ЧЕБЫШЕВСКИЙ СБОРНИК
Том 16 Выпуск 2 (2015)

УДК 512.77

СТРУКТУРА ДИСКРИМИНАНТНОГО МНОЖЕСТВА ВЕЩЕСТВЕННОГО МНОГОЧЛЕНА

А. Б. Батхин (г. Москва)

Аннотация

Проблема описания структуры дискриминантного множества вещественного многочлена часто возникает при решении различных прикладных задач, например, при описании множества устойчивости положений равновесия многопараметрических систем, при вычислении нормальной формы системы Гамильтона в окрестности положения равновесия в случае кратных частот. В работе рассматривается структура дискриминантного множества многочлена с вещественными коэффициентами. Предлагается два подхода к его изучению. Первый подход основан на исследовании нулей идеалов, образованных набором субдискриминантов исходного многочлена. Рассмотрены различные способы вычисления субдискриминантов. Во втором подходе предлагается исследовать особые точки дискриминантного множества. Методами компьютерной алгебры показано, что для малых значений степени исходного полинома оба подхода эквивалентны, но первый более предпочтителен из-за меньшего размера идеалов.

Предлагается конструктивный алгоритм построения полиномиальной параметризации дискриминантного множества в пространстве коэффициентов многочлена. С прикладной точки зрения наибольший интерес представляет описание компоненты коразмерности 1 дискриминантного множества. Именно эта компонента делит пространство коэффициентов на области с одинаковой структурой корней многочлена. Набор компонент различных размерностей дискриминантного множества имеет иерархическую структуру. Каждая компонента большей размерности может рассматриваться как некоторая касательная развертывающая поверхность, образованная линейными многообразиями соответствующей размерности. Роль направляющей при этом выполняет компонента дискриминантного множества, имеющая размерность на единицу меньше и на которой исходный многочлен обладает единственным кратным корнем, а остальные его корни простые. Начиная с одномерного алгебраического многообразия размерности 1, на котором исходный многочлен имеет единственный корень максимальной кратности, на следующем шаге алгоритма получаем описание многообразия, на котором многочлен имеет уже 2 корня — один

простой и один кратный. Повторяя последовательно шаги алгоритма, получаем в итоге параметрическое представление компоненты коразмерности 1 дискриминантного множества.

Приведены примеры дискриминантных множеств кубического многочлена и многочлена четвертой степени.

Ключевые слова: дискриминантное множество, особая точка, рациональная параметризация.

Библиография: 15 названий.

STRUCTURE OF DISCRIMINANT SET OF REAL POLYNOMIAL

A. B. Batkhin (Moscow)

Abstract

The problem of description the structure of the discriminant set of a real polynomial often occurs in solving various applied problems, for example, for describing a set of stability of stationary points of multiparameter systems, for computing the normal form of a Hamiltonian system in vicinity of equilibrium in the case of multiple frequencies. This paper considers the structure of the discriminant set of a polynomial with real coefficients. There are two approaches to its study. The first approach is based on the study of zeroes of ideals formed by the set of subdiscriminants of the original polynomial. Different ways of computing subdiscriminants are given. There is proposed to investigate the singular points of the discriminant set in the second approach. By the methods of computer algebra it is shown that for small values of the degree of the original polynomial, both approaches are equivalent, but the first one is preferred because of smaller ideals.

Proposed constructive algorithm for obtaining polynomial parameterization of the discriminant set in the space of coefficients of the polynomial. From the applied point of view the most interesting is the description of the components of codimension 1 of the discriminant set. It is this component divides the space of the coefficients into the domains with the same structure of the roots of the polynomial. The set of components of different dimensions of the discriminant set has a hierarchical structure. Each component of higher dimensions can be considered as some kind of tangent developable surface which is formed by linear varieties of respective dimension. The role of directrix of this component performs a variety of dimension one less than that on which the original polynomial has only multiple zero and the remaining zeroes are simple. Starting with a one-dimensional algebraic variety of dimension 1 on which the original polynomial has the unique zero of maximal multiplicity, in the next step of the algorithm we obtain the description of the variety on which the polynomial has a pair of zeroes: one simple and another multiple. Repeating sequentially the steps of the algorithm, the resulting parametric

representation of components of codimension 1 of the discriminant set can be obtained.

