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# О цепных дробях с рациональными неполными частными[1]

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**Аннотация**

Алгоритм Соренсона с левым сдвигом – один из быстрых алгоритмов вычисения наибольшего общего делителя двух натуральных чисел. В начале его работы фиксируется натуральное число $k > 2$, которое является параметром. На каждом шаге алгоритма выполняется поиск линейной комбинации входных чисел текущего шага, причем наименьшее из них предварительно домножается на параметр $k$, пока не начнет превосходить наибольшего. После этого наибольшее число замещается абсолютным значением линейной комбинации. Результатом работы алгоритма является наибольший общий делитель исходных чисел, умноженный на некоторое число, называемое побочным множителем. Для алгоритма Соренсона была доказана оценка числа шагов в худшем случае, приведен пример. Фиксация некоторой бесконечной последовательности $K$ натуральных чисел больших двух позволяет получить обобщенный алгоритм Соренсона. В нем на каждом шаге вместо числа $k$ будет задействовано определенное значение параметра $k_i \in K$, соответствующее текущему шагу алгоритма. В остальном алгоритмы полностью совпадают друг с другом.

Цепные дроби с рациональными неполными частными с левым сдвигом возникают в ходе применения к отношению натуральных чисел $a$, $b$ обобщенного $k$-арного алгоритма Соренсона с левым сдвигом. С ними связаны особые формы континуантов, то есть многочленов, при помощи которых выражаются числитель и знаменатель подходящей дроби. Для таких континуантов найдены формулы, позволяющие представить континуант $n$-го порядка в виде некоторой комбинации континуантов меньших порядков. Были найдены условия при которых последовательность континуантов увеличивающегося порядка является строго возрастающей. Также были найдены условия, при которых приближения рациональных чисел, выполненные при помощи цепных дробей с рациональными неполными частными, можно однозначно сравнивать.

*Ключевые слова:* цепные дроби, континуант, наибольший общий делитель, Диофантовы приближения.

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## On the continued fraction with rational partial quotients[2]

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**Abstract**

The Sorenson left shift $k$-ary gcd algorithm is one of the fastest greatest common divisor algorithms of two natural numbers. At the beginning a natural number $k > 2$ is fixed, which is a parameter of algorithm. At each step we multiply smaller of two input numbers of current step, until it does not become greater of the second number. Then we calculate linear combination between this number and the bigger of two input numbers. After that we replace the bigger of two input numbers by absolute value of the linear combination. At the end of the algorithm we obtain greatest common divisor of the two original numbers, which has been multiplied by some natural number. Spurious factor has appeared in the answer. We have proven estimation of the worst case of steps and obtained example of this case. Fixation of some endless sequence $K$ of natural numbers (each value is greater than 2) allows us to obtain the generalized Sorenson left shift $k$-ary gcd algorithm. There at $i$-th step the value of $k_i \in K$ is used instead of fixed parameter $k$. Both algorithms are completely coincide except this moment.

Continued fractions with rational partial quotients with left shift arise at applying of the generalized Sorenson left shift $k$-ary gcd algorithm to the ratio of two natural numbers $a$ and $b$. We can bind these continued fractions and polynomials of the special form, which called continuants. Numerator and denominator of such continued fractions can be expressed by continuants. Formulas have been found that allow us to express continuants of the $n$-th order as some combination of continuants of a smaller order. Conditions were found at which a sequence of continuants of increasing order is strictly increasing. We also found conditions that allow unambiguous comparison of convergents of rational numbers that had performed by using continued fractions with rational partial quotients.

*Keywords:* continued fraction, continuant, greatest common divisor, Diophantine approximation.

*Bibliography:* 17 titles.

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# 1. Introduction

The Euclidean algorithm is one of the most famous algorithms for calculating the greatest common divisor (gcd) of two natural numbers [3] $a$, $b$ (here and further $a > b > 1$). At each step

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[3]Natural numbers are the non-negative integers without zero.

input number $a$ is replaced by input number $b$, input number $b$ is replaced by smallest non-negative remainder $r$ from division of $a$ by $b$:

$$a = bt + r,\ t = \lfloor a/b \rfloor,\ 0 \leqslant r < t.$$

The algorithm runs until the second argument vanishes. Then first argument is equals to $\gcd(a, b)$. The classical Euclidean algorithm corresponds to the expansion of the number $a/b$ into a (regular) continued fraction

$$\frac{a}{b} = t_0 + \cfrac{1}{t_1 + \cfrac{1}{\ddots + \cfrac{1}{t_h}}}$$

of the length $h = h(a/b)$, where $t_0$ is integer, and the numbers $t_1, \cdots, t_h$ are natural, $t_i \geqslant 2$, $i \geqslant 1$. The integers $t_0$, $t_1$, ..., $t_h$ are called partial quotients. Note that continued fractions are related to many other mathematical objects (refer to [1], [2]).

In addition to the Euclidean algorithm, there are other algorithms that calculate gcd of two natural numbers. Among them, it is worth noting $k$-ary algorithms first introduced by Sorenson: the right-shift algorithm and the left-shift algorithm (refer to [3]). They quickly calculate the gcd, which is used in various mathematical algorithms (refer to [4, 5, 6]). Modification of the first algorithm allows us to increase its performance (refer to [7, 8]), calculate multiplicatively inverse elements in the ring of integers modulo a number (refer to [9]), and also allows to get rid of spurious factors that arise during execution of algorithm (refer to [10]).

In what follows we will need the Sorenson $k$-ary left-shift gcd algorithm. Here is its description. Let us fix some integer $k > 2$ and set $a_0 = a$, $b_0 = b$. At each step of this algorithm a pair of input numbers $(a_i, b_i)$ is replaced by new pair $(a_{i+1}, b_{i+1})$ by the next rule. First we find the integer $e_i$ from the following relation:

$$c_i \leqslant a_i < kc_i, \tag{1}$$

where $c_i = b_i k^{e_i}$. Further integers $x_i$ and $y_i$ are selected that satisfy the conditions

$$\gcd(x_i, y_i) = 1, 0 < y_i \leqslant k, \tag{2}$$

for which the inequality holds

$$\left| \frac{c_i}{a_i} - \frac{x_i}{y_i} \right| \leqslant \frac{1}{y_i(k+1)}. \tag{3}$$

After that we select a new pair of numbers

$$(a_{i+1}, b_{i+1}) = (b_i, |y_i c_i - x_i a_i|), \text{if } i \geqslant 0, \tag{4}$$

or

$$(a_{i+1}, b_{i+1}) = (b_{i+1}, a_{i+1}), \text{if } b_{i+1} > a_{i+1}. \tag{5}$$

We will assume that the number $x_i$ belongs to the interval $(0, k]$ because all other cases of choosing this number do not satisfy inequality (3). The choice of numbers $x_i$, $y_i$ is carried out by enumerating possible variants and checking the feasibility of this inequality. The existence of such numbers is guaranteed by Dirichlet's lemma on Diophantine approximations (refer to [11, chapter X lemma 2]).

The algorithm terminates when one of the arguments $a_i$, $b_i$ vanishes: in this case, the second argument becomes the answer. However, during algorithm execution at the $i$-th step a "spurious"

**Algorithm 1** Main loop of the left-shift $k$-ary gcd

**if** $a < b$ **then**
  $swap(a, b)$
**end if**
**while** $b \neq 0$ **do**
  compute $c = k^e b$ such that $c \leqslant a < ck$
  find x,y $\leqslant k$ such that $c/a \approx x/y$
  $a = |yc - xa|$
  **if** $a < b$ **then**
    $swap(a, b)$
  **end if**
**end while**

factor $\alpha_i = \gcd(b_i, x_i)$ may appear, therefore, the result of the Sorenson algorithm will be a certain number that differs from the $\gcd(a, b)$ by the factor $\prod_{i=1}^{n} \alpha_i$. In Sorenson's paper, this factor was removed using a special phase of the algorithm, performed after the main loop, called "trivial division". It consists of searching through possible divisors of the answer among all prime numbers from 2 to $k$ and then removing them. This phase was also performed at the beginning of the algorithm in order to find small common divisors of the input numbers. Subsequently, they were stored as a product by which each input number $a_0$, $b_0$ was divided. At the end, the saved product was added to the answer obtained as a result of running the "trivial division" phase again. This made it possible to save small common divisors of the input numbers because they could have been deleted regardless of the presence of a spurious factor. Here, usage of this phase of the algorithm is omitted, as well as precomputation of some numbers, which is used during the performing algorithm.

Instead of the precomputation phase, you can take the answer $a_n$ obtained during the main loop in $n$ steps and find gcd using the following scheme (this idea was proposed in modification of the Sorenson right shift $k$-ary gcd algorithm (refer to [12])):

$$\gcd(\gcd(a_0, a_n), b_0),$$

besides these two gcd calculation are performed using algorithms, in which there are no any spurious factors. For example, it is Euclidean algorithm or binary algorithm (refer to [13]).

Below is an example of how the algorithm works. Let us fix parameter $k$ to 7, input numbers $a_0$ and $b_0$ equals to 4415 and 60, respectively. At the zero step, first we find the value of $e_0$. It equals to 2. After that we select the values of the numbers $x_0$, $y_0$, so that inequality (3) is performed. Let us set them to 2 and 3, respectively. Further, we calculate a new pair of numbers $a_1$, $b_1$ according to rules (4), (5). It equals to $(60, 10)$. It is easy to see that the $\gcd(a_1, b_1)$ is equals to 10, although gcd of the input numbers $a_0$ and $b_0$ is equals to 5. Result does not match, due to the fact that at this step of the algorithm a spurious factor $\gcd(b_0, x_0)$ equals to 2 is appeared. The next step also begins with searching value of $e_1$. It equals to 0. The numbers $x_1 = 1$, $y_1 = 6$ satisfy inequality (3). At the end of this step search for a new pair of numbers $(a_2, b_2)$ is performed again. Value of $b_2$ is equals to 0, so the number $a_2$, which equals to 10, will be the answer after division by the spurious factor, which equals to 2.

