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О числе листов накрытий, определенных системами уравнений в n-мерных пространствах

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Аннотация

Накрытия в соновном рассматриваются в геометрии и анализе, и в некоторых случаях они не задаются явным образом. Задача определения накрытий в конкретной ситуации является очень важным. Накрытия возникают в теории многообразий,в особенности в связи с системами уравнений. Одним из действенных методов в этом направлении является использование теоремы о неявных функциях.

В настоящей статье мы изучаем эти вопросы во требуемом общем виде. Такой подход приводит проблему к рассмотрению основных понятий, которые были изучены классиками математики в последние два столетия. Этими математиками анализированы основные моменты теории, касающиеся поведению многообразий малых размерностей в многообразиях больших размерностей. Определение понятия кривой на плоскости является ярким примером того, как мы должны определить основные понятия, с которыми мы имеем дело, чтобы обеспечить необходимую свободу действий, не умаляя при этом необходимой общности. Bdедение квадрируемых кривых дает возможность развивать приемлемую теорию интегрирования в плоских областях. Однако, этого недостаточно, к примеру для установления теоремы Фубини в той общности, которая рассматривается в теории интегрирования Лебега. Здесь мы наталкиваемся на ограничения внесенные пересечениями многообразия с краем области. Поэтому, плодотоворную формулировку этой теоремы мы наблюдаем лишь в теории интергрирования Лебега. Это и есть один из множества вопросов, котрые связаны с поведением многообразий малых размерностей. Мы показываем, как нужно видоизменить некоторые понятия, чтобы преодолеть такие трудности. Мы установливаем, что обобщение понятия "неявного"поверхностного интеграла в некотором, отличном от традиционнного взглядя понимании, позволяет устранить возникающие трудности и решать поставленные задачи в достаточной общности.

В работе таким путьем удается свести вопрос об оценке числа листов накрытий, определяемых системами уравнений, к некоторым метрическим эадачам терии поверхностных интегралов.

Ключевые слова: накрытия, многообразие, системы уравнений, число листов, жорданова область.

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On number of sheets of coverings defined by a system of equations in n-dimensional spaces

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Abstract

The coverings are mostly used in geometry and analysis, and sometimes they are not given explicitly. The problem on defining of covering in concrete situation is substantive. Coverings arose in theory of manifolds, especially in connection with the system of equations. One of powerful methods in this direction is a theorem on implicit functions.

In the paper we study these questions in a necessary general form. Such a consideration lead the problems to the basic notions which were studied by classics of mathematics in last two centuries. By him it was analyzed the main points of the theory on behavior of manifolds of less dimensions in manifolds of higher dimensions. Defining of the notion of a curve in the plane is bright example showing how we can establish suitable properties of objects we deal with to get the necessary freedom of actions, does not avoiding simplest generality. Introducing of quadrable curves makes possible to develop an acceptable notion of the integral in the domains on the plane. But this is insufficient for establishing for example, the theorem of Fubini on repeated integrals in that form as in Lebesgue's theory. Here we rest to constraints brought by intersection of manifold with boundary. The useful formulation of this theorem is possible to get only in Lebesgue theory of integration. This is one of multiplicity of questions connected with behavior of manifolds of less dimensions. We show how some notions of the theory must be modified to avoid such difficulties. We establish that the generalization of a notion of "improper"surface integral in some different from the ordinary meaning, makes possible solve the problem in general.

In the prsent work we lead by such method the question on estimating of the number of sheets of covering to some metric relations connected with surface integrals.

Keywords: covering, manifold , system of equation, number of sheets, Jordan domain.

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1. Introduction

Let us consider two regular manifolds of equal dimensions M and N. Suppose that f: M→ N is some their map.

Definition 1. This map is called a covering, if following conditions are satisfied:

1) the Jacobian of the map f is distinct from zero in every point of the manifold M;

2) for every point $y \in N$ there exists a neighborhood $y \in U \subset N$ such that the preimage $f^{-1}(U) \subset M$ consists of finite or denumerable family of non-intersecting domains

$$f^{-1}(U) = V_1 \bigcup V_2 \bigcup \cdots ,$$

for which every map $f : V_j \to U$ is diffeomorphism;

3) the manifold is covered by finite or denumerable family of such domains U.

The manifold N is called a base of the covering, and M is called to be space of the covering. In the literature, an another definition equivalent to the introduced above is used. The condition a) sometimes can be omitted, if differentiability of the manifold is not of interest.

In literature another equivalent definition of the notion of covering [1] is widely used. Consider this definition.

Definition 2. The surjective continuous map π: X→ Y of linearly connected space X is called a covering:

1) if for every point a ∈ Y there exists a neighborhood V ⊂ Y for which it can be found a homeomorphism h : π^{-1} (V) → V × Γ , with a discrete space Γ.

2) If p: V × Γ → V is a natural projection then

$$\pi|_{\pi^{-1}(V)} = p \circ h.$$

The space X is called the space of covering; Y is called a base of covering.

The coverings are mostly used in geometry and analysis, and sometimes they are not given explicitly. The problem on defining of covering in concrete situation is substantive. Coverings arose in theory of manifolds, especially in connection with the system of equations. One of powerful methods in this direction is a theorem on implicit functions.

The number of sheets is an important characteristics of coverings and in many questions of analysis and geometry it arises the question on defining or estimating of the number of sheets. The number of sheets does not depend on the point of the base of covering, if the manifold is connected [1]. In general case this question is not easy for investigation. For one class of coverings this question allows solution by using of compactness [1]. Consider one of theorems of such kind.

Let M and N be smooth n-dimensional closed manifolds, and the map $f : M \to N$ is regular (with non-degenerating Jacobian) and surjective. Following theorem is true.

Theorem 1. The map $f : M \to N$ is a covering with finite number of sheets.

The condition on closeness of manifolds is substantive in this theorem. But this theorem does not give tools to advance any conclusions on the quantity of the number of sheets. In questions connected with algebraic or analytic manifolds this question allows solution due to connections of coverings with some groups of transformation of manifolds.

Coverings rather arose when the system of equations in multidimensional spaces are considered. In the present work we shall consider coverings defined by the system of equations in n-dimensional spaces $\mathbb{R}^n$. In some natural conditions we obtain bounds for the number of sheets.

