver recursion operators // J. Math. Phys. 1991. Vol. 32. P. 1733–1736.

10. Lou S. Y. Symmetries of the KdV equation and four hierarchies of the integrodifferential KdV equation // J. Math. Phys. 1994. Vol. 35. P. 2390–2396.

11. Qiao Z., Li J. Negative-order KdV equation with both solitons and kink wave solutions // Europhys. Lett. 2011. Vol. 94. P. 50003.

12. Zhijun Q., Engui F. Negative-order Korteweg–de Vries equations // Phys. Rev. E. 2012. Vol. 86. P. 016601.

13. Rodriguez M., Li J., Qiao Z. Negative Order KdV Equation with No Solitary Traveling Waves // Mathematics. 2022. Vol. 10. P. 48–58.

14. Urazboev G. U., Baltaeva I. I., Ismoilov O.B. Integration of the negative order Korteweg–de Vries equation by the inverse scattering method // Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki. 2023. Vol. 33. P. 523–533.

15. Kneser A. Belastete Integralgleihungen // Rendiconti del Circolo Mathematiko di Palermo. 1914. Vol. 38. P. 169–197.

16. Lichtenstein L. Vorlesungen uber einege Klassen nichtlinear Integral gleichungen und Integral differential gleihungen nebst // Anwendungen. Berlin: Springer. 1931.

17. Nakhushev A. M. The Darboux problem for a certain degenerate second order loaded integrodifferential equation // Differential Equations. 1976. Vol. 12. P. 103–108.

18. Kozhanov A. I. Nonlinear loaded equations and inverse problems // Zh. Vychisl. Mat. Mat. Fiz. 2004. Vol. 44. P. 1095–1097.

19. Baltaeva U., Baltaeva I., Agarwal P. Cauchy problem for a high-order loaded integro-differential equation // Mathematical Methods in the Applied Sciences. 2022. Vol. 45. P. 8115–8124.

20. Urazboev G. U., Baltaeva I. I. Integration of Camassa-Holm equation with a self-consistent source of integral type // Ufa Mathematical Journal. 2022. Vol. 14, pp. 77–86.

21. Yakhshimuratov A. B., Matyokubov M. M. Integration of a Loaded Korteweg-de Vries Equation in a Class of Periodic Functions // Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika. 2016. Vol. 69. P. 87–92

22. Urazboev G. U., Baltaeva I. I., Rakhimov I. D. The Generalized (G'/G)-Expansion Method for the Loaded Korteweg–de Vries Equation // Journal of Applied and Industrial Mathematics. 2021. Vol. 15. P. 679–685

23. Feckan M., Urazboev G., Baltaeva I. Inverse scattering and loaded Modified Korteweg-de Vries equation // Journal of Siberian Federal University. Mathematics and Physics. 2022. Vol. 15. P. 176–185.

REFERENCES

1. Gardner, C. S., Greene, J. M., Kruskal, M. D. & Miura, R. M., 1967. "Method for Solving Korteweg-Devries Equation", *Phys. Rev. Lett.*, vol. 19, pp. 1095–1097.

2. Lax, P. D., 1968. "Integrals of Nonlinear Equations of Evolution and Solitary Waves", *Communications on Pure and Applied Mathematics.*, vol. 21, pp. 467–490.

3. Leon, J. & Latifi, A., 1990. "Solution of an initial-boundary value problem for coupled nonlinear waves", *J. Phys. A.*, vol. 23, pp. 1385–1403.

4. Mel'nikov, V. K., 1988. "Exact solutions of the Korteweg-de Vries equation with a self-consistent source", *Phys. Lett.*, vol. 128, pp. 488–492.

5. Mel'nikov, V. K., 1990. "Integration of the Korteweg-de Vries equation with a source", *Inverse Problem.*, vol. 6, pp. 233–246.

6. Khasanov, A. B. & Khasanov, T. G., 2022. "Integration of a Nonlinear Korteweg–de Vries Equation with a Loaded Term and a Source", *J. Appl. Ind. Math.*, vol. 16, pp. 227–239.

7. Khasanov, A. B. & Urazboev, G. U., 2009. "Integration of the sine-Gordon equation with a self-consistent source of the integral type in the case of multiple eigenvalues", *Russ Math.*, vol. 53, pp. 45–55.

8. Urazboev, G. U. & Hasanov, M.M., 2022. "Integration of the negative order Korteweg-de Vries equation with a self-consistent source in the class of periodic functions", *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki.*, vol. 32, pp. 228–239.

9. Verosky, J. M., 1991. "Negative powers of Olver recursion operators", *J. Math. Phys.*, vol. 32, pp. 1733–1736.

10. Lou, S. Y., 1994. "Symmetries of the KdV equation and four hierarchies of the integrodifferential KdV equation", *J. Math. Phys.*, vol. 35, pp. 2390–2396.

11. Qiao, Z. & Li, J., 2011. "Negative-order KdV equation with both solitons and kink wave solutions", *Europhys. Lett.*, vol. 94, pp. 50003.

12. Zhijun, Q. & Engui, F., 2012. "Negative-order Korteweg–de Vries equations", *Phys. Rev. E.*, vol. 86, pp. 016601.

13. Rodriguez, M., Li, J. & Qiao, Z., 2022. "Negative Order KdV Equation with No Solitary Traveling Waves", *Mathematics.*, vol. 10, pp. 48–58.

14. Urazboev, G. U., Baltaeva, I. I. & Ismoilov, O.B., 2023. "Integration of the negative order Korteweg–de Vries equation by the inverse scattering method", *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki.*, vol. 33, pp. 523–533.

15. Kneser, A., 1914. "Belastete Integralgleihungen", *Rendiconti del Circolo Mathematiko di Palermo.*, vol. 38, pp. 169–197.

16. Lichtenstein, L., 1931. "Vorlesungen uber einege Klassen nichtlinear Integral gleichungen und Integral differential gleihungen nebst", *Anwendungen. Berlin: Springer.*

17. Nakhushev, A. M., 1976. "The Darboux problem for a certain degenerate second order loaded integrodifferential equation", *Differential Equations.*, vol. 12, pp. 103–108.

18. Kozhanov, A. I., 2004. "Nonlinear loaded equations and inverse problems", *Zh. Vychisl. Mat. Mat. Fiz.*, vol. 44, pp. 1095–1097.

19. Baltaeva, U., Baltaeva, I. & Agarwal, P., 2022. "Cauchy problem for a high-order loaded integro-differential equation", *Mathematical Methods in the Applied Sciences*, vol. 45, pp. 8115–8124.

20. Urazboev, G. U., Baltaeva, I. I., 2022. "Integration of Camassa-Holm equation with a self-consistent source of integral type", *Ufa Mathematical Journal*, vol. 14, pp. 77–86.

21. Yakhshimuratov, A. B. & Matyokubov, M. M., 2016. "Integration of a Loaded Korteweg-de Vries Equation in a Class of Periodic Functions", *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika.*, vol. 69, pp. 87–92

22. Urazboev, G. U., Baltaeva, I. I. & Rakhimov, I. D., 2021. "The Generalized (G'/G)-Expansion Method for the Loaded Korteweg–de Vries Equation", *Journal of Applied and Industrial Mathematics.*, vol. 15, pp. 679–685

23. Feckan, M., Urazboev, G. & Baltaeva, I., 2022. "Inverse scattering and loaded Modified Korteweg-de Vries equation", *Journal of Siberian Federal University. Mathematics and Physics.*, vol. 15, pp. 176–185.

Получено: 21.03.2024
Принято в печать: 28.06.2024