Sorenson showed that for pairs of numbers $(a, b)$, that are chosen according to the rules

$$a = \sum_{i=0}^{m} k^{2i}, b = 1, \tag{6}$$

the number of steps is equal to $m + 1$, and for the number $m$ the next estimate will be fair

$$m = \Omega(\log(ab)/\log(k)). \tag{7}$$

Estimate (7) has been proven only for numbers of type (6). Subsequently, for two numbers having $n$ binary digits, Sorenson proved the asymptotic of the number of steps in the worst case equal to $\Theta(n/\log(k))v$ using estimate (7). No worst-case examples of the algorithm work were given. Also, the question of the constant in the estimate bounded from above remained open until now. In paragraph 4, we give an example of the worst case of the algorithm, and prove the following result:

Theorem 1. *For arbitrary integers $a \geqslant b > 1$ the main loop of the Sorenson left-shift $k$-ary algorithm calculates gcd in no more than $\lfloor \log(a)/\log(k) \rfloor + \lfloor \log(b)/\log(k) \rfloor + 1$ steps.*

Using the Sorenson right-shift $k$-ary gcd algorithm, we can obtain continued fractions with rational partial quotients with right shift. For brevity, we call such fractions as continued fractions of the first kind. There are two main types of such fractions: continued fractions of the first type

$$\frac{y_0\gamma_0}{x_0\beta_0} + \cfrac{k_0}{\left(\cfrac{y_1 x_0\beta_0\gamma_1}{\gamma_0 x_1\beta_1} + \cfrac{k_1}{\left(\ddots + \cfrac{k_{n-1}}{\cfrac{y_n \prod\limits_{\substack{0\leqslant i<n,\\ i\not\equiv n \pmod 2}} x_i\beta_i \prod\limits_{\substack{0\leqslant t\leqslant n,\\ t\equiv n \pmod 2}} \gamma_t}{\prod\limits_{\substack{0\leqslant j\leqslant n,\\ j\equiv n \pmod 2}} x_j\beta_j \prod\limits_{\substack{0\leqslant m<n,\\ m\not\equiv n \pmod 2}} \gamma_m}}\right)}\right)},$$

and continued fractions of the second type

$$\frac{y_0\gamma_0}{x_0\beta_0} + \cfrac{k_0\gamma_0}{\left(\cfrac{x_0\beta_0}{\left(\cfrac{y_1\gamma_1}{x_1\beta_1} + \cfrac{k_1\gamma_1}{\ddots + \cfrac{k_{n-1}\gamma_{n-1}x_n\beta_n}{x_{n-1}\beta_{n-1}y_n\gamma_n}}\right)}\right)}.$$

The numerators and denominators of such fractions can be expressed using polynomials of a special kind called continuants. Previously, the properties of extreme values of such continuants with restrictions on the variables were studied, and a construction similar to the triangle of Fibonacci polynomials was obtained (refer to [14]).

This paper is introduced a generalized Sorenson left-shift $k$-ary gcd algorithm, finite continued fractions with rational partial quotients with left shift and corresponding continuants. We have obtained formulas expansion of continuants. We give conditions, under which it is possible to construct strictly increasing sequences of continuants, convergents of rational numbers. Also in this paper, we present an accurate estimate of the number of steps of the Sorenson left-shift $k$-ary gcd algorithm in the worst case.

## 2. The Generalized Sorenson algorithm

Consider the following generalized Sorenson algorithm. Instead of the number $k$, some infinite sequence of numbers $K = \{k_i\}_{i=0}^{\infty}$ is fixed, consisting of natural numbers $k_i \geqslant 2$. At the next step of the algorithm, a pair of numbers $(a_{i+1}, b_{i+1})$ is constructed using formulas that are obtained from (1) – (5) by replacing $k$ to $k_i$. If we multiply each half of inequality (3) by the number $y_i$, come to the one denominator on the left side, and then flip each parts of the inequality, then we get expression $a_i/|y_i c_i - x_i a_i| \geqslant (k_i+1)$. This fact ensures convergence of the generalized Sorenson algorithm to the solution.

Let $\overline{k} = \min(k_i)$, $k_i \in K$. Then the number of steps of the generalized Sorenson algorithm does not exceed the magnitude $\lfloor \log(a)/\log(\overline{k}) \rfloor + \lfloor \log(b)/\log(\overline{k}) \rfloor + 1$. The author suggests that this estimate, made by analogy with theorem 1, can be improved.

## 3. Finite continued fractions with rational partial quotients

The generalized Sorenson algorithm leads to a new expansion of the number $a/b$ into a (regular) continued fraction with rational partial quotients with a left shift [4], which for brevity we will call *continued fraction of the second kind.* Denote by $g_i$ the four numbers $(y_i, x_i, k_i, e_i)$. The number $y_0$ is integer, and $x_0$, $x_i$, $y_i$ are non-zero integers when $i$ is greater than one.

There are two main types of expansion of the number $a/b$ into a continued fraction of the second kind. For brevity, we will call them expansions of the third and fourth types. The third type continued fraction has the following form

$$\frac{y_0}{x_0}{k_0}^{e_0} + \cfrac{\delta_0}{\left(\frac{x_0 y_1}{x_1}{k_1}^{e_1} + \cfrac{\delta_1}{\left(\ddots + \cfrac{y_n {k_n}^{e_n} \prod\limits_{\substack{i<n,\\ i \not\equiv n \pmod 2}} x_i}{x_n \prod\limits_{\substack{j<n,\\ j \equiv n \pmod 2}} x_j}\right)}\right)} = [g_0; g_1 ..., g_n]_3\,, \tag{8}$$

where the value $\delta_i = \delta_i(a_i, c_i, x_i, y_i)$ is defined as

$$\delta_i = \begin{cases} -1, & \text{if } c_i y_i - x_i a_i \geqslant 0. \\ 1, & \text{if } c_i y_i - x_i a_i < 0. \end{cases} \tag{9}$$

Theorem 2. *Let the finite sequences of numbers $\{x_i\}_{i=0}^n$, $\{y_i\}_{i=0}^n$ have been obtained by applying the generalized Sorenson left-shift $k$-ary gcd algorithm for the input numbers $a$, $b$ and a pre-fixed infinite sequence $K$ of natural numbers greater than two. If the following inequalities are true for each $i$*

$$a_i > b_i > |y_i c_i - x_i a_i|, \tag{10}$$

*then the number $a/b$ can be represented as a third type continued fraction.*

The proof of the theorem and all following results are given in a separate section.

If continued fractions have been contained a large number of elements, and besides that each element needs to be shown, then it is not always convenient to represent such fractions with formula (8). In such cases it is convenient to use an alternative notation of the continued fraction, so that the sum of the elements will be written in a line:

$$\frac{y_0 {k_0}^{e_0}}{x_0} + \frac{\delta_0}{\dfrac{x_0 y_1 {k_1}^{e_1}}{x_1} +} \; \frac{\delta_1}{\dfrac{x_1 y_2 {k_2}^{e_2}}{x_2 x_0} +} \; \frac{\delta_2}{\dfrac{x_0 x_2 y_3 {k_3}^{e_3}}{x_3 x_1} +} \cdots$$

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[4]The left shift in the name is a reference to the Sorenson left-shift $k$-ary gcd algorithm, which is used when decomposing the number $a/b$. If we select $k = 2^s$, then multiplication by $k$ is the same as performing a bitwise left shift operation by $s$ positions (refer to [15]). This analogy is reflected in the name of the algorithm.

The fourth type continued fraction has the following form

$$\sum_{i=0}^{n-1} \frac{y_i {k_i}^{e_i}}{x_i} \prod_{j=0}^{i-1} \frac{\delta_j}{x_j} + \frac{x_n}{y_n {k_n}^{e_n}} \prod_{i=0}^{n-1} \frac{\delta_i}{x_i} = [g_0; g_1, ..., g_n]_4 \,, \tag{11}$$

where the value $\delta_i$ is defined as in theorem 2.

THEOREM 3. *Let the finite sequences of numbers $\{x_i\}_{i=0}^{n}$, $\{y_i\}_{i=0}^{n}$ have been obtained by applying the generalized Sorenson left-shift k-ary gcd algorithm for the input numbers $a$, $b$ and a pre-fixed infinite sequence $K$ of natural numbers greater than two. If the following inequalities are true for each $j$ under the condition $0 \leqslant j \leqslant n-1$*

$$b_j \leqslant |y_j c_j - x_j a_j| < a_j, \tag{12}$$

*and for $j$ equal to $n$, conditions (10) are satisfied, then the number $a/b$ can be represented as a fourth type continued fraction.*

The indices "3" and "4" after the square brackets to the right of the continued fraction in theorems 2, 3 mean the third and fourth types of expansion into continued fractions of the second kind. Note that at the last $n$-th step of the algorithm at expansion into a fourth type continued fraction formula (10) will be fulfilled instead of conditions (12).

The fourth type continued fraction has another form of notation:

$$\frac{y_0}{x_0}{k_0}^{e_0} + \cfrac{\delta_0}{\cfrac{x_0}{\left(\frac{y_1}{x_1}{k_1}^{e_1} + \cfrac{\delta_1}{\cfrac{x_1}{\left(\cfrac{\cdots}{\left(\frac{y_{n-1}k_{n-1}^{e_{n-1}}}{x_{n-1}} + \cfrac{\delta_{n-1}}{\cfrac{x_{n-1}y_n{k_n}^{e_n}}{x_n}}\right)}\right)}}\right)}}. \tag{13}$$

If we consider a part of this construction, which had obtained at the $i$-th step excluding the previous steps of the algorithm, then we can obtain the following:

$$\frac{y_i{k_i}^{e_i}}{x_i} + \cfrac{\delta_i}{\left(\cfrac{x_i}{\left(\cfrac{|y_ic_i - x_ia_i|}{b_i}\right)}\right)} = \frac{y_i{k_i}^{e_i}}{x_i} + \frac{\delta_i|y_ic_i - x_ia_i|}{x_ib_i}. \tag{14}$$

The right part is obtained by opening the brackets in the left part and writing all the values in a line. Then, continued fraction (13) can be represented as (11). Here and further we will use continued fraction (11) as more compact.

