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Об оценке сумм характеров с последовательностями Битти, связанными с составными модулями

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Аннотация

Неоднородные последовательности Битти играют важную роль в играх Витгофа и инвариантных играх, например, о том, как победить противника в играх Витгофа на трех фронтах, и придают свойства решению процедуры, опираясь только на несколько алгебраических тестов. В этой статье обсуждается мощность сумм характеров и их оценка относительно неоднородных последовательностей Битти $\beta_\alpha = \lfloor \alpha n + \beta : n = 1, 2, 3... \rfloor$, где β действительные числа и α положительное является иррациональным. Чтобы оценить мощность, используется измерения количества равномерного распределения последовательностей Битти. При оценке дробной части последовательностей Битти используется известный принцип «ячейки». При этом, неравенства Коши применяются для разложения сумм двойных характеров. Затем оценка сумм двойных характеров получается путем применения свойств сумм аддитивных и мультипликативных характеров. Результат оценки в этом исследовании по составным модулям является более общим по сравнению с предыдущими исследованиями, которые проводились только по простым модулям.

Ключевые слова: Мощность,оценка,конечные группы,сумма характеров, аддитивный характеры, мультипликативный характеры, последовательность битти, теория чисел, принцип «ячейки», рациональное число, иррациональные числа.

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On cardinality of character sums with Beatty sequences associated with composite modules

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Abstract

Non homogeneous Beatty sequences play important rules in Wythoff games and invariant games such as on how to beat your Wytoff games opponent on three fronts and give properties into a decision of the procedure relying only on a few algebraic tests. This paper discusses on the cardinality of character sums and their estimation with respect to non homogeneous Beatty sequences $\beta_\alpha = \lfloor \alpha n + \beta : n = 1, 2, 3... \rfloor$ where β in real numbers and α greater than zero is irrational. In order to estimate the cardinality, the discrepancy is used to measure the number of uniform distribution for Beatty sequences. Pigeonhole principle is discussed on the estimation of the fractional part of Beatty sequences involve. Meanwhile, Cauchy inequalities is applied to expand the double character sums. Then, the cardinality of double character sums is obtained by applying the extension properties of additive and multiplicative character sums. The result obtained is depend on the existing of identity of additive and multiplicative character sums and the uniformly distribution modulo 1. The result of the estimation in this study over composite modules is more general compared to previous studies, which only cover prime modules.

Keywords: cardinality, estimation, finite groups, sum of characters, additive characters, multiplicative character, Beatty sequences, number theory, pigeonhole principle, rational number, irrational numbers.

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1. Introduction

Beatty equences appear in various mathematics problems because of their versatility and arithmetic properties. There are two types of Beatty sequences which are homogeneous and nonhomogeneous cases. In this paper, non-homogeneous Beatty sequences are applied.

The sequences of integers $\lfloor \alpha n + \beta \rfloor$ where α, β be fixed reals numbers. All types o [1]-[3] and etc since the late 19th century. Nowadays, nonhomogeneous cases have been studied extensively by several authors such as [4]-[7] and etc.

In the game theory of Wythoff games, the properties of Fibonacci and Beatty sequences play important rules. Fraenkel [8] give some theory on how to beat your Wythoff games opponent on three fronts by applying Beatty sequences. In invariant game, Cassaigne et al. [9] apply Beatty sequences properties into a decision procedure relying only on a few algebraic tests.

The estimation on Beatty sequences is started in [10], [11] by using single character sums. The estimation of double character sums has been introduced by Friedlander [12] in the form of

$$\sum_{a \in A} \sum_{b \in B} \chi(a+b)$$

In reference [4], the results on the size of the least quadratic non-residue of the nonhomogeneous case are improved and a new approach is introduced to obtain the bounds of character sums of Beatty sequences associated with prime numbers. Therefore, in this paper, the estimation of the cardinality associated with composite modules is obtained by extending the bound of double character sums [4].

Furthermore, there is a slight difference in the properties when compared to those associated with odd primes. The method used closely follows Bank et al. [4] because they have improved the bounds on the size of the least quadratic nonresidue. The result yields explicit bounds on the error term.