2. Basic theorems on implicit functions

Following lemma is a general form of the theorem on implicit functions in normed spaces [6, 12, 13].

Lemma 1. Suppose we are given with normed spaces X, Y and Z. Let $\Phi : X \times Y \to Z$ be some differentiable map which at the points $a \in X$, $b \in Y$ satisfies the condition $\Phi(a, b) = 0$ and the linear operator $\frac{\partial \Phi(x,y)}{\partial y}$ (the operator of partial differentiation) is continuous and has an inverse in some neighborhood W of the point (a, b), $W = \{(x, y) | \, \|x - a\| < r, \|y - b\| < \rho\} \subset U$. Then there exists such a ball $U\rho = \{x | \, \|x - a\| < \rho < r\}$ of the point a, and a unique map $f : X \to Y$ such that:

1) f is continuous in the ball $U\rho$;

2) the equality $b = f(a)$ is true;

3) for every $x \in U\rho$ the equality $\Phi(x, f(x)) = 0$ is satisfied.

Uniqueness of the function $f(x)$ means: if there exist a pair of functions f_1 and f_2 defined in the balls $U\rho_1$ and $U\rho_1$, then they are coincident in the intersection $U = U_{\rho_1} \cap U_{\rho_2}$.

Let us formulate now the theorem on implicit functions in $\mathbb{R}^n$ for the system of equations.

Lemma 2. Suppose the conditions of Lemma 1 are satisfied. Consider the conditions:

1) There exists a point $(\bar{a}, \bar{b}) = (a_1, ..., a_{n-r}, b_1, ..., b_r)$ for which

$$\left.\begin{array}{r} f_1(a_1, ..., a_{n-r}, b_1, ..., b_r) = 0, \\ \cdots \quad \cdots \quad \cdots \\ f_r(a_1, ..., a_{n-r}, b_1, ..., b_r) = 0. \end{array}\right\};$$

2) There exists a neighborhood W of the point $(\bar{a}, \bar{b})$ in which the functions above have continuous partial derivatives

$$\frac{\partial f_i\left(x_1, ..., x_{n-r}, y_1, ..., y_r\right)}{\partial y_j}, \ i = 1, ..., r; \ j = 1, ..., r.$$

3) In the specified neighborhood of the point $(\bar{a}, \bar{b})$ the following determinant is distinct from zero:

$$\frac{\partial f(\bar{x}, \bar{y})}{\partial \bar{y}} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_r} \\ \cdots & \ddots & \cdots \\ \frac{\partial f_r}{\partial y_1} & \cdots & \frac{\partial f_r}{\partial y_r} \end{pmatrix}.$$

Then there exists a neighborhood U_δ of the point $\bar{a} = (a_1, ..., a_{n-r}) \in R^n$ and the unique system of continuous functions

$$\left.\begin{array}{r} y_1 = y_1(x_1, ..., x_{n-r}), \\ \cdots \quad \cdots \quad \cdots \\ y_r = y_r(x_1, ..., x_{n-r}) \end{array}\right\}$$

for which

$$\left.\begin{array}{r} y_1(a_1, ..., a_{n-r}) = b_1, \\ \cdots \quad \cdots \quad \cdots \\ y_r(a_1, ..., a_{n-r}) = b_r, \end{array}\right\}$$

and in the neighborhood U_δ identically satisfied the following equalities:

$$\left.\begin{array}{r} f_1(x_1, ..., x_{n-r}, y_1\left(x_1, ..., x_{n-r}\right), ..., y_r\left(x_1, ..., x_{n-r}\right)) \equiv 0, \\ \cdots \quad \cdots \quad \cdots \\ f_r(x_1, ..., x_{n-r}, y_1\left(x_1, ..., x_{n-r}\right), ..., y_r\left(x_1, ..., x_{n-r}\right)) \equiv 0. \end{array}\right\}$$

3. Posing of the problem and basic results

Let we are given with some bounded, one-connected, closed Jordan domain Ω, contained in other open domain Ω_0 given in n-dimensional space $\mathbb{R}^n$. Suppose that in the domain

Ω_0 some continuous function $f(\bar{x}) = f(x_1, \ldots, x_n)$ and continuously differentiable functions $f_j(\bar{x}) = f_j(x_1, \ldots, x_n), j = 1, ..., n, r < n$ be given; moreover, suppose that the Jacoby matrix

$$\frac{\partial(f_1, ..., f_n)}{\partial(x_1, ..., x_n)}$$

of this system of functions has in Ω_0 maximal rank. Let, further,$\bar{\xi}_0 = (\xi_1^0, ..., \xi_r^0)$ be a point of an image of the map $f : \bar{x} \mapsto (f_1, ..., f_r)$, and $\bar{x}_0$ be a point in Ω, such that

$$f_1(\bar{x}_0) = \xi_1^0, ..., f_r(\bar{x}_0) = \xi_r^0.$$

If we substitute the point $\bar{x}_0$ by a variable vector $\bar{x}$, we get some system of equations, which due to conditions above, defines some *n-r* dimensional surface in Ω. We assume that this surface has an *n-r* dimensional volume or area. This is possible when the intersection of the surface with the boundary of the domain Ω has *n-r* dimensional Jordan measure being equal to zero. This condition is always satisfied when the domain Ω is bounded by finite number of hyper surfaces of a view

$$\varphi : U \subset R^{n-1} \to V \subset R^n,$$

with continuously differentiable map φ of closed domains U and V. These hypersurfaces can intersect each with other by parts of their boundaries.

The map $f : \bar{x} \to (f_1(\bar{x}), \ldots, f_r(\bar{x}))$ with the system of equations $f_1(\bar{x}) = 0, \ldots, f_r(\bar{x}) = 0$ defines, in some natural conditions on the rank of Jacoby matrix, a covering in Ω. In many applications of analysis, it arises the question on the number of sheets of this covering, that is, the number of elementary surfaces defined by this system of equations. For definiteness, suppose that the Jacoby matrix of the given system of functions has non zero minor plased at first r columns, everywhere in Ω.

We formulate and prove two theorems on coverings defined by such system of equations. Bounding of the number of sheets we lead to some metric relations.