Examples of the discriminant set of a cubic and quartic polynomials are considered.

Keywords: discriminant set, singular point, rational parametrization.
Bibliography: 15 titles.

1. Introduction

Let

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0, \tag{1}$$

be a generic monic polynomial of degree n with real coefficients. The n-dimensional real space Π of its coefficients $(a_0, a_1, \ldots, a_{n-1})$ is called the *coefficient space*.

DEFINITION **1.** *The* ***discriminant set*** $\mathcal{D}(f)$ *of the polynomial* $f(x)$ *is called the set of all points of coefficient space* Π*, at which discriminant* $D(f)$ *is vanished.*

Investigation of the structure of the discriminant set $\mathcal{D}(f)$ is important for solution of many applied problems, for instance, for study stability of a stationary point of multiparameter mechanical systems (see [1, 2, 3]), for computing normal form of Hamiltonian system in vicinity of stationary point in the case of equal frequences [4].

The set $\mathcal{D}(f)$ contains the algebraic hypersurface $\mathcal{H}$ of codimension 1 in the space Π. This hypersurface divides the Π into $1 + [n/2]$ domains with fixed number of real zeroes of the polynomial $f(x)$ (see [5, Theorem 2]).

THEOREM 1. *In the space of coefficients* Π *the domains with the same number of real zeroes of the polynomial* $f(x)$ *are separated from each other by discriminant hypersurface* $\mathcal{H} \subset \mathcal{D}(f)$.

The goal of the presented paper is to give the description of the discriminant set $\mathcal{D}(f)$ of a generic monic polynomial with real coefficients and, in particular, to construct parametrization of the hypersurface $\mathcal{H}$ of codimension 1.

Two main approaches can be proposed for study of the discriminant set $\mathcal{D}(f)$.

- The first one is based on exploring the subdiscriminants of the polynomial (1).
- The second one is based on investigation of singular points of the discriminant set $\mathcal{D}(f)$.

The structure of the paper is as following. In Section 2 we give the definition of subdiscriminant of a polynomial and recall some its useful properties. In Section 3 we provide the description of the discriminant set $\mathcal{D}(f)$ based on subdiscriminant approach. In Section 4 we show that the aproach based on investigation of singular points of the set $\mathcal{D}(f)$ is equivalent to the approach based on subdiscriminant technique. Finally, some examples are provided in the last Section 5.

2. Subdiscriminant and its properties

Here we recall the definition of the subdiscriminant, which is used in Section 3 for discriminant set description. This section is mainly based on [6, 7]

DEFINITION **2.** *Let t_i, $i = 1, \ldots, n$ be the roots of the polynomial $f(x)$. The discriminant $D(f)$ of $f(x)$ is defined by formula*

$$D(f) = \prod_{1 \leqslant i \leqslant j \leqslant n} (t_i - t_j)^2. \tag{2}$$

Usually discriminant $D(f)$ is defined as a resultant computed on the polynomial $f(x)$ and its first derivative $f'(x)$ [7]:

$$D(f) = (-1)^{n(n-1)/2} R(f, f').$$

There are several ways for discriminant $D(f)$ computation:

1. as the determinant of Sylvester matrix (see [6, 7, 8] and below);
2. as the determinant of Bézout matrix (see [6, 7, 9]);
3. as the determinant of Hankel matrix constructed of Newtonian sums (see [6, 7]);
4. with the help of pseudo-remainders sequence.

The author's computational experiments demonstrated that the most effective way of the discriminant (and also subdiscriminant) computation is the way with the help of Bézout matrix.

We consider so called *Sylvester-Habicht matrix* [6, 10]

$$\operatorname{SylHab}(f, f') = \begin{pmatrix} 1 & a_{n-1} & a_{n-2} & \ldots\ldots\ldots & a_0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & a_{n-1} & \ldots\ldots\ldots & a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & & \ddots & & \\ 0 & 0 & \cdots & 1 & a_{n-1} & \ldots\ldots\ldots\ldots\ldots & & a_1 & a_0 \\ 0 & 0 & \cdots & 0 & n & (n-1)a_{n-1} & \ldots\ldots & 2a_2 & a_1 \\ \vdots & & \ddots & & & & \ddots & & \\ 0 & n & \ldots\ldots\ldots & 2a_2 & a_1 & 0 & \ldots\ldots\ldots & & 0 \\ n & \ldots\ldots\ldots & 2a_2 & a_1 & 0 & \ldots\ldots\ldots\ldots & & & 0 \end{pmatrix}$$

So, up to the sign $D(f) = \det \operatorname{SylHab}(f, f')$. The choice of Sylvester-Habicht matrix makes definition of subdiscriminant easier than using the original Sylvester matrix.