The properties of the character sums approach are capable to identify the number of solutions of equations over finite fields. In general, these sums can be formed by using the value of one or more characters.

The following results in Lidl et al. [13] discuss character sums associated with prime modulo. Let p is an odd prime number and $\mathbb{F}_p^*$ be a multiplicative group, where $\mathbb{F}_p^*$ is a cyclic subgroup of order $q-1$. The following propositions are the properties of character sums associated with primes.

Proposition 1.1 Let g be primitive elements of $\mathbb{F}_p$ with order $p-1$ and for each fixed integer of j, where $0 \leq j \leq p-1$. Then, a multiplicative character of $\mathbb{F}_p$ is

$$\chi_j(g^k) = e^{\frac{2\pi ijk}{p-1}} \quad \text{where} \quad k = 0, 1, ..., p-1.$$

Proposition 1.2 For additive character χ_a and χ_b where $a, b \in \mathbb{F}_p$. Then,

$$\sum_{c \in \mathbb{F}_p} \chi_a(c)\overline{\chi}_b(c) = \begin{cases} p+1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

For multiplicative character, if $a, b \in \mathbb{F}_p^*$, then

$$\sum_{\chi} \chi_c(a)\overline{\chi}_c(b) = \begin{cases} p & \text{if } a = b \\ 0 & \text{if } a \neq b, \end{cases}$$

where the sum is extended over all multiplicative character χ of $\mathbb{F}_p$ and $\overline{\chi}_c(b)$ character associated with to character $\chi_c(b)$.

The properties of character sums over composite modules are obtained in [14].

Beatty sequences have been used to investigate the availability of each movement in invariant games. The game can be played anywhere inside the game board. By using Beatty sequences, a wider class of pairs of complementary sequences and a process of generalization the notion of a subtraction game can be obtained in [15] The notations $U = O(V)$, $U \ll V$, and $V \gg U$ are applied equivalent to the assertion $|U| \leq cV$ for some constant $c > 0$. The constants symbols O, $\gg$ and $\ll$ may depend on the real number α. A function which tends to zero and depends only on α is denoted as $o(1)$. It is important to note that our bounds are uniform with respect to all of the other parameters, in particular, with respect to β.

Note that, the letters k, m, and n with or without subscripts are non-negative integers.

In this paper, non-homogeneous Beatty sequences with extended bounds of distributions associated with composite modules are considered. First, consider the sum of the form

$$S_m(\alpha,\beta,\chi;N) = \sum_{n\leq N} \chi(\lfloor \alpha n + \beta \rfloor), \tag{1}$$

where α is irrational and χ is a non-trivial multiplicative character modulo a composite number. We expect that the extended bounds of distributions on the cardinality of double character sums depend on the order of $\varphi(m)$. This result is more general cases compare to the prime case in previous studies.

2. PRELIMINARIES

In this section, a few related definitions and lemmas are listed.Let Q and $\bar{Q}$ be the set of rational and irrational numbers, respectively, i.e. $Q \cup \overline{Q} = \mathbb{Q}$ is the set of real numbers. Let $k \leq N$, N be a natural number and $\triangle \in (0, 1]$ be a rational number. Suppose that $\gamma \in \mathbb{Q}$. Then, we will obtain the following fractional part:

$$\mathcal{N}_\gamma = \{1 \leq n \leq N : \{\alpha n + \beta - \gamma\} < 1 - \triangle\},$$

$$\mathcal{K}_\gamma = \{1 \leq k \leq K : \{\alpha k + \gamma\} < \triangle\},$$

$$\mathcal{N}^c_\gamma = \{1, 2, 3, 5, 8 \ldots, N\} \backslash \mathcal{N}_\gamma.$$

Fix $\gamma \in \mathbb{Q}$, and the notation for the interval is as follows.

$$\mathcal{N} = \mathcal{N}_\gamma \quad \mathcal{N}^c = \mathcal{N}^c_\gamma \quad \text{and} \quad \mathcal{K} = \mathcal{K}_\gamma.$$

The definition of homogeneous and non-homogeneous Beatty sequences are given as follows.