Theorem 2. Take some cube $B \subset \mathbb{R}^{n-r}$ such that the covering with this base has a discrete space Γ. Then for the number of elements of Γ, the following inequality is satisfied:

$$|\Gamma| \leqslant M|B|^{-1} \int_{\Pi} \frac{ds}{\sqrt{G}},$$

where M denotes the maximal absolute value of minors of the Jacoby matrix in B, $|B|$ is a volume of B, and in the right hand side a surface integral taken along the surface Π, defined by the system of given equations, stands.

Theorem 3. In the conditions of the theorem 1 we have:

$$|\Gamma| \leqslant M|B|^{-1} \lim_{h \to 0} \frac{1}{h^r} \int_{\substack{0 < f_1 < h \\ \cdots \\ 0 < f_r < h}} dx_{r+1} \cdots dx_n.$$

where integration is taken along the subset of Ω , defined by inequalities indicated at the foot of the sign of integral.

In the paper we study these questions in a necessary general form. Such a consideration lead the problems to the basic notions which were studied by classics of mathematics in last two centuries. In [10] it was analyzed the main points of the theory on behavior of manifolds of less dimensions in manifolds of higher dimensions. Defining of the notion of a curve in the plane is bright example showing how we can establish suitable properties of objects we deal with to get the necessary freedom of actions, does not avoiding simplest generality. Introducing of quadrable curves makes possible to develop an acceptable notion of the integral in the domains on the plane. But this is insufficient for establishing of the theorem of Fubini on repeated integrals in that form as in Lebesgue's theory [11, 15]. Here we rest to constraints brought by intersection of manifold with boundary. The useful formulation of this theorem is possible to get only in Lebesgue theory of integration. This is one of multiplicity of questions connected with behavior of manifolds of less dimensions. We show how some notions of the theory must be modified to avoid such difficulties. We establish that the generalization of a notion of "improper"surface integral in the meaning of the work [4] makes possible solve the problem in general.

4. Main results and proof of theorems

Consider some statements on the system of equations by means of which we shall define some surfaces and study the number of sheets coverings, defined by this system of equations. The concrete covering connected with this system of equations we shall consider below. At first let us establish the following lemma, which has an independent interest (firstly this result with brief proof was given in [2, 3]; particular case was considered in [9, p. 319]).

LEMMA 3. Let we are given with some bounded, one-connected, closed Jordan domain Ω, contained in other open domain Ω_0 given in n-dimensional space $\mathbb{R}^n$. Suppose that in the domain Ω_0 some continuous function $f(\bar{x}) = f(x_1, \ldots, x_n)$ and continuously differentiable functions

$$f_j(\overline{x}) = f_j(x_1, \ldots, x_n), j = 1, ..., n, r < n$$

be given; moreover, suppose that the Jacoby matrix

$$\frac{\partial(f_1, ..., f_n)}{\partial(x_1, ..., x_n)}$$

of this system of functions has in Ω_0 maximal rank. Let, further,$\bar{\xi}_0 = (\xi_1^0, ..., \xi_r^0)$ be a point of an image of the map $f : \bar{x} \mapsto (f_1, ..., f_r)$, and $\bar{x}_0$ be a point in Ω, such that

$$f_1(\bar{x}_0) = \xi_1^0, ..., f_r(\bar{x}_0) = \xi_r^0.$$

Then in some neighborhood of the point $\bar{\xi}_0$ we have the equality

$$\frac{\partial^r}{\partial\xi_1 \cdots \partial\xi r} \int_{\Omega(\bar{\xi})} f(\bar{x}) d\bar{x} = \int_{M(\bar{\xi})} f(\bar{x}) \frac{ds}{\sqrt{G}},$$

where $\Omega(\bar{\xi})$ is a subdomain in Ω, defined by the system of inequalities $f_j(\bar{x}) \leqslant \xi_j$, and $M(\bar{\xi})$ is a surface defined by the system of equations, G is a Gram determinant of gradients of the functions $f_j(\bar{x})$, that is, $G = |(\nabla f_i, \nabla f_j)|$.

Befor starting the proof of Lemma 3, let us make some important remarks.

REMARK 1. As it was observed in [4] the surface integral arisen above is taken in some "improper"meaning which is distinct form ordinary improper integral, if we consider surface integral as a multidimensional integral after of substitution of the surface element ds by its expression given below. In the last case we would take the integral through the projection of the surface, considering that as $n - r$-dimensional domain. In this case defining of improper integral in ordinary manner demands that the boundary, that is the projection of an intersection of the surface with boundary, has zero Jordan measure. But, in general, this condition may not be satisfied. Below we overcome this difficulty considering the initial integral over Ω in improper meaning. Taking arbitrary positive small ε, we omit the open covering of the boundary of the domain Ω constructed as a union of cubes, with total measure less than ε. Since the remained closed domain is bounded by hyperplanes, then in the conditions of Lemma 1 the surface integral can be defined in ordinary meaning. Improper surface integral we define as a limit (which is existing, as it will be shown below) of the surface integral, as $\varepsilon \to 0$. So, we avoid the necessity to consider an intersection of the surface with the boundary of the domain. It is not difficult to show that the improper surface integrals in two different meanings are coincident when an intersection of the boundary has zero $n-r$-dimensional Jordan measure.

In Lebesgue theory of an integral, there best known Fubini's theorem which relatively easily reduces a multiple integral to repeated integrals. But in the Riemann's integral case, it required to establish an existence of integrals of less multiplicity for every fixed values of some variables. This difficulty, in many cases, is overcome by imposing additional conditions on boundary, besides Jordan measurability. In the light of the said above, from the consequence of Lemma 1 it follows that taking the intersection of the domain by $n-r$ -dimensional plane we see that Fubini's theorem for the Riemann's integral is as so powerful, as in the Lebesgue's integral case. Partial integrals of less dimension in this case must be taken in improper meaning.

Remark 2. When differentiating of the integral, passing to the limit performed from the interior of the domain $\Omega(\bar{\xi})$, which lead to left-hand (or write-hand) partial derivatives. The ordinary derivative and left-hand derivative (if are existing) are coincident everywhere, with exception for enumerable set of points, due to general property of derivative (theorem 11.43, of [11]). In applications, absence an ordinary partial derivative at a subset of zero Jordan measure does not affect main results.