If one wants to formulate a condition that the polynomial $f(x)$ has exactly k distinct zeroes, it is necessary to use *subdiscriminant* notion.

DEFINITION **3.** *The k-th* ***subdiscriminant*** $D^{(k)}(f)$ *of the polynomial $f(x)$ is defined by the following formula:*

$$D^{(k)}(f) = \sum_{\substack{I\subset\{1,\ldots,p\}\\ \#(I)=k}} \prod_{\substack{(j,l)\in I\\ l>j}} (t_j - t_l)^2.$$

Here $\#(I)$ is the cardinality of the set I.

The next proposition provides the way of subdiscriminant computation.

PROPOSITION 1 ([11]). *The determinant of a matrix, resulting from the Sylvester matrix* $\mathrm{SylHab}\,(f, f')$ *by deleting the first k and the last k rows, and the first k and the last k columns, is equal to the k-th subdiscriminant of the discriminant $D(f)$.*

When $k = 0$ one has the discriminant of the polynomial $f(x)$: $D^{(0)}(f) \equiv D(f)$. It is clear that k-th subdiscriminant $D^{(k)}(f)$ is the k-th inner [12] of the Sylvester-Habicht matrix.

Polynomial $f(x)$ has exactly $n-1$ non-trivial subdiscriminants each of which is a quasi homogeneous polynomial of its coefficients $a_0, \ldots, a_{n-1}$.

THEOREM 2. *The polynomial $f(x)$ has exactly d common zeroes with its derivative $f'(x)$ iff the following conditions take place*

$$\underbrace{D^{(0)}(f) = \ldots = D^{(d-1)}(f)}_{d} = 0, \quad D^{(d)}(f) \neq 0.$$

Multiple zeroes of the polynomial $f(x)$ are the zeroes of the polynomial $\mathrm{GCD}\,(f, f')$, *which can be written as follows:*

$$\mathrm{GCD}\,(f, f') = D^{(d)}\mu^d + \det M_d^{(1)}\mu^{d-1} + \cdots + \det M_d^{(d)},$$

where matrix $M_d^{(j)}$ is the d-th inner of the matrix which is obtained from the Sylvester – Habicht matrix by replacing $d+1$-th column counted from the right by the j-th column counted from the right.

The simple zeroes of the polynomial $\mathrm{GCD}\,(f, f')$ correspond to the double zeroes of the original polynomial $f(x)$. The zeroes of polynomial $\mathrm{GCD}\,(f, f')$ of multiplicity $m > 1$ correspond to zeroes of the polynomial $f(x)$ with multiplicity $m + 1$.

3. Subdiscriminant approach

Here we propose the following constructive algorithm for obtaining polynomial parameterization of the discriminant set in the space Π of coefficients of the polynomial $f(x)$. The main idea of the algorithm is to consider each variety $\mathcal{V}_i$ of dimension i as a tangent developable surface [13, 14] with the directrix defined by the variety $\mathcal{V}_{i-1}$.

1. We compute the variety $\mathcal{V}_1$ with parametrization

$$\mathcal{V}_1 : \left\{a_i = (-1)^{n-i} C_n^i t_1^{n-i},\ i = 0, \ldots, n-1\right\},$$

which solve the system $\left\{D^{(j)}(f) = 0\right\}, j = 0, \ldots, n-2$. The polynomial $f(x)$ has the only zero t_1 with multiplicity n on it.