Definition 2.1. Let $\alpha \in \bar{Q}$ and $n \in \mathbb{N}$. If $\alpha > 1$, then $\lambda = \frac{\alpha}{\alpha-1}$ is also an irrational number. $\mathbf{B}_\alpha = [\alpha], [2\alpha], [3\alpha], \ldots$, $\mathbf{B}_\lambda = [\lambda], [2\lambda], [3\lambda], \ldots$, satisfying the following condition $\frac{1}{\alpha} + \frac{1}{\lambda} = 1$. The condition gives a pair of complimentary Beatty sequences.

Definition 2.2. Let n be a positive integer and h be a real number. The non homogeneous Beatty sequences are defined by

$$\mathbf{B}_\alpha = [\alpha + h], [2\alpha + h], [3\alpha + h], \ldots$$

The complement of non-homogeneous Beatty sequences is of the form

$$\mathbf{B}_\lambda = [\lambda + H], [2\lambda + H], [3\lambda + H], \ldots$$

which satisfy

$$H = h(1 - \lambda) \text{ and } \lambda = \frac{\alpha}{\alpha - 1}.$$

Non-homogeneous Beatty sequence considered in this study is the set of

$$\mathbf{B}_{\alpha,\beta} = \{\lfloor \alpha n + \beta \rfloor : n = 1, 2, 3, ...\},$$

where α, β are real numbers.

Consider the functions of a real variable x implicitly ranging in the form $x \gg x_0$. Then, the following notations are defined.
$f(x) = O(g(x))$ is equivalent to $f(x) \ll g(x)$.
$h(x) \gg f(x)$ is applied equivalently to the assertion $|f(x)| \leq c \cdot h(x)$ for some constant $c > 0$.
The constants symbols O, $\gg$ and $\ll$ may be conditional on the real number α, but are otherwise absolute [16]. Moreover, $o(1)$ denotes a function that goes to zero and only depends on α. Non-negative integers are denoted by the letters k, m, and f, with or without subscripts. Next, the fractional part $\{x\}$ of a real number x is denoted in the definition as follows.

Definition 2.3. Let x be a real number and $[x]$ be an integral part of x. Then, the fractional part is

$$\{x\} = \lfloor x \rfloor - x.$$

It is the greatest integer less than or equal to x, similar to the distance notation from the real number x, which is denoted by

$$\|x\| = \min_{n \in \mathbb{Z}} |x - n|.$$

As an example in Beatty sequences from Definition 3, if $\{\alpha n + \beta - \gamma < 1 - \triangle\}$ and $\{\alpha k + \gamma < \triangle\}$, where $n \in \mathcal{N}$ and $k \in \mathcal{K}$, respectively. Then, we have

$$\begin{aligned} \lfloor \alpha(n+k) + \beta \rfloor &= \alpha(n+k) + \beta - \{\alpha(n+k) + \beta\} \\ &= (\alpha n + \beta - \gamma) + (\alpha k + \gamma) - \{\alpha n + \beta - \gamma\} - \{\alpha k + \gamma\} \\ &= (\alpha n + \beta - \gamma) - \{\alpha n + \beta - \gamma\} + (\alpha k + \gamma) - \{\alpha k + \gamma\} \\ &= \lfloor \alpha n + \beta - \gamma \rfloor + \lfloor \alpha k + \gamma \rfloor. \end{aligned} \tag{2}$$

The following illustration describes the above expression (2) on the fractional part of Beatty sequences.

Let $n = 23$, $\alpha = \sqrt{3}$, $\beta = 1.1$, and $\gamma = 0.3$. Then, $\alpha n + \beta - \gamma = 19.3526$ and the fractional of $\{\alpha k + \gamma\} < 0.3628$. Suppose $k = 11$, then we will have

$$\lfloor \alpha(n+k) + \beta \rfloor = \lfloor 59.6897 \rfloor = 59$$

$$\lfloor \alpha n + \beta - \gamma \rfloor + \lfloor \alpha k + \gamma \rfloor = \lfloor 40.6372 \rfloor + \lfloor 19.3526 \rfloor = 59.$$

Thus, it satisfies (2). Additionally, we use the following notation: $\#A$- cardinality set A, K- some natural number, $\Delta \in (0, 1]$-rational numbers.