Proof of the lemma 3. Consider a graph Γ of the function $\bar{u} = \bar{f}(\bar{x})$, $\bar{x} \in \Omega$ (here $\bar{f} = (f_1, \ldots, f_r)$), that is, the set of all such pairs $(\bar{x}, \bar{u})$. From the closeness of Ω and continuousness of the given functions it follows that the graph Γ is closed, and, therefore, is compact. Really, if $((\bar{x}_k, \bar{u}_k))_{k\geqslant 1}$ is some sequence from Γ, then due to boundedness and compactness of Ω, it follows that all of limit points of the sequence $((\bar{x}_k))$ belong to the set Ω. Let $\bar{x}$ be a limit point of this sequence corresponding some converging subsequence of the sequence $((\bar{x}_k))$. From continuousness of given functions it follows that for every limit point $\bar{u}$ of the sequence $\left(\bar{u}_k = \bar{f}(\bar{x}_k)\right)$ (specifically for the limit point of taken subsequence also) the point $(\bar{x}, \bar{u})$ belongs to the graph of considered map, that is, the graph is closed.

Suppose that the point $(\bar{x}, \bar{u})$ be any point of the graph. Consider the system of equations $\bar{f}(\bar{x}) - \bar{u} = 0$. Explicitly, this system can be written as follows:

$$\begin{gathered} f_1(\bar{x}_1, \bar{x}_2) - u_1 = 0, \\ \vdots \\ f_r(\bar{x}_1, \bar{x}_2) - u_r = 0. \end{gathered}$$

Let the minor of the Jacoby matrix constructed of first r columns be distinct from zero at a point $\left(\bar{x}_1^0, \bar{x}_2^0\right)$. By the theorem on implicit functions [46, p. 309], there exist open cubic neighborhoods (interiors of closed cubes) U of the point $\bar{u}$, X of the point $\bar{x}_1^0$ and W of the point $\bar{x}_2^0$ such that in the parallelepiped $X \times W \times U$ the graph of the function consists of triples of a view $(\bar{x}_1 = \bar{\varphi}(\bar{x}_2, \bar{u}), \bar{x}_2, \bar{u})$; moreover the function $\bar{x}_1 = \bar{\varphi}(\bar{x}_2, \bar{u})$ (being a diffeomorphism $(\bar{x}_1, \bar{x}_2) \leftrightarrow (\bar{u}, \bar{x}_2)$) is unique, that is, the system is solvable in regard to $\bar{x}_1$ by a unique way. This function is also continuous with respect to pair of variable vectors. In regard to the said above, the graph Γ is a closed set. From the compactness it follows an existence of a finite number of parallelepipeds $P_1, \ldots, P_m$ such that their union covers Γ. Note that the union of their projections into the cube $X \times W$ covers Ω. Applying the known arguments, we may construct, using these neighborhoods, a sequence of such closed Jordan sets $V_1, \ldots, V_m$ which intersect each with any other, possible by pieces of their boundaries, and the union of them overlaps the graph Γ. Moreover, in each of these sets the considered system of equations has a unique solution given by equality of the type $\bar{x}_1 = \varphi_i(\bar{x}_2, \bar{u})$ (thus we do not assume that one of these solutions is a continuation of the solution found in other set); here $(\bar{x}_1, \bar{x}_2) \in R^n$ is some transposition of the components of the vector $(x_1, \ldots, x_n)$, because in various parallelepipeds minors of Jacoby matrix being distinct from zero are taken with respect different set of variables.

Consider one of such closed sets, denoting it as V. V is a Jordan subset in some product $P_i = X_i \times W_i \times U_i$. Let Δ be a projection of V into coordinate space $(\overline{x}_1, \overline{x}_2)$, that is, a projection into $X_i \times W_i$. In consent with the said above, for every fixed $\bar{x}_2 = (x_{r+1}, ..., x_n) \in W_i$ the unique solution of the system of equations $f_j(\bar{x}) - u_j = 0$ (here we denote $\phi\left(\bar{x}_1\right) = \phi(\bar{x}_1; \bar{x}_2) = \bar{f}\left(\bar{x}_1,\ \bar{x}_2\right)$ for fixed $\bar{x}_2$) will be an inverse function $\bar{x}_1 = \phi^{-1}(\bar{u}) = \phi^{-1}(\bar{u}; \bar{x}_2), \bar{x}_2 \in W_i, \bar{u} \in U_i,\ j = 1, ..., r$, such that

$$\bar{f}^{-1}\left(\bar{u}\right) = \left(\phi^{-1}\left(\bar{u}; \bar{x}_2\right), \bar{x}_2\right),\ \bar{f} = (f_1, \ldots,\ f_r).$$

From the said above it follows that the closed sets $\Omega(\bar{\xi})$ and $I(\bar{\xi}) = [m_1, \xi_1] \times \cdots \times [m_r, \xi_r]$ (here m_j denotes a minimal value of the function $f_j(\bar{x})$) are Jordan sets. Replacing in reasoning above the domain Ω by $\Omega(\bar{\xi})$, we obtain a finite family of products of a view $X_0 \times W_0 \times U_0$, containing subsets P_i, the union of which covers the product $\Omega(\bar{\xi}) \times I(\bar{\xi})$. Let us perform the change of variables in the integral

$$J_0(\bar{\xi}) = \int_{X_0 \times W_0 \cap \Omega(\bar{\xi})} f d\bar{x},$$

by using formulae:

$$u_j = \{ \begin{array}{c} f_j(\bar{x}),\ \text{if}\ j = 1, ..., r \\ u_j = x_j,\ \text{if}\ j > r \end{array}$$

(obviously, by the said above, this a bijective map, for each fixed $\bar{x}_2$). Jacobian of this exchange is equal to $|J|^{-1}$, where $|J|$ is a determinant of corresponding minor. Really,

$$det \frac{\partial(x_1, ..., x_n)}{\partial(u_1, ..., u_n)} = \left(det \frac{\partial(u_1, ..., u_n)}{\partial(x_1, ..., x_n)}\right)^{-1} = |J|^{-1}.$$

So,

$$J_0\left(\bar{\xi}\right) = \int_{\left(I\left(\bar{\xi}\right) \cap U_0\right) \times W_0, \left(\phi^{-1}(\bar{u}; \bar{x}_2), \bar{x}_2\right) \in \Omega\left(\bar{\xi}\right)} f|J|^{-1} d\bar{u} dx_{r+1} \cdots dx_n,$$

and for the representation of this integral as a repeated integral of a view

$$\int_{\chi_1}^{\xi_1} du_1 \cdots \int_{\chi_r}^{\xi_r} du_r \int_{W_0, (\phi^{-1}(\bar{u}; \bar{x}_2), \bar{x}_2) \in \Omega(\bar{\xi})} f|J|^{-1} dx_{r+1} \cdots dx_n \tag{1}$$

it suffices existence of the inner integral

$$\int_{W_0, (\phi^{-1}(\bar{u}; \bar{x}_2), \bar{x}_2) \in \Omega(\bar{\xi})} f|J|^{-1} dx_{r+1} \cdots dx_n,$$

for every $\bar{u} \in U_0$. Proof of an existance of this integral, in some improper meaning, will be made below.