We need to make a small remark regarding the last $n$-th step in theorem 3. Condition (10) is fulfilled instead of condition (12). If $n$ steps were not enough to expand the number $a/b$ into a fourth type continued fraction and at least one more step is required, on which condition (10) must be satisfied, then in this case we will assume that we are dealing with a combination of fractions of the fourth and third types.

Expansion into continued fractions of the third and fourth types is ambiguous. This is due to the fact that at each step of the algorithm there can be several pairs of numbers $(x, y)$ that satisfy inequality (3). For example, the inequality $|8/13 - x/y| < 1/(4y)$ is satisfied by pairs $(1, 2)$, $(2, 3)$.

Example 1. Consider an example of expansion into a third type continued fraction. The following table shows the calculation results:

| Step № | $a_i$ | $b_i$ | $\lvert y_i c_i - x_i a_i \rvert$ | $k_i$ | $e_i$ | $x_i$ | $y_i$ | $\delta_i$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1117 | 505 | 107 | 3 | 0 | 1 | 2 | 1 |
| 1 | 505 | 107 | 30 | 5 | 0 | 1 | 5 | -1 |
| 2 | 107 | 30 | 17 | 3 | 1 | 1 | 1 | 1 |
| 3 | 30 | 17 | 5 | 5 | 0 | 3 | 5 | 1 |
| 4 | 17 | 5 | 2 | 3 | 1 | 1 | 1 | 1 |
| 5 | 5 | 2 | 0 | 7 | 0 | 2 | 5 | -1 |

Using this table and formula (8), we can get at a fraction

$$\frac{1117}{505} = 2 + \cfrac{1}{5 - \cfrac{1}{3 + \cfrac{1}{\frac{5}{3} + \cfrac{1}{3\left(3 + \cfrac{1}{\left(\frac{5}{2}\right)}\right)}}}}$$

Example 2. Consider an example of expansion into a fourth type continued fraction. The following table shows the calculation results:

| Step № | $a_i$ | $b_i$ | $\lvert y_i c_i - x_i a_i \rvert$ | $k_i$ | $e_i$ | $x_i$ | $y_i$ | $\delta_i$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 291 | 11 | 54 | 4 | 2 | 2 | 3 | 1 |
| 2 | 54 | 11 | 12 | 3 | 1 | 1 | 2 | -1 |
| 3 | 12 | 11 | 1 | 11 | 0 | 1 | 1 | 1 |
| 4 | 11 | 1 | 0 | 13 | 0 | 1 | 11 | -1 |

Using this table and formula (11), we can get at a fraction

$$\frac{291}{11} = \frac{3}{2} \times 16 + \frac{1}{2} \times \left(3 \times 2 - \left(1 + \frac{1}{11}\right)\right).$$

This expansion ended at the third step, since at the fourth step $b_4 = 1$ and $a_4 < k_4$. In general, when given a certain sequence $K$, the expansion into a continued fraction can end much earlier. If in this example at the second step we take the number 54 as the parameter $k_i$, and define the pair $(x_i, y_i)$ as $(11, 54)$, then expansion into a continued fraction will be finished at the second step. This means that the ratio of the numbers 54/11 will not actually expand into any continued fraction. In this case, the last element in formula (11) will disappear and it will take the following form:

$$\sum_{i=0}^{n} \frac{y_i {k_i}^{e_i}}{x_i} \prod_{j=0}^{i-1} \frac{\delta_j}{x_j}.$$

Such cases of "stopping" number expansion into a continued fraction are not considered in the article.

For given integers $a$, $b$, a combination of continued fractions of type (8), (11) is observed, when conditions (10) and (12) alternate subject to condition $i \neq j$. Expansion of the number $a/b$ into a second kind continued fraction is not the only one, since at the next step of the algorithm there may be several pairs of numbers $x_i$, $y_i$ that satisfy conditions (1) – (3).

The numerator and denominator of a continued fraction of the second kind can be expressed using *continuant*. A continuant of the third type is defined as a determinant

$$\langle g_0, g_1, ..., g_n\rangle_3 = \det\begin{pmatrix} y_0{k_0}^{e_0} & \delta_0 & 0 & 0 & 0\cdots 0 \\ -x_1 & y_1{k_1}^{e_1} & \delta_1 & 0 & 0\cdots 0 \\ 0 & -x_2 & y_2{k_2}^{e_2} & \delta_2 & 0\cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0\cdots 0 & 0 & 0 & -x_n & y_n{k_n}^{e_n} \end{pmatrix}, \tag{15}$$

where $x_i$, $y_i$, $k_i$, $e_i$ are elements of corresponding continued fracion, and the value of $\delta_i$ is determined according to rule (9). In particular, $\langle g_0\rangle_3 = y_0{k_0}^{e_0}$, $\langle g_0, g_1\rangle_3 = y_0y_1{k_0}^{e_0}{k_1}^{e_1} + \delta_0x_1$. Moreover, by definition we assume $\langle\ \rangle_3 = 1$.

Lemma 1. *Let $[g_0; g_1, g_2, ..., g_n]_3$ be the expansion of the number $a/b$ into a third type continued fraction, with $n \geqslant 3$. Then the following formulas are true:*

1. $\langle g_0, ..., g_n\rangle_3 = y_n{k_n}^{e_n}\langle g_0, ..., g_{n-1}\rangle_3 + \delta_{n-1}x_n\langle g_0, ..., g_{n-2}\rangle_3$;
2. $\langle g_0, g_1, ..., g_n\rangle_3 = y_0{k_0}^{e_0}\langle g_1, ..., g_n\rangle_3 + \delta_0x_1\langle g_2, ..., g_n\rangle_3$;
3. $\langle g_0, ..., g_n\rangle_3 = \langle g_0, ..., g_j\rangle_3\langle g_{j+1}, ..., g_n\rangle_3 + \delta_jx_{j+1}\langle g_0, ..., g_{j-1}\rangle_3\langle g_{j+2}, ..., g_n\rangle_3$, *where the number $j$ satisfies the condition $1 \leqslant j \leqslant n$.*

*Moreover, the following equality is true:*

$$\frac{a}{b} = \frac{\langle g_0, g_1, g_2, ..., g_n\rangle_3}{x_0\langle g_1, g_2, ..., g_n\rangle_3}.$$

A continuant of the fourth type is defined as a determinant

$$\det\begin{pmatrix} \delta_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (-1)^{n+1}w_0 & 0 \\ x_1 & \delta_1 & 0 & 0 & 0 & \cdots & 0 & 0 & (-1)^{n+2}w_1 & 0 \\ 0 & x_2 & \delta_2 & 0 & 0 & \cdots & 0 & 0 & (-1)^{n+3}w_2 & 0 \\ 0 & 0 & x_3 & \delta_3 & 0 & \cdots & 0 & 0 & (-1)^{n+4}w_3 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \delta_{n-3} & 0 & (-1)^{2n-2}w_{n-3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & x_{n-2} & \delta_{n-2} & (-1)^{2n-1}w_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & x_{n-1} & w_{n-1} & \delta_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -x_n & w_n \end{pmatrix}, \tag{16}$$

where $w_i = y_i{k_i}^{e_i}$. Again $x_i$, $y_i$, $k_i$, $e_i$ are elements of continued fraction, and the function $\delta_i$ is defined according to rule (9). Continuants of the fourth type are denoted as $\langle g_0, ..., g_n\rangle_4$. In particular, $\langle g_0\rangle_4 = y_0{k_0}^{e_0}$, $\langle g_0, g_1\rangle_4 = y_0y_1{k_0}^{e_0}{k_1}^{e_1} + \delta_0x_1$, $\langle\ \rangle_4 = 1$.

Lemma 2. *Let $[g_0; g_1, g_2, ..., g_n]_4$ be the expansion of the number $a/b$ into a fourth type continued fraction, with $n \geqslant 3$. Then the following formulas are true:*

$$\langle g_0, ..., g_n\rangle_4 = \delta_0\langle g_1, ..., g_n\rangle_4 + \langle g_0\rangle_4\langle g_n\rangle_4\prod_{i=1}^{n-1}x_i.$$

*Moreover, the following equality is true:*

$$\frac{a}{b} = \frac{\langle g_0, g_1, g_2, \cdots g_n\rangle_4}{\langle g_n\rangle_4\prod_{i=0}^{n-1}x_i}.$$

Due to the fact that matrix (16) is not diagonal, a continuant of the fourth type of $n$-th order can not be represented as a sum of products of continuants of lower orders by performing an arbitrary partition of the original continuant following the example of paragraph 3 of lemma 1.

LEMMA 3. *If at least one of the conditions is performed*

- $\delta_i = 1$ *for any* $0 \leqslant i < n$,
- $\delta_j = -1$ ***and*** $y_i {k_i}^{e_i} - x_i > 1$ *for any* $0 \leqslant i \leqslant n, 0 \leqslant j < n$, (17)
- $\delta_{n-1} = -1$ ***and*** $y_n {k_n}^{e_n} - x_n > 1$, *and for other values* $\delta_i = 1, 0 \leqslant i < n-1$,

*then for an arbitrary* $n$ *the following inequalities are true:*

$$0 < \langle g_0 \rangle_3 < \langle g_0, g_1 \rangle_3 < ... \langle g_0, g_1, ..., g_n \rangle_3. \tag{18}$$

If one of three conditions (17) is performed at expansion into the third type continued fraction, then the third type continuants, with the help of which the numerators and denominators are expressed, will strictly increase.

Example 3. Consider an example of an increasing sequence of the third type continuants. The following table shows the calculation results:

| Step № | $a_i$ | $b_i$ | $\|y_i c_i - x_i a_i\|$ | $k_i$ | $e_i$ | $x_i$ | $y_i$ | $\delta_i$ | Condition |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 117520 | 1371 | 173 | 7 | 2 | 4 | 7 | -1 | true |
| 1 | 1371 | 173 | 13 | 2 | 3 | 1 | 1 | -1 | true |
| 2 | 173 | 13 | 5 | 3 | 2 | 2 | 3 | -1 | true |
| 3 | 13 | 5 | 0 | 13 | 0 | 5 | 13 | -1 | true |

The following values of the continuants are obtained: $\langle g_0 \rangle_3 = 7^3 = 343$, $\langle g_0, g_1 \rangle_3 = 7^3 \times 8 - 1 =$ $= 2743$, $\langle g_0, g_1, g_2 \rangle_3 = 2743 \times 9 \times 3 - 2 \times 343 = 73375$, $\langle g_0, g_1, g_2, g_3 \rangle_3 = 73375 \times 13 - 5 \times 2743 = 940160$. The last column in the table shows the fulfillment of inequality (3), which corresponds to the “true” value. Hence we get that $\langle g_0 \rangle_3 < \langle g_0, g_1 \rangle_3 < \langle g_0, g_1, g_2 \rangle_3 < \langle g_0, g_1, g_2, g_3 \rangle_3$.