By applying equation (2) in equation (1), we obtained the following equation.

$$W = \sum_{n \in N} \sum_{k \in K} \chi(\lfloor \alpha(n+k) + \beta \rfloor). \tag{3}$$

The sequence discrepancy is introduced as a quantity that measures the sequence's deviation from an ideal distribution. The following Definitions 2.4 and 2.5 define the sequence's discrepancy, D, and the uniform distribution modulo 1. Their examples can be obtained in [17].

Definition 2.4. The discrepancy, D, is defined as

$$D = \sup_{I \subseteq [0,1)} \left| \frac{V(I, M)}{M} - |I| \right|,$$

where $V(I, M)$ is the cardinality of the set $\{1 \leq m \leq M : \gamma_m \in I\}$, $|I|$ is the length of I and the supremum is taken over all subintervals $I = (a, b)$ of the interval $[0, 1)$. For an ideal distribution of sequences, the discrepancy can measure the number of uniform distribution sequences. The discrepancy, D of M sequences is not necessarily distinct since $\{\gamma_1, \gamma_2, ..., \gamma_M\} \in [0, 1)$ are real numbers.

Definition 2.5. Let $w = x_n$ be a sequence of a real number and $A(E; N; w)$ is denoted by the number of terms in $x_n \in E$. The sequences w is said to be uniformly distributed modulo 1 if and only if

$$\lim_{N\to\infty} \frac{A([a, b); N; w)}{N} = b - a$$

for all half-open interval $[a, b)$ with $0 \leq a < b \leq 1$.

The following elementary statement and the proof of Lemma 2.1 are from [4]. They discuss the cardinality of the fractional part of Beatty's sequences and apply Pigeonhole principle.

Lemma 2.1 Let α be a fixed irrational number. Then, for every positive integer M and real number $\delta \in (0, 1]$, there exists a real number γ such that

$$\#\{m \leq M : \{\alpha m + \gamma\} < \delta\} \geq 0.5M\delta.$$

2.1.Properties of Character Sums Associated With Composite Modules

Suppose that m is a composite number from primitive elements. The following lemmas are the features of character sums extended to composite modules.

Lemma 2.1.1. Let $\mathbb{F}_m$ be a finite group of order $\varphi(m)$ and g be a cyclic subgroup of $\mathbb{F}_m$ of order μ. A multiplicative character of $\mathbb{F}_m^*$ is

$$\chi_j(g^k) = e^{\frac{2\pi ijk}{\mu}} \quad \text{where} \quad k = 0, 1, ..., \varphi(m) - 1,$$

where g is a fixed primitive element of $\mathbb{F}_m$ with order μ and j is a fixed integer, $0 \leq j \leq \varphi(m) - 1$.

Proof. The proof of this result is given in [14].

When orthogonality relations are applied to additive or multiplicative characters sums to $\mathbb{F}_m$, the following fundamental identities are obtained:

Lemma 2.1.2. For additive character χ_a and χ_b,

$$\sum_{c\in\mathbb{F}_m} \chi_a(c)\overline{\chi}_b(c) = \begin{cases} \varphi(m) + 1, & \text{if } a = b \\ 0, & \text{if } a \neq b. \end{cases}$$

For multiplicative character, if $c, d \in \mathbb{F}_m^*$, then

$$\sum_{\chi} \chi_c(a)\overline{\chi}_c(b) = \begin{cases} \varphi(m), & \text{if } a = b \\ 0, & \text{if } a \neq b, \end{cases}$$

where the sum is extended over all multiplicative character χ in $\mathbb{F}_m$.

Proof. The proof of this result is given in [14].

Lemma 2.1.3. is established by using the function from Lemma 2.1.1,

$$\chi_j(q_i^k) = e^{\frac{2\pi ijk}{\mu}} \quad \text{where} \quad k = 0, 1, ..., \varphi(m) - 1, \tag{4}$$

which provide all additive and multiplicative characters of $\mathbb{F}_m$ for any value composite modules as stated in Lemma 2.1.2.