In the explicit form the function under the inner integral along W_0 can be found by substituting the variable $\bar{x}_1$ in the function $f\,|J|^{-1}$ by its values found from the system: $\bar{x}_1 = \phi^{-1}(\bar{u}) =$ $= \phi^{-1}(\bar{u}; \bar{x}_2), \bar{x}_2 \in W_0$. Now we note that lower bounds of variation of the variable ξ_i has a view $\chi_i = \chi_i(u_1, ..., u_{i-1})$, for $i \geq 1$. Differentiating with respect to upper variables ξ_i, we obtain:

$$j_0 = \frac{\partial^r}{\partial \xi_1 \cdots \partial \xi_r} J_0\left(\bar{\xi}\right) = \int_{W_0, \left(\phi^{-1}\left(\bar{\xi}; \bar{x}_2\right), \bar{x}_2\right) \in \Omega\left(\bar{\xi}\right)} f|J|^{-1} dx_{r+1} \cdots dx_n,$$

when the integral in the right hand side is continuous with respect to $\bar{\xi}$. Thus, $\bar{x}_1 = \phi^{-1}(\bar{\xi}) =$ $= \phi^{-1}(\bar{\xi}; \bar{x}_2), \bar{x}_2 \in W_0$, and it means that the variable $\bar{x}$ lies on the surface $f(\bar{x}) = \bar{\xi}$. From uniqueness of the solution it follows that the number of these parts (surface integrals) does not exceed the number of neighborhoods of the form $X_0 \times W_0 \times U_0$. Let us transform the integral j_0 into the surface integral. From the told above it follows that at fixed $\xi_1, ..., \xi_r$ for every solution $\overline{x} = (x_1, \dots \ , x_n)$ of the system, defining the surface $M(\bar{\xi})$, the variables $x_1, ..., x_r$ are defined by equalities

$$x_1 = \varphi_1 \left(x_{r+1}, \dots x_n\right),$$
$$\dots\dots\dots$$
$$x_r = \varphi_r \left(x_{r+1}, \dots, x_n\right),$$

in W_0, for some smooth functions $\varphi_1, \dots, \varphi_r$. Consider the surface defined by the map φ, given as

$$\bar{\varphi}(x_{r+1}, \dots, x_n) = \begin{pmatrix} \varphi_1 \left(x_{r+1}, \dots, x_n\right) \\ \vdots \\ \varphi_r \left(x_{r+1}, \dots, x_n\right) \\ x_{r+1} \\ \vdots \\ x_n \end{pmatrix}.$$

The image of the indicated neighborhood of the point $(x_{r+1}^1 \dots x_n^1)$ in such mapping will be just a part M_0 of the surface $M(\bar{\xi})$, the projection of which a part of W_0 serves. Then, in accordance with [13, pp. 292, 327], denoting by D_j minors of an order $n - r$ of the matrix

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial u_{r+1}} & \frac{\partial \varphi_2}{\partial u_{r+1}} & \cdots & \frac{\partial \varphi_r}{\partial u_{r+1}} & 1 & 0 & \cdots & 0 \\ \frac{\partial \varphi_1}{\partial u_{r+2}} & \frac{\partial \varphi_2}{\partial u_{r+2}} & \cdots & \frac{\partial \varphi_r}{\partial u_{r+2}} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_1}{\partial u_n} & \frac{\partial \varphi_2}{\partial u_n} & \cdots & \frac{\partial \varphi_r}{\partial u_n} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

the$(n-r)$ -dimensional volume of an element of the surface (or “surface element”) can be represented as

$$ds = \sqrt{D_1^2 + \dots + D_l^2} dx_{r+1} \cdots dx_n, l = \begin{pmatrix} n \\ r \end{pmatrix}$$

It is easy to observe that this matrix has a block-view: $(\Phi|I)$, and here the matrix Φ is a transposed Jacoby matrix of the system of functions $\varphi_1, \dots, \varphi_r$, and I is a unite matrix of the order $n-r$. It is clear that $\Phi = -J^{-1}F$, where F is a Jacoby matrix of the system of functions $f_j(\bar{x})$, with respect to variables $x_{r+1}, \dots, x_n$, and by J, sometimes, one denotes a matrix of the considered minor, also. So, we have the equality $(-\Phi|I) = J^{-1}\left(F|J\right)$. Therefore, using the symbol t on the top from the left over the matrix to indicate a transposition, we obtain

$$(-\Phi|I) \left(\frac{-{}^t\Phi}{I} \right) = J^{-1} \left(\frac{{}^tF}{{}^tJ} \right) \cdot J^{-1}.$$

Then the surface element can be represented in the view

$$ds = \sqrt{det \left((-\Phi|I) \left(\frac{-{}^t\Phi}{I} \right) \right)} dx_{r+1} \dots dx_n =$$

$$= |detJ|^{-1} \sqrt{det \left((F|J) \left(\frac{{}^tF}{{}^tJ} \right) \right)} dx_{r+1} \dots dx_n =$$

$$= \sqrt{G} |J|^{-1} dx_{r+1} \dots dx_n.$$

By this reason, we have

$$j_0 = \frac{\partial^r}{\partial\xi_1\cdots\partial\xi_r} J_0\left(\bar{\xi}\right) =$$

$$= \int_{W_0, (\phi^{-1}(\bar{\xi},\bar{x}_2),\bar{x}_2)\in\Omega(\bar{\xi})} f|J|^{-1} dx_{r+1}\cdots dx_n = \int_{M_0} \frac{f ds}{\sqrt{G}}, \tag{2}$$

where at the end of the chain of equalities a surface integral stands. We have proved this relation on M_0. It does not depend on the minor which at the beginning of reasoning was accepted to be distinct from zero. So, these conclusions hold true for all products of a view $X_0 \times W_0 \times U_0$, which cover the product $\Omega(\bar{\xi}) \times I(\bar{\xi})$. Summing now over all such products, preferably representing the sums as a sum of surface integrals taken along the parts of the surface does not having intersections by their inners, we find a needed result.