Growth the sequence of continuants of the fourth type is equivalent to fulfillment several inequalities, as indicated by the following

LEMMA 4. *If at least one of conditions (17) is satisfied for zero- and first-order continuants, if* $\langle g_0, g_1, g_2 \rangle_4 > \langle g_0, g_1 \rangle_4$, $\delta_{-1} = 1$, *and also for all* $2 < i \leqslant n$ *the following inequality holds*

$$\delta_0 \delta_1 \cdots \delta_{i-2} (\langle g_{i-1}, g_i, g_{i+1} \rangle_4 - \langle g_{i-1}, g_i \rangle_4) > (\langle g_i \rangle_4 - x_i \langle g_{i+1} \rangle_4) \sum_{j=0}^{i-1} \delta_{j-1} y_j {k_j}^{e_j} \prod_{z=j+1}^{i-1} x_z, \tag{19}$$

*then for an arbitrary* $n$ *the following inequalities are true:*

$$0 < \langle g_0 \rangle_4 < \langle g_0, g_1 \rangle_4 < ... \langle g_0, g_1, ..., g_n \rangle_4. \tag{20}$$

Example 4. Consider an example of an increasing sequence of the fourth type continuants. The following table shows the calculation results:

| Step № | $a_i$ | $b_i$ | $\|y_i c_i - x_i a_i\|$ | $k_i$ | $e_i$ | $x_i$ | $y_i$ | $\delta_i$ | Condition |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 518 | 11 | 221 | 3 | 2 | 1 | 3 | 1 | false |
| 1 | 221 | 11 | 112 | 6 | 1 | 2 | 5 | 1 | false |
| 2 | 112 | 11 | 2 | 5 | 1 | 1 | 2 | 1 | true |
| 3 | 11 | 2 | 0 | 11 | 0 | 2 | 11 | -1 | true |

Using this table, we will find the corresponding values of the continuants. So, we get the following values: $\langle g_0\rangle_4 = 27$, $\langle g_0, g_1\rangle_4 = 3\times9\times5\times6+2 = 812$, $\langle g_0, g_1, g_2\rangle_4 = (5\times6\times2\times5+1)+3\times3^2\times2\times2\times5 = 841$. Now find the value of the fourth-order continuant. So, we get the following values: $\langle g_2, g_3\rangle_4 = 2\times5\times11+2=112$, $\langle g_1, g_2, g_3\rangle_4=112+5\times6\times11=442$, $\langle g_0, g_1, g_2, g_3\rangle_4=442+3\times3^2\times2\times11 = 1036$. We get that $\langle g_0\rangle_4 < \langle g_0, g_1\rangle_4 < \langle g_0, g_1, g_2\rangle_4 < \langle g_0, g_1, g_2, g_3\rangle_4$.

In example 4 condition (3) is not always taken into account. This means that a pair of numbers $(x_i, y_i)$ may be not the best approximation to the fraction $b_i{k_i}^{e_i}/a_i$ in the general case. The last column "Condition" indicates that this condition is fulfilled (true) or not fulfilled (false). The question of the existence of an increasing sequence of continuants of arbitrary length ($n \geqslant 3$) remains open, provided that formula (3) is fulfilled.

Definition 1. *The rational numbers*

$$\frac{p_{i,(0,0)}}{q_{i,(0,0)}} = [g_0]_i, \frac{p_{i,(0,1)}}{q_{i,(0,1)}} = [g_0; g_1]_i; \frac{p_{i,(0,2)}}{q_{i,(0,2)}} = [g_0; g_1, g_2]_i, \cdots, \frac{p_{i,(0,n)}}{q_{i,(0,n)}} = [g_0; g_1, \cdots, g_n]_i$$

*are called the convergents of i-th type of the number $p/q$, expressed using the fraction $[g_0; g_1, \cdots, g_n]_i$.*

The first numeral in the index of convergent of the number $p_{i,(0,j)}/q_{i,(0,j)}$ indicates the appropriate type of continued fraction. For example, when $i = 3$, we use third type continued fraction. If some part of the continued fraction is considered, for example, $[g_l; g_{l+1}, \cdots, g_n]_i$, then the corresponding convergents also begin from the $l$-th element: $p_{i,(l,l)}/q_{i,(l,l)}$, $p_{i,(l,l+1)}/q_{i,(l,l+1)}$, $p_{i,(l,l+2)}/q_{i,(l,l+2)}$ etc.

The following lemma allows us to calculate convergents of the third type without resorting to calculating the values of the continuants. You just need to perform the product of matrices.

Lemma 5. *Let the number $n > 1$ and the product of matrices $A = \prod_{i=0}^{n} A_i$ are set, where*

$$A_i = \begin{pmatrix} 0 & 1 \\ \dfrac{\delta_0}{x_0} & \dfrac{y_i{k_i}^{e_i}}{x_i} \end{pmatrix}, A = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix}, \tag{21}$$

*then*

$$P = \frac{\delta_n\langle g_1, \cdots, g_{n-1}\rangle_3}{\prod_{i=1}^{n} x_i}, P' = \frac{\langle g_1, \cdots, g_n\rangle_3}{\prod_{i=1}^{n} x_i},$$
$$Q = \frac{\delta_n\langle g_0, \cdots, g_{n-1}\rangle_3}{\prod_{i=0}^{n} x_i}, Q' = \frac{\langle g_0, \cdots, g_n\rangle_3}{\prod_{i=0}^{n} x_i}, \det(A) = \prod_{i=0}^{n}\frac{\delta_i}{x_i}.$$

Corollary 1.

$$\frac{Q}{P} = \frac{p_{3,(0,n-1)}}{q_{3,(0,n-1)}}, \frac{Q'}{P'} = \frac{p_{3,(0,n)}}{q_{3,(0,n)}}.$$

The proof of corollary 1 follows automatically from lemmas 1, 5.

Consider convergents of rational numbers, which had been obtained using (regular) continued fractions. Let's number them in order, adding the corresponding index to them. All convergents with even indices form a strictly increasing sequence, and all convergents with odd indices form a strictly decreasing sequence (refer to [16]). But despite this, in the general case nothing similar can be done for convergents $p_{i,(0,0)}/q_{i,(0,0)}, p_{i,(0,1)}/q_{i,(0,1)}, \ldots$. Look at example 1 again. We calculate all convergents of the number 1117/505. Then we get

$$\frac{p_{3,(0,0)}}{q_{3,(0,0)}} = 2, \frac{p_{3,(0,1)}}{q_{3,(0,1)}} = \frac{10+1}{5} = \frac{11}{5}, \frac{p_{3,(0,2)}}{q_{3,(0,2)}} = \frac{3\cdot 11-2}{5\cdot 3-1} = \frac{31}{14}, \frac{p_{3,(0,3)}}{q_{3,(0,3)}} = \frac{5\cdot 31+3\cdot 11}{5\cdot 14+3\cdot 5} = \frac{188}{85},$$

$$\frac{p_{3,(0,4)}}{q_{3,(0,4)}} = \frac{3\cdot 188+31}{85\cdot 3+14} = \frac{595}{269}, \frac{p_{3,(0,5)}}{q_{3,(0,5)}} = \frac{5\cdot 595+2\cdot 188}{5\cdot 269+2\cdot 85} = \frac{1117}{505}.$$

Let's compare all convergents with each other:

$$\frac{p_{3,(0,0)}}{q_{3,(0,0)}} < \frac{p_{3,(0,1)}}{q_{3,(0,1)}} < \frac{p_{3,(0,3)}}{q_{3,(0,3)}} < \frac{p_{3,(0,5)}}{q_{3,(0,5)}} < \frac{p_{3,(0,4)}}{q_{3,(0,4)}} < \frac{p_{3,(0,2)}}{q_{3,(0,2)}}.$$

It is possible to identify conditions, under which successive convergents will be strictly greater or lesser than each other. Comparison of convergents of the third type $n$ and $n+1$ order $p_{3,(0,n)}/q_{3,(0,n)}$, $p_{3,(0,n+1)}/q_{3,(0,n+1)}$ is equivalent to comparing magnitudes $\delta_0 q_{3,(1,n)}/p_{3,(1,n)}$, $\delta_0 q_{3,(1,n+1)}/p_{3,(1,n+1)}$. Their calculation can be simplified. To do this, you need to perform a certain number of expansions of the continuants in numerators and denominators, grouping the common parts together. This idea underlies the following results. Moreover, in some cases it is not even necessary to completely calculate the value of continuant.