LEMMA 2.1.3 Let a, b in $\mathbb{F}_m$ and c in $\mathbb{F}_m^*$. If $c + a = d_1$ and $c + b = d_2$. Then, the nontrivial multiplicative character of $\mathbb{F}_m$ is given by,

$$\sum_{\chi} \chi(d_1)\overline{\chi}(d_2) = \begin{cases} 0, & \text{if } d_1 \neq d_2 \\ \varphi(m) + 1, & \text{if } d_1 = d_2. \end{cases}$$

PROOF. The proof of this result is given in [14]. Lemma 2.1.3 gives two possible results depending on conditions of d_1 and d_2. The number of elements of character sums will be different because the inner sum has an additive identity of the character. Then, $\sum_{\chi} \chi(d_1)\overline{\chi}(d_2) = \varphi(m) + 1$ for $d_1 = d_2$; zero otherwise.

3. RESULT AND DISCUSSION

The section gives the result on the cardinality of double character sums of Beatty sequence associated with composite modules given in the following theorem.

THEOREM 3.1. Let α be a fixed irrational number, β be any real number, n be any natural number and $\varphi(m)$ be an order of m. For any positive integers $N \leq m$ and non-trivial multiplicative characters $\chi(\text{mod m })$, the following bound holds

$$|W_3|^2 = \# \left\{ \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} \chi(\lfloor \alpha(n+k) + \beta \rfloor)\overline{\chi}(\lfloor \alpha(n+k) + \beta \rfloor) \right\} \ll \varphi(m)N(\#K).$$

PROOF. From expressions (2) and (3), we substitute n with natural numbers. Then, we have

$$W_3 = \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} \chi(\lfloor \alpha(n+k) + \beta \rfloor) = \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} \chi(\lfloor \alpha n + \beta - \gamma \rfloor + \lfloor \alpha k + \gamma \rfloor).$$

Suppose $N \leq n$, where n is a natural number. The following expression gives the cardinality of $\lfloor \alpha n + \beta - \gamma \rfloor$, such that

$$\#\{n \in \mathcal{N} : \lfloor \alpha n + \beta - \gamma \rfloor \equiv s(\bmod m)\} = O(1),$$

where $s \in \mathbb{Z}$. Then, by applying the Cauchy inequality, we have

$$|W_3|^2 \ll N \sum_{n \in \mathcal{N}} \left| \sum_{k \in \mathcal{K}} \chi(\lfloor \alpha n + \beta - \gamma \rfloor + \lfloor \alpha k + \gamma \rfloor) \right|^2 \ll N \sum_{s=1}^{m} \left| \sum_{k \in \mathcal{K}} \chi(s + \lfloor \alpha k + \gamma \rfloor) \right|^2 =$$

$$= N \sum_{k,l \in \mathcal{K}} \sum_{s=1}^{m} \chi(s + \lfloor \alpha k + \gamma \rfloor)\overline{\chi}(s + \lfloor \alpha l + \gamma \rfloor). \quad (5)$$

Expanding the following double sums yield

$$\sum_{k \in \mathcal{K}} \sum_{s=1}^{m} \chi(s + \lfloor \alpha k + \gamma \rfloor)\overline{\chi}(s + \lfloor \alpha \ell + \gamma \rfloor).$$

Then, we will have all the number of elements as follows.

$$\chi(1 + \lfloor \alpha k_1 + \gamma \rfloor)\overline{\chi}(1 + \lfloor \alpha \ell_1 + \gamma \rfloor) + \chi(1 + \lfloor \alpha k_1 + \gamma \rfloor)\overline{\chi}(1 + \lfloor \alpha \ell_2 + \gamma \rfloor) + \dots$$
$$\dots + \chi(1 + \lfloor \alpha k_1 + \gamma \rfloor)\overline{\chi}(1 + \lfloor \alpha K + \gamma \rfloor)$$

$$\chi(1+\lfloor \alpha k_2+\gamma\rfloor)\overline{\chi}(1+\lfloor \alpha \ell_1+\gamma\rfloor)+\chi(1+\lfloor \alpha k_2+\gamma\rfloor)\overline{\chi}(1+\lfloor \alpha \ell_2+\gamma\rfloor)+\dots$$
$$\dots+\chi(1+\lfloor \alpha k_2+\gamma\rfloor)\overline{\chi}(1+\lfloor \alpha K+\gamma\rfloor)$$