We have got the relation (2) in condition on continuousness of the function, indicated above, with respect to $\bar{\xi}$. To complete the proof of the theorem we must make spent above calculations without assumption on continuousness of the inner integral in (1).

By the conditions of the theorem, the theorem on implicit functions delivers the solution of considered system in some wider domain, than the Ω. First of all, consider the expression (1) in the case $r = 1$, noting that the general case can be settled by an analogy. Let us denote

$$g(\xi) = \int_{\chi}^{\xi} du \int_{W_0, (\phi^{-1}(u;\bar{x}_2),\bar{x}_2)\in\bar{\Omega}(\bar{\xi})} f|J|^{-1} dx_{r+1}\cdots dx_n, \tag{3}$$

considering the case $r = 1$. We cannot, generally, state that the function under the integral is continuous with respect to u.

It is clear that the function $g(\xi)$ defined by the formula (3) is a function of bounded variation. Then, by the consequence 2 of Theorem 6, [11, p. 206], this function has at most enumerable number of points of discontinuity, moreover at every point ξ of discontinuity both limits below are exist

$$g\left(\xi+0\right) = \lim_{x\to\xi+0} g(\ x),\quad g\left(\xi-0\right) = \lim_{x\to\xi-0} g(\ x).$$

In other hand, this function is differentiable, and therefore is continuous at every point ξ at which the function under the integral in (3), that is, the function

$$\rho\left(u\right) = \int_{W_0, (\phi^{-1}(u;\bar{x}_2),\bar{x}_2)\in\Omega(\bar{\xi})} f|J|^{-1} dx_{r+1}\cdots dx_n,$$

is continuous at the point $u = \xi$; here the minor $|J|$ coincides with the modulus of partial derivative of the function f_1 with respect to, say x_1. Therefore, we need to investigate the function $\rho\left(u\right)$ under the integral, without any assumptions on continuity. Note that the special case of the integral $g\left(\xi\right)$, when the boundary is constructed of algebraic surfaces of a special form, the question is studied in [5]. In general case, we establish below the same result with the natural condition that derivative must be taken left- or right- hand meaning.

We had dissected the initial domain, using the theorem on implicit functions, into domains in every of which performed change of variables is one to one. This change of variables makes possible to introduce in a new system of coordinates: correspondence $(x_1,\ x_2,\ldots,x_n) \leftrightarrow (u,\ x_2,\ldots,x_n)$ is one to one. Moreover, the image of this map is a Jordan domain also. Do not destroying the generality, we assume that in whole domains Ω and Ω_0 performed change is bijective. So, we can instead of covering for the boundary of the domain Ω in coordinates $(x_1,\ x_2,\ldots,x_n)$ take a covering in coordinates $(u,\ x_2,\ldots,x_n)$. The last is more suitable, because in this case the ribs of covering is parallel to the coordinate axes.

There are close relations between these two coverings. Let us estimate variation of the function $u = f\left(x_1, x_2, \ldots, x_n\right)$ when the variables $x_2, \ldots, x_n$ are fixed. Take the values of this function in two neighbor points x_1' and x_1''. By the theorem on mean values, there exists a point $\theta, x_1'' > \theta > x_2'$ such that we may write

$$|u_1 - u_2| = \left| \frac{\partial f}{\partial x_1}(\theta, x_2, \ldots, x_n) \left(x_1'' - x_2'\right) \right| \leqslant K \left|x_1'' - x_2'\right|,$$

where

$$K = \max_{\bar{x} \in \Omega} |\partial f / \partial x_1(\bar{x})| .$$

This relation shows that every cube of a given covering (in the first system of coordinates) is possible recover by no more than $2[K]+1$ cubes in new coordinates. Then every covering of the domain Ω with total measure $\leqslant \varepsilon$ is possible substitute by covering in coordinates $(u,\ x_2, \ldots, x_n)$, with total measure $\leqslant (2[K]+1)\varepsilon$. So, we assume that the covering is given in coordinates $(u,\ x_2, \ldots, x_n)$, with total measure $\leqslant \varepsilon$.

Let $L(\varepsilon)$, for every positive ε, denote the union of cubes containing the boundary of $\Omega\left(\bar{\xi}\right)$, with total measure not exceeding ε. Consider the set of all vertex points of the cubes of this covering. Taking the set of all i-th coordinates of these vertices denote it as $A_i, i = 1, ..., n$. Taking hyperplains parallel to coordinate axes, containing the points from the sets A_i, we get a dissection of the domain Ω by parallelepipeds among which the cubes of covering are taking part. Since

$$J_0(\xi) = \int_{X_0 \times W_0 \cap \Omega(\xi)} f d\bar{x} = \lim_{\varepsilon \to 0} \int_{X_0 \times W_0 \cap \Omega(\xi) \setminus L(\varepsilon)} f d\bar{x},$$

then

$$\frac{d}{d\xi} \int_{\Omega(\xi)} f(\bar{x}) d\bar{x} = \lim_{\lambda \to 0} \lambda^{-1} \lim_{\varepsilon \to 0} \int_{\xi - \lambda < f_1(\bar{x}) \leqslant \xi, \bar{x}_2 \in \Omega(\xi) \setminus L(\varepsilon)} f(\bar{x}) d\bar{x}$$

Below we show for every u, that the integral $\rho(u)$ has left-hand (or right-hand) derivative. An existence of that integral follows from the reasoning above, in improper meaning, passing to the limit in (3), after of substituting there the domain of integration $\Omega(\bar{\xi})$ by the domain $\Omega(\bar{\xi}) \setminus L(\varepsilon)$.