Theorem 4. *If the following inequality is true*

$$\frac{\langle g_{n-1}, g_n\rangle_3}{x_{n-1}\langle g_n\rangle_3} > \frac{\langle g_{n-1}, g_n, g_{n+1}\rangle_3}{x_{n-1}\langle g_n, g_{n+1}\rangle_3}, \tag{22}$$

*$\delta_j = 1$ for all $0 \leqslant j < n$ and the number $n$ is odd, then*

$$\frac{p_{3,(0,n)}}{q_{3,(0,n)}} > \frac{p_{3,(0,n+1)}}{q_{3,(0,n+1)}}. \tag{23}$$

*If condition (22) is performed, $\delta_j = 1$ for all $0 \leqslant j < n$ and the number $n$ is even, then*

$$\frac{p_{3,(0,n)}}{q_{3,(0,n)}} < \frac{p_{3,(0,n+1)}}{q_{3,(0,n+1)}}.$$

Theorem 5. *If inequality (22) is true, $\delta_j = -1$ for all $0 \leqslant j < n$, then inequality (23) is true.*

Theorem 6. *If for all positive integers $t$ the equalities $\delta_{2t} = 1$, $\delta_{2t+1} = -1$ are satisfied, condition (22) is true, and for a natural number $n$ and a positive integer $i$, $0 \leqslant i \leqslant n-1$, condition*

- $n-1 \equiv 0 \pmod 4$ *or* $n-1 \equiv 2 \pmod 4$ *is true, then for* $i \equiv 0 \pmod 4$ *and* $i \equiv 1 \pmod 4$ *inequality*

$$\frac{p_{3,(n-1-i,n)}}{q_{3,(n-1-i,n)}} > \frac{p_{3,(n-1-i,n+1)}}{q_{3,(n-1-i,n+1)}} \tag{24}$$

  *is true, and for* $i \equiv 2 \pmod 4$ *and* $i \equiv 3 \pmod 4$ *inequality (24) is satisfied with a minus sign "<".*

- $n-1 \equiv 1 \pmod 4$ *or* $n-1 \equiv 3 \pmod 4$ *is true, then for* $i \equiv 0 \pmod 4$ *and* $i \equiv 2 \pmod 4$ *inequality (24) is true, and for* $i \equiv 1 \pmod 4$ *and* $i \equiv 3 \pmod 4$ *inequality (24) is satisfied with a minus sign "<".*

Theorem 6 can also be used to compare two convergents. To do this, you only need to know the remainder of dividing the number $n-1$ by 4 and use one of the conditions.

If in condition (22) sign will change to the opposite, then signs of comparisons in inequalities of theorems 4, 6, 5 will also change.

It is quite difficult to obtain results similar to theorems 4, 6, 5 for convergents of the fourth type. Nevertheless, this task can be reduced to comparing the following magnitudes.

PROPOSITION 1. *The task of comparing convergents of the fourth type*

$$\frac{p_{4,(0,n)}}{q_{4,(0,n)}} \text{ and } \frac{p_{4,(0,n+1)}}{q_{4,(0,n+1)}}$$

*is equivalent to comparing magnitudes*

$$\frac{\delta_0\langle g_1,\cdots,g_n\rangle_4}{\langle g_n\rangle_4} \text{ and } \frac{\delta_0\langle g_1,\cdots,g_n\rangle_4}{x_n\langle g_{n+1}\rangle_4}.$$

## 4. Estimation of the number of steps in the Sorenson algorithm

PROOF OF THEOREM 1.

Proof of this theorem, as in the Sorenson theorem (refer to [3], lemma 3.2) is based on the method of mathematical induction. Let $s$, $m$ are positive integers, $s \geqslant m$, and the numbers $a$, $b$ are defined as

$$a=\sum_{i=0}^{s} t_i k^i,\ b=\sum_{j=0}^{m} q_j k^j, \tag{25}$$

where $t_i, q_i \in \{1,2,\cdots,k-1\}$. Hence the inequality $0<b\leqslant a<k^{s+1}$ is true. Let's prove that $\gcd(a,b)$ is calculated at $\lfloor \log(a)/\log(k)\rfloor + \lfloor \log(b)/\log(k)\rfloor + 1$ step.

Let the numbers $a$, $b$ take any values on the interval $[1,k-1]$, then $m$, $s$ are equal to zero. The value of $e$ is equal to zero, and as a pair of numbers $(x,y)$, satisfying inequality (3), we can take $(b,a)$. If they are not coprime (refer to formula (2)), then divide both numbers $x$, $y$ by their gcd. Thus, it will take 1 step to calculate $\gcd(a,b)$.

Let the inequalities $0<m\leqslant M$, $0<s\leqslant M$ are true. Suppose that for all pairs of numbers $(a,b)$ defined using formula (25), the induction assumption is true. Then the number of steps is $2m+1$ in the worst case. We prove the induction hypothesis for pairs of numbers $(a',b')$ defined using the magnitudes $m=M+1$, $s=S$, $m<s$:

$$a'=\sum_{i=0}^{S} t'_i k^i, b'=\sum_{j=0}^{M+1} q'_j k^j, 1\leqslant t'_i, q'_j \leqslant k, a'\geqslant b'. \tag{26}$$

After the value $e'$ has been calculated and the numbers $(x',y')$ has been choosen that satisfy inequality (3), we obtain

$$a''=|b'k^{e'}y'-a'x'|<\frac{a'}{k+1}<\left\lfloor\frac{a'}{k}\right\rfloor=\sum_{i=0}^{S} t'_{i+1}k^i, a''>b'. \tag{27}$$

Further, for the input numbers $(a'',b')$, we calculate $e'$, select the numbers $(x',y')$ and find a linear combination, as in formula (27), where instead of $a'$ under formula module (27) there will be $a''$. This procedure continues for no more than $S-M-1$ step, including the one described. After that the number $a''$ can be represented as the sum of the powers of $k$, and each of them will be multiplied by a number from 0 to $k$, as in formula (26), but the summation will be carried out up to $M+1$. If $a'' < b'$, then swap their values. After that we again calculate $e'$, select the numbers $(x', y')$ and find a linear combination of the numbers $a''$, $b'$, as in formula (27):

$$a'' = |b'k^{e'}y' - a''x'|. \tag{28}$$

The new value of $a''$ is $\sum_{i=0}^{M} t_i''k^i$, and the inequality $0 \leqslant t_i'' < k$ is true. If at this step $a'' < b'$, then swap their values again and run another step of the procedure described above.

As a result, we obtain that the numbers $a''$, $b'$ are expressed as the sums $\sum_{i=0}^{M} \bar{t}_i k^i$, $\sum_{j=0}^{M} t_j''k^j$, moreover the inequality $0 \leqslant \bar{t}_i, t_j''k$ is true. And for such numbers, according to the assumption of induction, the number of steps does not exceed $2m+1$.

We have done no more than $S-M-1$ linear combinations after that we could perform a total of 2 exchanges and linear combinations. As a result, the total number of steps does not exceed $S-M-1+2+2M+1 = S+(M+1)+1$. The proof is done. □

Here is an example of how the algorithm works in the worst case. Let the following numbers be given: $a = 1000351$, $b = 38530$, $k = 25$. The following table shows the calculation results. Columns named "poly(a)", "poly(b)", means representation of the numbers $a$, $b$ as sum of the powers of $k$. The fourth column, named $r$, contains all the values $|ybk^e - ax|$. Note that the last step, where one of the numbers is zero, is not included in the total number of steps, since no calculations occur there. In the end there were 8 steps.

| $a$ | $b$ | $r$ | $poly(a)$ | $poly(b)$ | $e$ | $x$ | $y$ |
|---|---|---|---|---|---|---|---|
| 1000351 | 38530 | 37101 | $2k^4+13k^3+25k^2+14k+1$ | $2k^3+11k^2+16k+5$ | 1 | 1 | 1 |
| 38530 | 37101 | 1429 | $2k^3+11k^2+16k+5$ | $2k^3+9k^2+9k+1$ | 0 | 1 | 1 |
| 37101 | 1429 | 1376 | $2k^3+9k^2+9k+1$ | $2k^2+7k+4$ | 1 | 1 | 1 |
| 1429 | 1376 | 53 | $2k^2+7k+4$ | $2k^2+5x+1$ | 0 | 1 | 1 |
| 1376 | 53 | 51 | $2k^2+5x+1$ | $2k+3$ | 1 | 1 | 1 |
| 53 | 51 | 2 | $2k+3$ | $2k+1$ | 0 | 1 | 1 |
| 51 | 2 | 1 | $2k+1$ | 2 | 1 | 1 | 1 |
| 2 | 1 | 0 | 2 | 1 | 0 | 1 | 2 |

In example, at each step of algorithm, ratio between the numbers $a$, $bk^e$ is approximately equal to one. The numbers $x$, $y$ are equal to one (except for the last step of algorithm). All this leads to the fact that at each step ratio of largest of the numbers and result of the linear combination is approximately equal to $k$, which is consistent with inequality (3): $|byk^e - xa| \leqslant a/(k+1)$.

## 5. Proofs

All proofs in this article is presented in full without abbreviations.

Proof of theorem 2. Let $a_0$ and $b_0$ are the input numbers of the generalized Sorenson left-shift gcd algorithm, and the magnitude $s_i$ is equal to $|y_i c_i - x_i a_i|$. Using rules (4), (5) and determining magnitude (9), you can get the formula $\delta_0 s_0 + y_0 c_0 = x_0 a_0$. Division both parts by $x_0 b_0$ leads to the expression

$$\frac{a_0}{b_0} = \frac{y_0 {k_0}^{e_0}}{x_0} + \frac{\delta_0}{x_0\left(\dfrac{b_0}{s_0}\right)}. \tag{29}$$

At the first step of the algorithm, we move on to a pair of numbers $(a_1, b_1) = (b_0, s_0)$, moreover $b_0 > s_0$. In view of this, at expansion of the numbers $a_1$, $b_1$ into a continued fraction of the third type, expression (29) is represented as

$$\frac{a_0}{b_0} = \frac{y_0 {k_0}^{e_0}}{x_0} + \frac{\delta_0}{\dfrac{y_1 x_0 {k_1}^{e_1}}{x_1} + \dfrac{\delta_1}{\left(\dfrac{x_1}{x_0} \times \left(\dfrac{b_1}{s_1}\right)\right)}}. \tag{30}$$

Further, the process of expansion into a continued fraction of the third type continues similarly to the previous steps. We will assume that formula (8) is correct at the $t$ steps of algorithm. Using the principle of mathematical induction, we prove that it is also true at the $(t+1)$-th step, where the value of $t+1$ does not exceed $n$. Consider expansion of the number $a_t/b_t$ taking into account all the multipliers $x_i$ obtained in the previous steps of the algorithm:

$$\frac{y_t {k_t}^{e_t} \prod\limits_{\substack{i<t,\\ i\not\equiv t \pmod 2}} x_i}{x_t \prod\limits_{\substack{j<t,\\ j\equiv t \pmod 2}} x_j} + \frac{\delta_n}{\left(\dfrac{b_t x_t \prod\limits_{\substack{j<t,\\ j\equiv t \pmod 2}} x_j}{s_t \prod\limits_{\substack{i<t,\\ i\not\equiv t \pmod 2}} x_i}\right)} \tag{31}$$