2. Using the variety $\mathcal{V}_1$ as a directrix we obtain the variety $\mathcal{V}_2$ with parametrization

$$\mathcal{V}_2 : \left\{a_i = (-1)^{n-i} C_n^i t_1^{n-i} + (-1)^{n-i} C_{n-1}^i t_1^{n-1-i} t_2,\ i = 0, \ldots, n-1\right\},$$

which solve the system $\left\{D^{(j)}(f) = 0\right\}, j = 0, \ldots, n-3$. The polynomial $f(x)$ has two distinct zeroes on it: zero t_1 with multiplicity $n-1$ and simple zero $t_1 + t_2$. The variety $\mathcal{V}_2$ is a tangent developable surface of dimension 2 in the space Π.

3. Considering the variety $\mathcal{V}_2$ as an envelope of two-dimensional planes one gets variety $\mathcal{V}_3$, on which the polynomial $f(x)$ has a zero of mupltiplicity $n-2$ and a pair of simple zeroes.

4. Finally, repeating the steps described above one obtains the variety $\mathcal{V}_{n-1}$, which is the tangent develope hypersurface $\mathcal{H}$ in Π. On this hypersurface polynomial $f(x)$ has one double zero and $n-2$ simple zeroes.

Remark **1.** *There are other varieties in the discriminant set $\mathcal{D}(f)$, on which polynomial $f(x)$ has zeroes with other scheme of multiplicity. It is easy to show that in the generic position case these varieties have codimension more then* 1 *and therefore they cannot be the parts of the hypersurface $\mathcal{H}$.*

Proposition 2. *The subset of codimension* 1 *of the discriminant set $\mathcal{D}(f)$ of a generic monic polynomial with real coefficients is a tangent develope hypersurface $\mathcal{H}$ in the space* Π. *Therefore it has polynomial paramerization.*

4. Singular points approach

It is possible to describe the mentioned above varieties $\mathcal{V}_i$, $i = 1, \ldots, n-1$ as singular points of the discriminant set $\mathcal{D}(f)$ of orders $n-1-i$ correspondingly.

Definition **4.** *Let $\varphi(X)$ be a polynomial of $X = (x_1, \ldots, x_n)$. The point $X = X^0$ of the set $\varphi(X) = 0$ is said to be a* ***singular point*** *of the* **k-th order** *if all the partial derivatives of $\varphi(X)$ with respect to $x_1, \ldots, x_n$ of order k, inclusive, vanish and at least one partial derivative of order $k+1$ does not vanish.*

Compose the ideal $\mathcal{J}_k$ from the discriminant $D(f)$ and all its partial derivatives up to the order k inclusive. The ideal $\mathcal{J}_k$ includes all singular points of order k and below. The direct computations show that ideal $\mathcal{J}_k$ for certain value $k \leqslant n-2$ does not equal to the ideal $\mathcal{I}_k \equiv \left\{D^{(j)}(f) | j = 0, \ldots, k\right\}$, which include the variety $\mathcal{V}_{n-k-1}$. With the help of computer algebra algorithms (Gröbner basis and procedures for polynomial ideals [15]) it was shown that $\operatorname{Rad}\mathcal{J}_k = \operatorname{Rad}\mathcal{I}_k$ for the following values of $n = 3, 4, 5$, where $\operatorname{Rad}\mathcal{P}$ is the radical of ideal $\mathcal{P}$. It is quite possible that this statement takes place for any $n \in \mathbb{N}$.

REMARK **2.** *If statement, that for any finite value k radicals of the ideals $\mathcal{J}_k$ and $\mathcal{I}_k$ coincide, is right, then the subdiscriminant approach is equal to the singular point approach. But it is clear that the first approach is easer than the second one as far as the number of polynomials in the ideals $\mathcal{I}_k$ depends linear on the degree n and the number of polynomials in the ideals $\mathcal{J}_k$ grows as n^k.*

5. Examples

EXAMPLE 1. *Monic cubic polynomial*

Let consider monic cubic polynomial $f_3(x) = x^3 + a_2x^2 + a_1x + a_0$. Its subdiscriminants are:

$$D^{(0)}(f_3) = -4a_2^3a_0 + a_2^2a_1^2 + 18a_2a_1a_0 - 4a_1^3 - 27a_0^2,$$
$$D^{(1)}(f_3) = a_2^2 - 3a_1.$$

The case of the only zero of multiplicity 3 is described by the ideal

$$\mathcal{J}_3^{(1)} \stackrel{\text{def}}{=} \{D^{(0)}(f_3), D^{(1)}(f_3)\}$$

with Hilbert dimension equal to one. Normalization the polynomial $\operatorname{GCD}(f_3, f_3')$ on the ideal $\mathcal{J}_3^{(1)}$ gives

$$\operatorname{GCD}(f_3, f_3') = 3x^2 + 2a_2 + a_1 = (3x + a_2)^2/3,$$

which is a full square.