From the said above, we obtain a following representation for the integral

$$\rho(u) = \int_{W_0, (\phi^{-1}(u; \bar{x}_2), \bar{x}_2) \in \Omega(\bar{\xi})} f |J|^{-1} dx_{r+1} \cdots dx_n :$$

$$\rho(u) = \lim_{\varepsilon \to 0} \Delta_1(u, \varepsilon),$$

where

$$\Delta_1(u, \varepsilon) = \int_{W_0, (\phi^{-1}(u; \bar{x}_2), \bar{x}_2) \in \Omega(\bar{\xi}) \setminus L(\varepsilon)} f |J|^{-1} dx_{r+1} \cdots dx_n,$$

is an integral taken in the domain $\Omega\left(\bar{\xi}\right) \setminus L(\varepsilon)$. Let's examine the integral

$$\Phi(\lambda, \varepsilon) = \lambda^{-1} \int_{\xi - \lambda}^{\xi} \Delta_1(u, \varepsilon) \, du. \tag{4}$$

Then, considered above derivative is possible represent as a repeated limit

$$\lim_{\lambda \to 0} \lim_{\varepsilon \to 0} \Phi(\lambda, \varepsilon) \quad ,$$

which in consent with the said above, coincides with left-hand derivative of $g(\xi)$, that is with $g'(\xi - 0)$.

Making the change of variable $\xi - u = \lambda t$, we may write

$$\Phi(\lambda, \varepsilon) = \int_0^1 \Delta_1(\xi - \lambda t, \varepsilon) \, dt =$$

$$= \int_0^1 \left(\int_{f_1(\bar{x})=\xi-\lambda t,\ \Omega(\bar{\xi})\setminus L(\varepsilon)} f|J|^{-1} dx_2 \cdots dx_n \right) dt.$$

Considering inner integral in Lebesgue sense, one can represent it as a difference of two integrals with non-negative functions under integration, as below:

$$\int_{f_1(\bar{x})=\xi-\lambda t,\Omega(\bar{\xi})\setminus L(\varepsilon)} f|J|^{-1} dx_2 \cdots dx_n =$$

$$= \int_{f_1(\bar{x})=\xi-\lambda t,\Omega(\bar{\xi})\setminus L(\varepsilon),\ f\geqslant 0} f|J|^{-1} dx_2 \cdots dx_n -$$

$$- \int_{f_1(x)=\xi-\lambda t,\Omega(\bar{\xi})\setminus L(\varepsilon),\ f<0} (-f)|J|^{-1} dx_2 \cdots dx_n$$

Both integrals are monotone and bounded with respect to $\varepsilon \to 0$. Therefore, $\Phi(\lambda, \varepsilon)$ tends to some limit function $\Phi(\lambda)$ as $\varepsilon \to 0$. The integrals in the right hand side of the last equality have the same view and can be investigated by similar way. Take some sequence of positive numbers $\varepsilon_1 > \varepsilon_2 >, ...,$ tending to 0, such that $L(\varepsilon_{m+1}) \subset L(\varepsilon_m)$, for $m \geq 1$. Let us consider the sequence of functions:

$$s_m(\lambda)(\bar{x}) = f|J|^{-1},\ if \bar{x} \in \Omega\left(\bar{\xi}\right) \setminus L(\varepsilon_m) \wedge f_1(\bar{x}) = \xi - \lambda t,$$

$s_m(\lambda)(\bar{x}) = 0$, otherwise. This sequence is monotonic and positive which satisfies the conditions of the theorem 10.82 of the book [11]. Then we have

$$\lim_{m\to 0} s_m(\lambda) = \Phi^+(\lambda).$$

The analogical relation is valid for the function $\Phi^-(\lambda)$, got by substituting in the above integral the condition $f \geqslant 0$ by the condition $-f > 0$. Therefore, the sum $\Phi^+(\lambda) + \Phi^-(\lambda) = \Phi(\lambda)$ is existing. But this sum is equal to

$$\int_{f_1(\bar{x})=\xi-\lambda t,\Omega(\xi)} f|J|^{-1} dx_2 \cdots dx_n,$$

in improper meaning.

Prove that the convergence $\Phi(\lambda, \varepsilon) \to \Phi(\lambda)$ is uniform with respect to λ. In consent with above, in the domain $\Omega(\xi) \setminus L(\varepsilon)$ the equation $f_1(\bar{x}) = u$ has a solution $\psi(u, x_2, ..., x_n)$ continuously depending on u in every parallelepiped, constructed above by hyperplanes, having non-empty intersection with hyperplane $u = u_0$. Take arbitrarily small $\eta > 0$. For taken $u = \xi - t\lambda$ there is a number $\delta > 0$ such that

$$|\psi(u, x_2, ..., x_n) - \psi(u_0, x_2, ..., x_n)| < \eta,$$

when $|u - u_0| < \delta$.

Recalling the definition of the function $\Phi^+(\lambda)$ let us estimate the difference $|\Phi^+(\lambda) - \Phi^+(\lambda_0)| < \varepsilon$ when $|\lambda) - \lambda_0)| < \delta$. Note that from the boundedness of the domain Ω it follows that the integrals

$$\int_{f_1(\bar{x})=\xi-\lambda t,\Omega(\xi)} dx_2 \cdots dx_n$$

are bounded by some positive constant $D > 0$, for all t and λ. The function under the integral is continuous in the closed domain Ω. Then it is uniform continuous. Therefore, for given positive κ there exists η such that

$$|\int_{f_1=u} f|J|^{-1} dx_2 \cdots dx_n - \int_{f_1=u_0} f|J|^{-1} dx_2 \cdots dx_n| \leqslant D\kappa,$$

when $|\psi(u, x_2, ..., x_n) - \psi(u_0, x_2, ..., x_n)| < \eta$. Changing u_0 we see that the system of intervals of a view $|u - u_0| < \delta$ covers the all segment $[m, M]$ of variation of the variable u. From the compactness of this segment it follows existence of a finite number of values of u_0 for which the union of these neighborhoods covers the segment $[m, M]$. So, from notes above it follows that the convergence $\Phi(\lambda, \varepsilon) \to \Phi(\lambda)$ is uniform with respect to λ, because $\xi - u = \lambda t \leqslant$. Since the function $\Phi(\lambda, \varepsilon)$ is continuous at $\varepsilon = 0$ (because the value of the function is defined above as an existing limit), then, by the notes on the page 86 of the book [6], from these conclusions one decides that

$$\lim_{\lambda \to 0} \lim_{\varepsilon \to 0} \Phi(\lambda, \varepsilon) = \lim_{\varepsilon \to 0} \lim_{\lambda \to 0} \Phi(\lambda, \varepsilon).$$