Now let's perform expansion of the number $a_{t+1}/b_{t+1} = b_t/s_t$. The denominator of the right term can be written as

$$\frac{y_{t+1} {k_{t+1}}^{e_{t+1}} \prod\limits_{\substack{j<t+1,\\ j\not\equiv t+1 \pmod 2}} x_j}{x_{t+1} \prod\limits_{\substack{i<t,\\ i\equiv t+1 \pmod 2}} x_i} + \frac{\delta_t}{\left(\dfrac{b_{t+1} x_{t+1} \prod\limits_{\substack{i<t,\\ i\equiv t+1 \pmod 2}} x_i}{s_{t+1} \prod\limits_{\substack{j<t+1,\\ j\not\equiv t+1 \pmod 2}} x_j}\right)}. \tag{32}$$

Thus, we obtained the necessary product of the elements $x_i$ in numerators and denominators of formula (32). The proof is done. □

Proof of theorem 3. The zero step of the algorithm completely coincides with its description in the proof of theorem (2). As a result of this step, we can obtain the expression

$$\frac{a_0}{b_0} = \frac{y_0 {k_0}^{e_0}}{x_0} + \frac{\delta_0 s_0}{x_0 b_0}. \tag{33}$$

At the first step of the algorithm, we get a pair of numbers $(a_1, b_1) = (s_0, b_0)$, moreover $s_0 \geqslant b_0$. In view of this, at expansion the numbers $a_1$, $b_1$ into a continued fraction of the fourth type, expression (33) is represented in the form

$$\frac{a_0}{b_0} = \frac{y_0 {k_0}^{e_0}}{x_0} + \frac{\delta_0}{x_0}\left(\frac{y_1 {k_1}^{e_1}}{x_1} + \frac{\delta_1 s_1}{x_1 b_1}\right). \tag{34}$$

Then the process of expansion into a fourth type continued fraction continues in the same way as at the previous steps. Let's assume that formula (11) is correct at $t$ steps of the algorithm. Using the principle of mathematical induction, we prove that it's also true at the $(t+1)$-th step, where the value of $t+1$ does not exceed $n$. Consider expansion of the number $a_t/b_t$ taking into account all the multipliers $x_i$ that had been obtained in the previous steps of the algorithm:

$$\sum_{i=0}^{t-1} \frac{y_i {k_i}^{e_i}}{x_i} \prod_{j=0}^{i-1} \frac{\delta_j}{x_j} + \frac{s_t}{b_t} \prod_{i=0}^{t} \frac{\delta_i}{x_i} \tag{35}$$

Now let's perform expansion of number $a_{t+1}/b_{t+1} = b_t/s_t$. Fraction (35) can be written as

$$\sum_{i=0}^{t-1} \frac{y_i {k_i}^{e_i}}{x_i} \prod_{j=0}^{i-1} \frac{\delta_j}{x_j} + \prod_{i=0}^{t-1} \frac{\delta_i}{x_i} \left( \frac{y_t {k_t}^{e_t}}{x_t} + \frac{\delta_t s_t}{x_t b_t} \right). \tag{36}$$

If $t+1$ is the last step of the algorithm, then this means that conditions (10) are performed instead of conditions (12). Then, the number $b_t/s_t$ will be represented as $y_{t+1}{k_{t+1}}^{e_{t+1}}/x_{t+1}$ and substituted into formula (36). This proves formula (11). □

T. Muir introduced continuant (refer to [17]), as a determinant of the tridiagonal matrix $E_{1,n}$:

$$\det(E_{1,n}) = \det \begin{pmatrix} h_1 & l_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ m_1 & h_2 & l_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & h_3 & l_3 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & m_{n-2} & h_{n-1} & l_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & m_{n-1} & h_n \end{pmatrix}. \tag{37}$$

He obtained a formula for the expansion of a continuant of arbitrary order (refer to [17, p. 518]). We write this formula in terms of the expansion of determinant of the matrix $E_{1,n}$ of $n$-th order:

$$\det(E_{1,n}) = \det(E_{1,r}) \det(E_{r+1,n}) - l_{r-1} m_{r-1} \det(E_{1,r-1}) \det(E_{r+2,n}). \tag{38}$$

The matrix $E_{i,j}$ is defined in the same way as the matrix $E_{1,n}$:

$$\det(E_{i,j}) = \det \begin{pmatrix} h_i & l_i & 0 & 0 & 0 & 0 & \cdots & 0 \\ m_i & h_{i+1} & l_{i+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & m_{i+1} & h_{i+2} & l_{i+2} & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & m_{j-2} & h_{j-1} & l_{j-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & m_{j-1} & h_j \end{pmatrix}.$$

Proof of lemma 1. It is easy to see that the formulas of points 1-3 are obtained by simply substituting the corresponding values of the matrix elements from formula (15) to formula (38).

Since third type continued fractions are connected to the generalized Sorenson left-shift algorithm, then to prove the last point of the lemma we use integer sequences $\{a_i\}_{i=0}^n$, $\{b_i\}_{i=0}^n$ defined using the algorithm and principle of mathematical induction. At the penultimate step of the algorithm, we can obtain formula (29), in which all indices are equal to $n-1$. Expanding the number $b_{n-1}/s_{n-1}$ into a continued fraction we obtain the expression

$$\frac{a_{n-1}}{b_{n-1}} = \frac{y_{n-1}{k_{n-1}}^{e_{n-1}}}{x_{n-1}} + \frac{\delta_{n-1}}{\dfrac{y_n x_{n-1} {k_n}^{e_n}}{x_n}} = \frac{\langle g_{n-1}, g_n \rangle_3}{x_{n-1} \langle g_n \rangle_3}. \tag{39}$$

Consider the ratio of the magnitudes $a_t$, $b_t$ at the $(t+1)$-th step of algorithm. Let condition $0 < t < n$ is satisfied and the formula

$$\frac{a_t}{b_t} = \frac{\langle g_t, ..., g_n \rangle_3}{x_t \langle g_{t+1}, ..., g_n \rangle_3}$$

is true. Consider the ratio of the magnitudes $a_{t-1}$, $b_{t-1}$ at the $t$-th step of algorithm:

$$\frac{a_{t-1}}{b_{t-1}} = \frac{y_{t-1}{k_{t-1}}^{e_{t-1}}}{x_{t-1}} + \frac{\delta_{t-1}}{\left(\dfrac{x_{t-1}a_t}{b_t}\right)} = \frac{y_{t-1}{k_{t-1}}^{e_{t-1}}}{x_{t-1}} + \frac{\delta_{t-1}x_t\langle g_{t+1},...,g_n\rangle_3}{x_{t-1}\langle g_t,...,g_n\rangle_3}. \tag{40}$$

After several mathematical operations, we obtain the same denominator for both fractions in formula (40) and obtain expression

$$\frac{a_{t-1}}{b_{t-1}} = \frac{\langle g_{t-1},...,g_n\rangle_3}{x_{t-1}\langle g_t,...,g_n\rangle_3}.$$

The lemma is proved. □

Proof of lemma 2. It is easy to see that the formula of continuant expansion is obtained from the formula of matrix determinant expansion (16) by the first row.

As in the proof of lemma (1), here we again use integer sequences $\{a_i\}_{i=0}^n$, $\{b_i\}_{i=0}^n$, which are determined using algorithm, and also the principle of mathematical induction. At the penultimate step of the algorithm, expanding the number $b_{n-1}/s_{n-1}$ into a fraction, we obtain formula (39). It‘s equals to $\langle g_{n-1},g_n\rangle_4/(x_{n-1}\langle g_n\rangle_4)$.

Consider the ratio of the magnitudes $a_t$, $b_t$ at the $(t+1)$-th step of algorithm. Let condition $0 < t < n$ is satisfied and the formula

$$\frac{a_t}{b_t} = \frac{\langle g_t,g_{t+1},g_{t+2},\cdots,g_n\rangle_4}{\langle g_n\rangle_4\prod_{i=t}^{n-1}x_i}$$

is true. Consider the ratio of the magnitudes $a_{t-1}$, $b_{t-1}$ at the $t$-th step of algorithm:

$$\frac{a_{t-1}}{b_{t-1}} = \frac{y_{t-1}{k_{t-1}}^{e_{t-1}}}{x_{t-1}} + \frac{\delta_{t-1}a_t}{x_{t-1}b_t} = \frac{y_{t-1}{k_{t-1}}^{e_{t-1}}}{x_{t-1}} + \frac{\delta_{t-1}\langle g_t,g_{t+1},g_{t+2},\cdots g_n\rangle_4}{x_{t-1}\langle g_n\rangle_4\prod_{i=t}^{n-1}x_i} = .$$

$$\frac{\langle g_{t-1}\rangle_4\langle g_n\rangle_4\prod_{i=t}^{n-1}x_i + \delta_{t-1}\langle g_t,g_{t+1},g_{t+2},\cdots,g_n\rangle_4}{x_{t-1}\langle g_n\rangle_4\prod_{i=t}^{n-1}x_i} = \frac{\langle g_{t-1},\cdots,g_n\rangle_4}{\langle g_n\rangle_4\prod_{i=t-1}^{n-1}x_i}$$

The lemma is proved. □

Proof of lemma 3. The numbers $x_i$, $y_i$ are positive (refer to introduction), and $x_i \leqslant y_i$ (refer to inequality (3)). It follows that for $\delta_i = 1$ and for all $\leqslant i < n$ continuants of the third and fourth types are strictly positive and form a strictly increasing sequence. If $\delta_i = -1$, then the situation is already ambiguous, and each case needs to be analyzed separately. At the beginning, instead of a certain comparison sign, we will put a question mark, which will be replaced by a comparison sign as the proof progresses.