...

$$\chi(1+\lfloor \alpha K+\gamma\rfloor)\overline{\chi}(1+\lfloor \alpha \ell_1+\gamma\rfloor)+\chi(1+\lfloor \alpha K+\gamma\rfloor)\overline{\chi}(1+\lfloor \alpha \ell_2+\gamma\rfloor)+\dots$$
$$\dots+\chi(1+\lfloor \alpha K+\gamma\rfloor)\overline{\chi}(1+\lfloor \alpha K+\gamma\rfloor)$$

$$\chi(2+\lfloor \alpha k_1+\gamma\rfloor)\overline{\chi}(2+\lfloor \alpha \ell_1+\gamma\rfloor)+\chi(2+\lfloor \alpha k_1+\gamma\rfloor)\overline{\chi}(2+\lfloor \alpha \ell_2+\gamma\rfloor)+\dots$$
$$\dots+\chi(2+\lfloor \alpha k_1+\gamma\rfloor)\overline{\chi}(2+\lfloor \alpha K+\gamma\rfloor)$$

...

$$\chi(3+\lfloor \alpha k_2+\gamma\rfloor)\overline{\chi}(3+\lfloor \alpha \ell_1+\gamma\rfloor)+\chi(1+\lfloor \alpha k_2+\gamma\rfloor)\overline{\chi}(3+\lfloor \alpha \ell_2+\gamma\rfloor)+\dots$$
$$\dots+\chi(3+\lfloor \alpha k_2+\gamma\rfloor)\overline{\chi}(3+\lfloor \alpha K+\gamma\rfloor)$$

...

$$\chi(m+\lfloor \alpha K+\gamma\rfloor)\overline{\chi}(m+\lfloor \alpha \ell_1+\gamma\rfloor)+\chi(m+\lfloor \alpha K+\gamma\rfloor)\overline{\chi}(1+\lfloor \alpha \ell_2+\gamma\rfloor)+\dots$$
$$\dots+\chi(m+\lfloor \alpha K+\gamma\rfloor)\overline{\chi}(m+\lfloor \alpha K+\gamma\rfloor).$$

The total number of elements is $\varphi(m)(\#K)^2$ since there are $\varphi(m)$ elements with K pairs and K times elements for each $\varphi(m)$.

If $\mathbb{F}_m^*$ is a cyclic subgroup of $\mathbb{F}_m$ with order $\varphi(m)$. Then, the character sums can be easily determined by applying Lemma 2.1.3. Also remark that since $N \le m$, the inner sum has just two potential values, as seen in equation (4).

The congruence $\lfloor \alpha k+\gamma\rfloor \equiv \lfloor \alpha l+\gamma\rfloor)(\mod m)$ occurs for at most $O(K)$ pairs $k, l \in \mathcal{K}$ since $K \le m$. Therefore, (5) will be

$$N\sum_{k,l\in\mathcal{K}}\sum_{s=1}^{m}\chi(s+\lfloor \alpha k+\gamma\rfloor)\overline{\chi}(s+\lfloor \alpha l+\gamma\rfloor) \ll N(\varphi(m)(\#K)^2)$$
$$\ll N((\#K)^2+\phi(m)(\#K)) \ll \varphi(m)N(\#K).$$

Thus, the theorem holds.

4. Conclusion

The cardinality of character sums over natural numbers n with respect to non-homogeneous Beatty sequences $[\alpha n+\beta]$ is as follows.

$$\#\left\{\sum_{n\in\mathcal{N}}\sum_{k\in\mathcal{K}}\chi(\lfloor \alpha(n+k)+\beta\rfloor)\overline{\chi}(\lfloor \alpha(n+k)+\beta\rfloor)\right\} \ll \varphi(m)N(\#K).$$

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