One-parametric variety $\mathcal{V}_1$ has the following parametrization

$$\mathcal{V}_1 : \left\{a_0 = -t_1^3,\ a_1 = 3t_1^2,\ a_2 = -3t_1\right\},$$

and polynomial $f_3(x) = (x - t_1)^3$ on it.

The case of two zeroes with multiplicities 2 and 1 correspondingly is described by the ideal $\mathcal{J}_3^{(0)} \stackrel{\text{def}}{=} \{D^{(0)}(f_3)\}$, which has Hilbert dimension 2 and its zeroes forms two-parametric variety $\mathcal{V}_2$ of codimension 1

$$\mathcal{V}_2 : \left\{a_0 = -t_1^3 - t_1^2t_2,\ a_1 = 3t_1^2 + 2t_1t_2,\ a_2 = -3t_1 - t_2\right\},$$

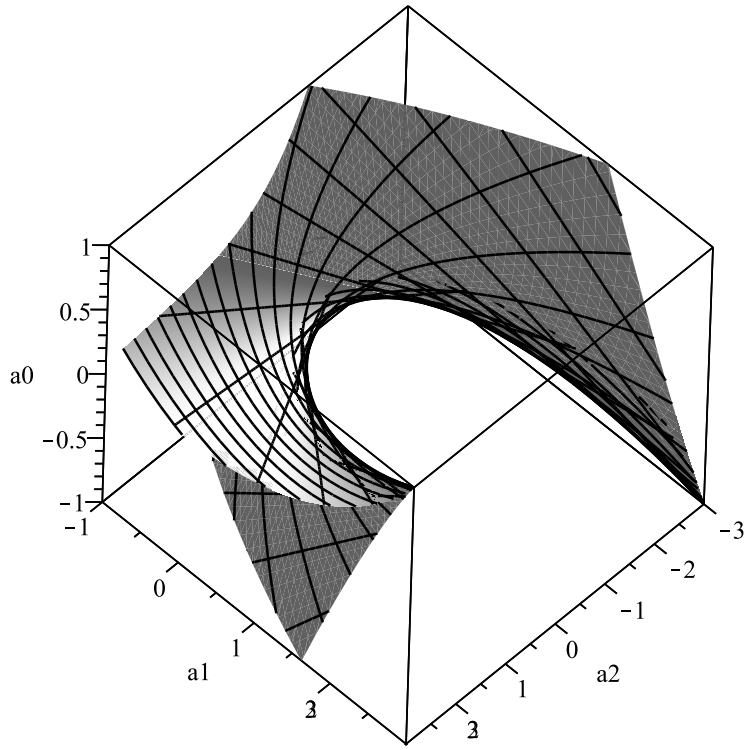

Рис. 1: Discriminant surface for monic cubic polynomial

and polynomial $f_3(x) = (x - t_1)^2(x - (t_1 + t_2))$ on it. This variety is a tangent developable surface in $\Pi = \mathbb{R}^3$ with envelope $\mathcal{V}_1$ and is shown in Fig. 1.

So, variety $\mathcal{V}_2$ is the hypersurface $\mathcal{H}$ dividing coefficient space Π into two domains with different numbers of real zeroes.