To prove inequalities (18), we use the method of mathematical induction. Obviously, $\langle g_0\rangle_3 > 0$. Now let's compare the zeroth and first order continuants:

$$y_0y_1{k_0}^{e_0}{k_1}^{e_1} - x_1 \ ? \ y_0{k_0}^{e_0}$$

Division by $y_0{k_0}^{e_0}$ will allow us to compare $y_1{k_1}^{e_1} - x_1/(y_0{k_0}^{e_0})$ ? 1. If the second condition from system (17) is satisfied, then instead of the sign "?" you can put a ">" sign. Further, assume that for all continuants up to the $n$-th order, formula (18) is true (here it is possible to fulfill the conditions $\delta_j = 1$, $0 \leqslant j < n$ or $y_i{k_i}^{e_i} - x_i > 1$ for all $0 \leqslant i \leqslant n$ ).

Division by $y_0{k_0}^{e_0}$ will allow us to compare $y_1{k_1}^{e_1} - x_1/(y_0{k_0}^{e_0})$ ? 1. If the second condition from system (17) is satisfied, then instead of the sign "?" you can put a ">" sign. Further, assume that for all continuants up to the $n$th order, formula (18) is true (here it is possible to fulfill the conditions $\delta_j = 1$, $0 \leqslant j < n$ or $y_i{k_i}^{e_i} - x_i > 1$ for all $0 \leqslant i \leqslant n$ ). We will prove this formula for continuants of $(n+1)$-th order by comparing a continuant of $(n+1)$-th order with a continuant of $n$-th order, simultaneously performing an expansion of the first of them (the magnitude $\delta_n$ is equal to -1 according to the initial guess):

$$y_{n+1}{k_{n+1}}^{e_{n+1}}\langle g_0, ..., g_n\rangle_3 - x_{n+1}\langle g_0, ..., g_{n-1}\rangle_3 \ ?\ \langle g_0, ..., g_n\rangle_3.$$

Since the induction assumption is valid for all previous continuants, then division both sides by $\langle g_0, ..., g_n\rangle_3 > 0$, we obtain the expression

$$y_{n+1}{k_{n+1}}^{e_{n+1}} - x_{n+1}\frac{\langle g_0, ..., g_{n-1}\rangle_3}{\langle g_0, ..., g_n\rangle_3} \ ?\ 1.$$

Condition 2 from formula (17) is satisfied. This allows you to replace the "?" sign to the ">" sign. □

Proof of lemma 4. Consider the continuants of $i+1$ and $i$ order, compare them with each other and simultaneously perform an expansion of each of them. As in previous proofs, we will put a question mark at the beginning when comparing two magnitudes. So,

$$\langle g_0, ..., g_{i+1}\rangle_4 \ ?\ \langle g_0, ..., g_i\rangle_4$$

$$\delta_0\langle g_1, ..., g_{i+1}\rangle_4 + \langle g_0\rangle_4\langle g_{i+1}\rangle_4\prod_{j=1}^{i} x_j \ ?\ \delta_0\langle g_1, ..., g_i\rangle_4 + \langle g_0\rangle_4\langle g_i\rangle_4\prod_{z=1}^{i-1} x_z$$

Perform another expansion of the continuants, grouping them with each other. We get expression

$$\delta_0\delta_1(\langle g_2, ..., g_{i+1}\rangle_4 - \langle g_2, ..., g_i\rangle_4) \ ?\ \langle g_0\rangle_4\prod_{z=1}^{i-1} x_z(\langle g_i\rangle_4 - x_i\langle g_{i+1}\rangle_4) + \delta_0\langle g_1\rangle_4\prod_{z=2}^{i-1} x_z(\langle g_i\rangle_4 - x_i\langle g_{i+1}\rangle_4)$$

Performing further expansion of continuants, taking into account condition $\delta_{-1} = 1$, allows us to obtain expression

$$\delta_0\delta_1...\delta_{i-2}(\langle g_{i-1}, g_i, g_{i+1}\rangle_4 - \langle g_{i-1}, g_i\rangle_4) \ ?\ \sum_{j=0}^{i-1}\delta_{j-1}\langle g_j\rangle_4\prod_{z=j+1}^{i-1} x_z(\langle g_i\rangle_4 - x_i\langle g_{i+1}\rangle_4). \qquad (41)$$

If instead of the sign "?" we put ">", then the original continuant of the $(i+1)$-th order will be greater than continuant of the $i$-th order. Execution of formula (41) for all $2 < i \leqslant n$, together with the other conditions, which is specified in condition of the lemma, allows us to construct a strictly increasing sequence of continuants. □

Proof of lemma 5. At the beginning, we will prove formulas for the magnitudes $P$, $Q$, $P'$, $Q'$. Product of the matrices $A_0$, $A_1$ is equal to

$$\begin{pmatrix} \dfrac{\delta_1}{x_1} & \dfrac{y_1{k_1}^{e_1}}{x_1} \\ \dfrac{y_0{k_0}^{e_0}\delta_1}{x_0x_1} & \dfrac{y_0y_1{k_0}^{e_0}{k_1}^{e_1} + \delta_0x_1}{x_0x_1} \end{pmatrix} \qquad (42)$$

It is easy to verify veracity of the statement for resulting matrix. Now let's assume that for the first $n$ matrices, statement of the theorem is true. We prove this for $n+1$ products of matrices. We get expressions

$$P=\frac{\delta_{n+1}\langle g_1,\cdots,g_n\rangle_3}{\prod_{i=1}^{n+1}x_i},\quad P'=\left(\frac{\delta_n\langle g_1,\cdots,g_{n-1}\rangle_3}{\prod_{i=1}^{n}x_i}+\frac{y_{n+1}{k_{n+1}}^{e_{n+1}}}{x_{n+1}}\frac{\langle g_1,\cdots,g_n\rangle_3}{\prod_{i=1}^{n}x_i}\right),$$
$$Q=\frac{\delta_{n+1}\langle g_0,\cdots,g_n\rangle_3}{\prod_{i=0}^{n+1}x_i},\quad Q'=\left(\frac{\delta_n\langle g_0,\cdots,g_{n-1}\rangle_3}{\prod_{i=0}^{n}x_i}+\frac{y_{n+1}{k_{n+1}}^{e_{n+1}}}{x_{n+1}}\frac{\langle g_0,\cdots,g_n\rangle_3}{\prod_{i=0}^{n}x_i}\right). \tag{43}$$

Usage of lemma 1 makes it easy to prove formulas for the magnitudes $P'$, $Q'$.

Now we can prove the formula for the determinant of the matrix. In the case of matrix (42) determinant is equal to $\delta_0/x_0$. Let the formula for the determinant obtained by multiplying $n$ matrices be fulfilled. Let's find the determinant of matrix (43). It‘ equals to

$$\frac{\delta_{n+1}(\langle g_1,\cdots,g_n\rangle_3\langle g_0,\cdots,g_{n+1}\rangle_3-\langle g_0,\cdots,g_n\rangle_3\langle g_1,\cdots,g_{n+1}\rangle_3)}{\left(\prod_{i=1}^{n+1}x_i\right)\left(\prod_{j=0}^{n+1}x_j\right)}$$

Performing the expansion of $(n+1)$-th order continuants, we obtain the expressions

$$\frac{\delta_{n+1}\delta_n(\langle g_1,\cdots,g_n\rangle_3\langle g_0,\cdots,g_{n-1}\rangle_3-\langle g_0,\cdots,g_n\rangle_3\langle g_1,\cdots,g_{n-1}\rangle_3)}{\left(\prod_{i=1}^{n+1}x_i\right)\left(\prod_{j=0}^{n}x_j\right)}.$$

Further, we decompose the $n$th order continuants and obtain the expressions

$$\frac{\delta_{n+1}\delta_n\delta_{n-1}(\langle g_1,\cdots,g_{n-2}\rangle_3\langle g_0,\cdots,g_{n-1}\rangle_3-\langle g_0,\cdots,g_{n-2}\rangle_3\langle g_1,\cdots,g_{n-1}\rangle_3)}{\left(\prod_{i=1}^{n+1}x_i\right)\left(\prod_{j=0}^{n-1}x_j\right)}.$$

Continuing this procedure, we obtain the final formula for the determinant of matrix (43). □

Proof of theorem 4. Instead of some comparison sign, for now we will put the sign "?". We will define the comparison sign at the end of the proof. Let us decompose numerators and denominators of convergents. We get inequality

$$\frac{y_0{k_0}^{e_0}}{x_0}+\frac{\delta_0x_1\langle g_2,...,g_n\rangle_3}{x_0\langle g_1,...,g_n\rangle_3}\ ?\ \frac{y_0{k_0}^{e_0}}{x_0}+\frac{\delta_0x_1\langle g_2,...,g_{n+1}\rangle_3}{x_0\langle g_1,...,g_{n+1}\rangle_3}.$$

Subtract $y_0{k_0}^{e_0}/x_0$ from each part, multiply by $x_0$, invert the ratio of continuants and obtain the inequality

$$\frac{\delta_0}{\left(\frac{\langle g_1,...,g_n\rangle_3}{x_1\langle g_2,...,g_n\rangle_3}\right)}\ ?\ \frac{\delta_0}{\left(\frac{\langle g_1,...,g_{n+1}\rangle_3}{x_1\langle g_2,...,g_{n+1}\rangle_3}\right)}.$$

Further we will perform $n-2$ decompositions of continuants and $n-2$ upheavals. We will get continued fractions

$$\frac{\delta_0}{\left(\frac{y_1{k_1}^{e_1}}{x_1}+\frac{\delta_1}{\left(\frac{x_1y_2{k_2}^{e_2}}{x_2}+\frac{\delta_2}{\left(\ddots+\frac{\delta_{n-2}}{\left(\frac{\langle g_{n-1},g_n\rangle_3\prod\limits_{\substack{0\leqslant i<n-1,\\ i\not\equiv n-1\ (\mathrm{mod}\ 2)}}x_i}{x_{n-1}\langle g_n\rangle_3\prod\limits_{\substack{0\leqslant j<n-1,\\ j\equiv n-1\ (\mathrm{mod}\ 2)}}x_j}\right)}\right)}\right)}\right)} \tag{44}$$

and

$$\frac{\delta_0}{\left(\frac{y_1{k_1}^{e_1}}{x_1}+\frac{\delta_1}{\left(\frac{x_1y_2{k_2}^{e_2}}{x_2}+\frac{\delta_2}{\left(\ddots+\frac{\delta_{n-2}}{\left(\frac{\langle g_{n-1},g_n,g_{n+1}\rangle_3\prod\limits_{\substack{0\leqslant i<n-1,\\ i\not\equiv n-1 \pmod 2}}x_i}{x_{n-1}\langle g_n,g_{n+1}\rangle_3\prod\limits_{\substack{0\leqslant j<n-1,\\ j\equiv n-1 \pmod 2}}x_j}\right)}\right)}\right)}\right)} \tag{45}$$