So, we have showed that the left-hand derivative $g'(\xi - 0)$ exists and has a representation (in improper meaning, said about above):

$$g'(\xi - 0) = \int_{f_1(\bar{x}) = \xi,\ \Omega(\xi)} f|J|^{-1} dx_2 \cdots dx_n.$$

By an analogy, considering the limit

$$\Phi(\lambda, \varepsilon) = \lambda^{-1} \int_{\xi}^{\xi + \lambda} \Delta_1(u, \varepsilon)\, du,$$

we prove an existence of the right hand side derivative $g'(\xi + 0)$.

It is known ([14]) that the set of points ξ at which $g'(\xi - 0) \neq g'(\xi + 0)$ is finite or at most enumerable, we see that in the formulation of the lemma it is sufficient consider the left-hand derivative. In applications the set of points at which $g'(\xi - 0) \neq g'(\xi + 0)$ is not substantive. At such points, for definiteness we can accept the value of the derivative to be equal to $g'(\xi - 0)$ or $g'(\xi + 0)$, or more symmetrically, to $\frac{g'(\xi - 0) + g'(\xi + 0)}{2}$. The proof of Lemma 3 in the case of $r = 1$ is finished. The general case can be considered by an analogy. Lemma 3 is proved.

Note. The lemma 1 remains true, if the point $\bar{\xi}$ is a point of the boundary.

Consequence.Let conditions of Lemma 3 be satisfied. Then we have

$$\int_{\Omega} f(\bar{x}) d\bar{x} = \int_{m_1}^{M_1} \cdots \int_{m_r}^{M_r} du_1 \cdots du_r \int_{M} f(\bar{x}) \frac{ds}{\sqrt{G}},$$

where m_j and M_j, correspondingly, denote minimal and maximal values of $f_j(\bar{x}), j = 1, ..., r$, $M = M(\bar{u})$ denotes a surface in Ω defined by the system of equations $f_j = u_j$, $j = 1, ..., r$, and G is a Gram's determinant of gradients of functions defining M.

The statement of this consequence easily follows from Lemma 1 by integration, and it required to notice that at some values of $\bar{u}$ the surface $M = M(\bar{u})$ can degenerate into empty set. Obviously, that the statement of Lemma 1 is possible extend to the case of non-isolated point $\bar{\xi}_0$ belonging to the boundary. This consequence has many applications ([2, 3]). In [5, p.278], is given a generalization of the consequence of Lemma 1, named as co-area formula.

Now we can now prove Theorems 2-3. We assume that all of conditions imposed in the section 3 (that is the conditions of Lemma 3) are satisfied.

Proof of Theorem 2. Take any inner point $\bar{x}_0 \in \Omega$, and suppose that the image of this point by the map $f : \bar{x} \mapsto (f_1, ..., f_r)$ is an inner point also. Then Lemma 1 on implicit functions is applicable. We find such an open ball $S(\bar{x}_0, r)$ with the center at the point $\bar{x}_0$ and radius $r > 0$, contained in Ω_0 in which a given system of equations has, in imposed conditions, a solution $\bar{x}_1 = \varphi(\bar{x}_2)$, $(\bar{x}_1, \bar{x}_2) \in S(\bar{x}_0, r)$, "defined"in the ball $S(\bar{x}_0, r)$. Take a continuation of this solution along continuous curves lying in the interior of the domain Ω_0. We get unique solution $\bar{x}_1 = \varphi(\bar{x}_2)$ of the given system with the conditions: 1) the point $(\bar{x}_1, \bar{x}_2)$ is inner point of the domain Ω_0; 2) the point $\bar{x}_0$ belongs to the graph of the solution, that is $\bar{x}_0 = (\bar{x}_1, \bar{x}_2)$ and $f(\bar{x}_0) = 0$. If there is a solution of the given system does not satisfying above two conditions then we, repeating reasoning spent above, can define uniquely another solution with two conditions, lying in the interior of the domain Ω_0. And so on. From the compactness of tha domain Ω it follows finiteness of the number of such solutions. In the conditions of Theorem 2 consider covering defined by given system of equations with the base B. If $\bar{x}_1^1 = \varphi^1(\bar{x}_2), ..., \bar{x}_1^N = \varphi^N(\bar{x}_2)$ are all solutions a domain of definition of which contained the cube B, then some of solutions $\bar{x}_1 = \varphi(\bar{x}_2)$ of the system can not be defined in B.

Applying Lemma 3, with $f \equiv 1$ we obtain:

$$\frac{\partial^r}{\partial \xi_1 \cdots \partial \xi r} \int_{\Omega(\bar{0})} f(\bar{x}) d\bar{x} = \int_{M(\bar{0})} \frac{ds}{\sqrt{G}},$$

where $\Omega(\bar{0})$ is a subdomain in Ω, defined by the system of inequalities $f_j(\bar{x}) \leqslant 0$, and $M(\bar{0})$ is a surface defined by the system of equations $f_j(\bar{x}) = 0$, G is a Gram determinant of gradients of the functions $f_j(\bar{x})$, that is, $G = |(\nabla f_i, \nabla f_j)|$. Take the subdomain of Ω the projection of which into the space $\mathrm{R}^{\mathrm{n-r}}$ is B. Then we have

$$\int_{M(\bar{0})} \frac{ds}{\sqrt{G}} \geq N \int_B \frac{d\bar{x}_2}{|J|} \geq NBM^{-1}.$$

Statemant of Teorem 1 follows from here, if we note that $|\Gamma| = N$.

Proof of Theorem 3. Integral standing in the right part of the relation of Theorem 2 is possible represent as a limit

$$\lim_{h \to 0} \frac{1}{h^r} \int_{\substack{0 < f_1 < h \\ \cdots \\ 0 < f_r < h}} dx_{r+1} \cdots dx_n.$$

The proof of Theorem 3 is finished.

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