EXAMPLE 2. *Monic quartic polynomial.*

Let $f_4(x) \stackrel{\text{def}}{=} x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be a monic quartic polynomial. Its subdiscriminants are the following:

$$\begin{aligned}
D^{(0)}(f_4) &= -27a_3^4a_0^2 + 18a_3^3a_2a_1a_0 - 4a_3^3a_1^3 - 4a_3^2a_2^3a_0 + a_3^2a_2^2a_1^2 + \\
&\quad + 144a_3^2a_2a_0^2 - 6a_3^2a_1^2a_0 - 80a_3a_2^2a_1a_0 + 18a_3a_2a_1^3 + 16a_2^4a_0 - \\
&\quad - 4a_2^3a_1^2 - 192a_3a_1a_0^2 - 128a_2^2a_0^2 + 144a_2a_1^2a_0 - 27a_1^4 + 256a_0^3, \\
D^{(1)}(f_4) &= -6a_3^3a_1 + 2a_3^2a_2^2 - 12a_3^2a_0 + 28a_3a_2a_1 - 8a_2^3 + 32a_2a_0 - 36a_1^2, \\
D^{(2)}(f_4) &= 3a_3^2 - 8a_2.
\end{aligned}$$

One-parameter variety $\mathcal{V}_1$ has parametrization

$$\mathcal{V}_1 : \left\{a_0 = t_1^4,\ a_1 = -4t_1^3,\ a_2 = 6t_1^2,\ a_3 = -4t_1\right\}.$$

On this variety $f_4(x) = (x - t_1)^4$.

Discriminant set $\mathcal{D}(f_4)$ contains two varieties, on which the polynomial $f_4(x)$ has only two distinct zeroes with multiplicities $(3,1)$ and $(2,2)$ correspondingly. The variety $\mathcal{V}_2^{(1)}$ is a tangent developable surface in coefficient space $\Pi = \mathbb{R}^4$ as was shown in Section 3. Its parametrization is the following

$$\mathcal{V}_2^{(1)} : \{\, a_0 = t_1^4 + t_1^3 t_2, \quad a_1 = -4t_1^3 - 3t_1^2 t_2, \quad a_2 \ = 6t_1^2 + 3t_1 t_2, a_3 \ = -4t_1 - t_2, \}$$

on which $f_4(x) = (x - t_1)^3(x - t_1 - t_2)$. Considering variety $\mathcal{V}_2^{(1)}$ as an envelope of hyperplanes one can obtain the parametrization of those part of variety $\mathcal{V}_3^{(1)}$ of codimension 1, on which the polynomial $f_4(x)$ has only 3 distinct real zeroes. The parametriation of this veriety can be given in the following form:

$$\mathcal{V}_3^{(1)} : \{a_0 = t_1^2\left(t_2^2 - t_3^2\right), a_1 = 2t_1\left(t_3^2 - t_2^2 - t_1 t_2\right), \\ a_2 = t_1^2 + 4t_1 t_2 + t_2^2 + t_3^2, a_3 = -2(t_1 + t_2)\}, \quad (3)$$

on which $f_4(x) = (x - t_1)^2(x - (t_2 + t_3))(x - (t_2 - t_3))$. Setting in (3) $t_3 = 0$ one can get the parametric representation of two-dimensional variety $V_2^{(2)}$, on which the polynomial $f_4(x)$ has a pair of real distinct zeroes with multiplicity two. Moreover, setting $t_3 = it_3$ one gets parametrization of 3-dimensional variety $\mathcal{V}_3^{(2)}$ on which the polynomial $f_4(x)$ has one real zero t_1 with multiplicity 2 and a pair of simple complex mutually conjugated zeroes $t_2 \pm it_3$. So, both varieties $\mathcal{V}_3^{(1)}$ and $\mathcal{V}_3^{(2)}$ form the hypersurface $\mathcal{H} \subset \mathcal{D}(f_4)$ dividing the coefficient space Π into 3 domains with different numbers of real zeroes.

Let us make a linear Tschirnhaus transformation $x = y - a_3/4$ and we obtain polynomial $\widetilde{f}_4(y) = y^4 + b_2 y^2 + b_1 y + b_0$ with the coefficient space $\widetilde{\Pi} = \mathbb{R}^3$. Then the variety $\widetilde{\mathcal{V}}_1$ shrinks to the origin, the variety $\widetilde{\mathcal{V}}_2^{(1)} : \{b_0 = -3t_1^4,\ b_1 = 8t_1^3,\ b_2 = -6t_1^2\}$, on which $\widetilde{f}_4(y) = (y - t_1)^3(y + 3t_1)$, becomes one-dimensional, and, finally, the variety

$$\widetilde{\mathcal{V}}_3 : \left\{b_0 = t_1^2\left(t_1^2 - t_3\right), b_1 = 2t_1 t_3, b_2 = -2t_1^2 - t_3, \right\}$$

which is a tangent developable surface, on which

$$f_4(y) = (y - t_1)^2(y - (t_1 + t_3))(y - (t_2 - t_3)),$$

and the variety $\widetilde{\mathcal{V}}_2(2) : \{b_0 = t_2^2,\ b_1 = 0,\ b_2 = -2t_2\}$.