In the resulting formulas, only the terms differ

$$\frac{\langle g_{n-1},g_n\rangle_3\prod\limits_{\substack{0\leqslant i<n-1,\\ i\not\equiv n-1 \pmod 2}}x_i}{x_{n-1}\langle g_n\rangle_3\prod\limits_{\substack{0\leqslant j<n-1,\\ j\equiv n-1 \pmod 2}}x_j} \text{ and } \frac{\langle g_{n-1},g_n,g_{n+1}\rangle_3\prod\limits_{\substack{0\leqslant i<n-1,\\ i\not\equiv n-1 \pmod 2}}x_i}{x_{n-1}\langle g_n,g_{n+1}\rangle_3\prod\limits_{\substack{0\leqslant j<n-1,\\ j\equiv n-1 \pmod 2}}x_j}. \tag{46}$$

Taking into account condition (22), and the sequence $\{\delta_i\}$ fixed by condition, and necessity to flip over the ratio of continuants, we can conclude that

$$\frac{p_{3,(n-2,n)}}{q_{3,(n-2,n)}} < \frac{p_{3,(n-2,n+1)}}{q_{3,(n-2,n+1)}}.$$

Further comparing the convergents $p_{3,(n-3,n)}/q_{3,(n-3,n)}$ and $p_{3,(n-3,n+1)}/q_{3,(n-3,n+1)}$, we get that they will change the sign of comparison:

$$\frac{p_{3,(n-3,n)}}{q_{3,(n-3,n)}} > \frac{p_{3,(n-3,n+1)}}{q_{3,(n-3,n+1)}}.$$

After a few steps, in a similar way the ratio of continuants will be considered, in which the first element will be $g_1$. We perform their comparison, after which we change its sign to the opposite one due to the necessity reverse the ratio of continuants. This ratio is multiplied by $\delta_0$ (refer to formulas (44), (45)). Therefore, the comparison sign will depend only on the parity or oddness of $n$ under conditions specified above. □

Proof of theorem 5. The proof is similar to the proof of theorem 4. After $(n-1)$ expansion of continuant and upheaval of the ratio of continuants, formulas (44), (45) will be obtained. The only different terms will be elements of formula (46).

Taking into account condition (22), and fixed sequence $\{\delta_i\}$ by condition, as well as the need to reverse the ratio of continuants, we can conclude that

$$\frac{p_{3,(n-2,n)}}{q_{3,(n-2,n)}} > \frac{p_{3,(n-2,n+1)}}{q_{3,(n-2,n+1)}}.$$

Further, comparing the convergents $p_{3,(n-3,n)}/q_{3,(n-3,n)}$ and $p_{3,(n-3,n+1)}/q_{3,(n-3,n+1)}$, we obtain that they have the same comparison sign again:

$$\frac{p_{3,(n-3,n)}}{q_{3,(n-3,n)}} > \frac{p_{3,(n-3,n+1)}}{q_{3,(n-3,n+1)}}.$$

Following this procedure, the desired result is obtained. □

Proof of theorem 6. Let $n-1\equiv 0 \pmod 4$. Condition (22) is satisfied. Let us denote by $f_{s,t}$ result of the following expression:

$$\frac{\langle g_s,\cdots,g_t\rangle_3 \prod\limits_{\substack{0\leqslant i<s,\\ i\not\equiv s \pmod 2}} x_i}{x_s\langle g_{s+1},\cdots,g_t\rangle_3 \prod\limits_{\substack{0\leqslant i<s,\\ i\equiv s \pmod 2}} x_j} = \frac{y_s \prod\limits_{\substack{0\leqslant i<s,\\ i\not\equiv s \pmod 2}} x_i}{x_s \prod\limits_{\substack{0\leqslant i<s,\\ i\equiv s \pmod 2}} x_j} + \frac{\delta_s}{f_{s+1,t}}. \tag{47}$$

We can compare the elements of the sequences of magnitudes $\{f_{s,t}\}$, $\{f_{s,t+1}\}$ with each other. Taking into account the results of lemma 1, the following inequalities are true:

$$f_{n-1,n} > f_{n-1,n+1},\ f_{n-2,n} > f_{n-2,n+1},\ f_{n-3,n+1} < f_{n-3,n+1},$$

$$f_{n-4,n} < f_{n-4,n+1},\ f_{n-5,n} > f_{n-5,n+1}.$$

Let the following inequalities are true for natural $v$, $1 < v < (n-4)/4$:

$$f_{n-4v-1,n} > f_{n-4v-1,n+1},\ f_{n-4v-2,n} > f_{n-4v-2,n+1},$$

$$f_{n-4v-3,n+1} < f_{n-4v-3,n+1}, f_{n-4v-4,n} < f_{n-4v-4,n+1},$$

Consider the following 4 values of the elements of each of the sequences $\{f_{s,t}\}$, $\{f_{s,t+1}\}$ separately. According to the definition of magnitude $f_{s,t}$, it can be expressed in terms of magnitude $f_{s+1,t}$. Inasmuch as $n-1 \equiv 0 \pmod 4$, then $n-1 \equiv 0 \pmod 2$. Hence, $n-4v-5$ is even number, so $\delta_{n-4v-5} = 1$. Therefore, comparing magnitudes $f_{n-4v-5,n}$ and $f_{n-4v-5,n+1}$, we obtain inequality

$$\frac{y_{n-4v-5} \prod\limits_{\substack{0\leqslant i<n-4v-5,\\ i\not\equiv n-4v-5 \pmod 2}} x_i}{x_{n-4v-5} \prod\limits_{\substack{0\leqslant j<n-4v-5,\\ j\equiv n-4v-5 \pmod 2}} x_j} + \frac{1}{f_{n-4v-4,n}} > \frac{y_{n-4v-5} \prod\limits_{\substack{0\leqslant i<n-4v-5,n,\\ i\not\equiv n-4v-5 \pmod 2}} x_i}{x_{n-4v-5,n} \prod\limits_{\substack{0\leqslant j<n-4v-5,\\ j\equiv n-4v-5 \pmod 2}} x_j} + \frac{1}{f_{n-4v-4,n+1}}.$$

The number $n-4v-6$ is odd, so therefore, expanding the value of magnitudes $f_{n-4v-6,n}$, $f_{n-4v-6,n+1}$, we get inequality

$$\frac{y_{n-4v-6} \prod\limits_{\substack{0\leqslant i<n-4v-6,\\ i\not\equiv n-4v-6 \pmod 2}} x_i}{x_{n-4v-6} \prod\limits_{\substack{0\leqslant j<n-4v-6,\\ j\equiv n-4v-6 \pmod 2}} x_j} - \frac{1}{f_{n-4v-5,n}} > \frac{y_{n-4v-6} \prod\limits_{\substack{0\leqslant i<n-4v-6,n,\\ i\not\equiv n-4v-6 \pmod 2}} x_i}{x_{n-4v-6,n} \prod\limits_{\substack{0\leqslant j<n-4v-6,\\ j\equiv n-4v-6 \pmod 2}} x_j} - \frac{1}{f_{n-4v-5,n+1}}.$$

The following inequality can also be proven by comparison magnitudes $1/f_{n-4v-6,n}$ and $1/f_{n-4v-6,n+1}$. Hence, the inequality $f_{n-4v-7,n} < f_{n-4v-7,n+1}$ is obtained. Comparing expressions with each other $-1/f_{n-4v-7,n}$ and $-1/f_{n-4v-7,n+1}$, we get inequality $f_{n-4v-8,n} < f_{n-4v-8,n+1}$. This proves the first point of the theorem. The remaining points can be proved in the same way. □

Proof of proposition 1. Let us write down fourth type continued fractions corresponding to the convergents under consideration, and also perform an expansion of continuants. We get expressions

$$\frac{\delta_0\langle g_1,\cdots,g_n\rangle_4 + \langle g_0\rangle_4\langle g_n\rangle_4\prod_{i=1}^{n-1}x_i}{\langle g_n\rangle_4\prod_{i=0}^{n-1}x_i} \text{ and } \frac{\delta_0\langle g_1,\cdots,g_{n+1}\rangle_4 + \langle g_0\rangle_4\langle g_{n+1}\rangle_4\prod_{i=1}^{n}x_i}{\langle g_{n+1}\rangle_4\prod_{i=0}^{n}x_i}$$

Multiply both parts by $\prod_{i=0}^{n-1}x_i$, then we get expression

$$\frac{\delta_0\langle g_1,\cdots,g_n\rangle_4}{\langle g_n\rangle_4}+\langle g_0\rangle_4\prod_{i=1}^{n-1}x_i \text{ and } \frac{\delta_0\langle g_1,\cdots,g_{n+1}\rangle_4}{x_n\langle g_{n+1}\rangle_4}+\langle g_0\rangle_4\prod_{i=1}^{n-1}x_i.$$

Reducing common parts, we obtain the required values. □

## 6. Conclusion

In this paper, the generalized Sorenson left-shift gcd algorithm was introduced. It coincides with the original Sorenson algorithm, except that instead of the parameter $k$, an infinite sequence of natural numbers $K$ is fixed, each element of which is greater than two. For the original Sorenson algorithm, an estimate of the number of steps in the worst case is obtained, an example is given. An evaluation of the generalized algorithm is also given. However, the author believes that it can be improved.

Also in this article, continued fractions with rational partial quotients with a left shift were introduced, and continuants, with the help of which one can express the numerator and denominator of such fractions. The question of search conditions, under which the sequence of continuants with increasing order is strictly increasing was investigated. Conditions have been found, under which convergents of rational numbers made using continued fractions with rational partial quotients can be unambiguously compared. Applying these conditions to all obtained convergents allows us to determine whether such sequence will be strictly increasing or not.

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