Variety $\widetilde{\mathcal{V}}_2^{(1)}$ is one-parameter set of singular points of the first order, because they form the set of casps. The part of variety $\widetilde{\mathcal{V}}_2^{(2)}$ is one-parameter set of singular points of the first order, because one branch of the parabola is isolated (for $t_2 > 0$) and the other branch is the curve of selfintersection of the variety $\widetilde{\mathcal{V}}_3$. The last variety is hypersurface $\mathcal{H}$ of codimention 1 in the coefficient space $\widetilde{\Pi}$ and it divides this space into three domains with different numbers of real zeroes of the polynomial $\widetilde{f}_4(y)$. All the varieties described above are shown in Figure 2.

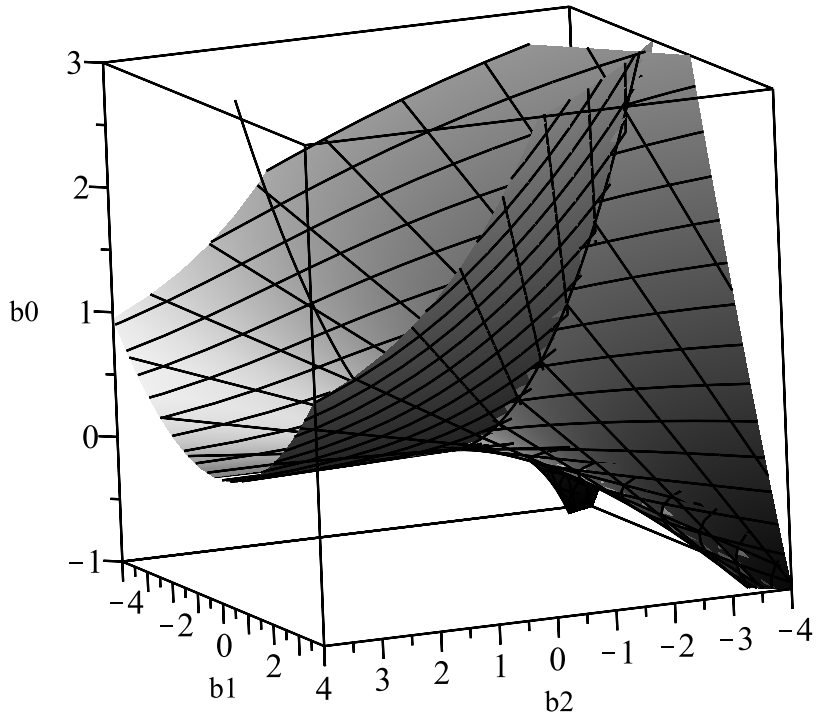

Рис. 2: Discriminant surface for monic quartic polynomial $\widetilde{f}_4(y)$.

6. Resume

The described in Section 3 algorithm of polynomial parametrization of the discriminant set $\mathcal{D}(f)$ of the real polynomial $f(x)$ allows to represent this parametrization in a such form that it is possible to obtain the structure of the discriminant set $\mathcal{D}(f)$ as a finite set of algebraic varieties $\mathcal{V}_i$ of different dimensions from 1 to $n-1$.

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Батхин, А. Б., Брюно А. Д., Варин В. П. Множества устойчивости многопараметрических гамильтоновых систем // ПММ. 2012. Т. 76, № 1. С. 80—133.

2. Батхин А. Б. Устойчивость одной многопараметрической системы Гамильтона // Препринт № 69. М.: ИПМ им. М.В.Келдыша. 2011. 28 с.

3. Грязина Е. Н., Поляк Б. Т., Тремба А. А. Современное состояние метода D-разбиения // Автоматика и Телемеханика. 2008. № 12. С. 3—40.

4. Маркеев А. П. Точки либрации в небесной механике и космодинамике. М.: Наука. 1978.

5. Нейман Н. Н. Некоторые задачи распределения нулей многочленов // УМН. 1949. Т. 4, № 6(34). С. 154—188.

6. Basu S., Pollack R., Roy M.-F. Algorithms in Real Algebraic Geometry. 2006. Springer-Verlag. Berlin Heidelberg New York.

7. Калинина Е. А., Утешев А. Ю. Теория исключения: Учеб. пособие. Спб.: Изд-во НИИ химии СПбГУ. 2002. 72 с.

8. Sylvester J. J. On a theory of syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's function // Trans. Roy. Soc. London (1853).

9. Bézout É. Théorie générale des Équations Algébrique. P.-D. Pierre. Paris. 1779.

10. Habicht W. Eine Verallgemeinerung des Sturmschen Wurzelzählverfahrens // Comm. Math. Helvetici. 1948. Vol. 21, pp. 99–116.

11. Uteshev A. Yu., Cherkasov T. M. The search for the maximum of a polynomial // J. Symbolic Computation. 1998. Vol. 25. no 5. pp. 587–618.

12. Джури Э. Инноры и устойчивость динамических систем. М.: Наука. 1979. 304 с.

13. Oprea J. Differential Geometry and its Applications. The Mathematical Assosiation of America. 2007.

14. Фиников С. П. Теория поверхностей. М.: ГТТИ. 1934. 203 с.

15. Кокс Д., Литтл Д., О'Ши Д. Идеалы, многообразия и алгоритмы. Введение в вычислительные аспекты алгебраической геометрии и коммутативной алгебры. М.: Мир. 2000. 687 с.

REFERENCES

1. Batkhin, A. B., Bruno, A. D. & Varin, V. P. 2012, "Stability sets of multi-parameter Hamiltonian systems" *J. Appl. Math. Mech.*, vol. 76, no. 1, pp. 56–92. doi: 10.1016/j.jappmathmech.2012.03.006

2. Batkhin, A. B. 2011, "Stability of the certain multiparameter Hamiltonian system", Preprint No. 69, *KIAM, Moscow.*

3. Gryazina, E. N., Polyak, B. T., & Tremba, A. A. 2008, "D-decomposition technique state-of-the-art", *Automation and Remote Control*, vol. 69, no. 12, pp. 1991–2026.

4. Markeev, A. P. 1978, "Libration Points in Celestial Mechanics and Cosmodynamics", *Nauka, Moscow.*

5. Neiman, N. N. 1949, "Some problems on the distributions of the zeroes of polynomials", *Uspekhi Mat. Nauk*, vol. 4, no. 6(34), pp. 154–188. (in Russian)

6. Basu, S, Pollack, R & Roy, M-F 2006, "Algorithms in Real Algebraic Geometry", *Springer-Verlag, Berlin Heidelberg New York.*

7. Kalinina, E. A. & Uteshev, A. Yu. 2002, "Elimination theory", *Izd-vo NII Khimii SPbGU, Saint-Petersburg.*

8. Sylvester, J. J. 1853, "On a theory of syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's function", *Trans. Roy. Soc., London.*

9. Bézout, É. 1779, "Théorie générale des Équations Algébrique", *P.-D. Pierre. Paris.*

10. Habicht, W. 1948, "Eine Verallgemeinerung des Sturmschen Wurzelzählverfahrens", *Comm. Math. Helvetici*, vol. 21, pp. 99–116.

11. Uteshev, A. Yu. & Cherkasov, T. M. 1998, "The search for the maximum of a polynomial", *J. Symbolic Computation*, vol. 25, no 5. pp. 587–618.

12. Jury, E. 1974, "Inners and stability of dynamic systems", *John Wiley and Sons.*

13. Oprea, J. 2007, "Differential Geometry and its Applications", *The Mathematical Assosiation of America.*

14. Finikov, S. P. 1934, "Theory of Surfaces", *GTTI, Moscow.*

15. Cox, D., Little, J. & O'Shea, D. 1997, "Ideals, varieties and algorithms: an introduction to computational algebraic geometry and commutative algebra", *Undergraduate Texts in Mathematics, Springer-Verlag, New York.*

Институт прикладной математики им. М. В. Келдыша.
Поступило 30.04.2015