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# Коэффициент и арифметическая сложность объединения $n$!

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**Аннотация**

В этой статье мы покажем, что факторная сложность бесконечного слова $\mathfrak{F}_b$ определяемая путем объединения базовых $b$ представлений $n!$ полна. Затем мы покажем, что арифметическая сложность этого слова также является полной. С другой стороны, $\mathfrak{F}_b$ это дизъюнктивное слово. В теории чисел такой вид слов называется *богатыми цифрами.*

*Ключевые слова:* факторная сложность, равнораспределенная по модулю 1, критерий Вейля, цифровые задачи, факториалы.

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# Factor and arithmetic complexity of concatenating the $n$!

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**Abstract**

In this paper, we show that factor complexity of the infinite word $\mathfrak{F}_b$ is defined by concatenating base-$b$ representations of the $n$! is full. Then we show that the arithmetic complexity of this word is full as well. On the other hand, $\mathfrak{F}_b$ is a disjunctive word. In number theory, this kind of words is called *rich numbers.*

*Keywords:* factor complexity, equidistributed modulo 1, Weyl's criterion, digital problems, factorials.

*Bibliography:* 5 titles.

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# 1. Introduction and preliminaries

Applying combinatorial analysis to finite or infinite words is the domain of word combinatorics, a field at the intersection of theoretical computer science and discrete mathematics. This area of mathematics evolved from number theory, group theory, probability, and of course combinatorics. The basics of word combinatorics are given for instance in [5].

## 1.1. Combinatorics on words

A nonempty finite set $\Sigma$ is called an *alphabet*. The elements of the set $\Sigma$ are called *letters*. The alphabet consisting of $b$ symbols from 0 to $b-1$ will then be denoted by $\Sigma_b = \{0, \ldots, b-1\}$. A *word* $\mathbf{w}$ is a sequence of letters. The *finite word* $\mathbf{w}$ can be considered as a function of $\mathbf{w} : \{1, \cdots, |\mathbf{w}|\} \to \Sigma$, where $\mathbf{w}[i]$ is the letter in the $i^{\text{th}}$ position. The *length* of the word $|\mathbf{w}|$ is the number of letters contained in it. The *empty word* is denoted by $\varepsilon$. Then we introduce *infinite words* as functions $\mathbf{w} : \mathbb{N} \to \Sigma$. The set of all finite words over $\Sigma$ is denoted by $\Sigma^*$, and $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$; the set of all infinite words is denoted by $\Sigma^{\mathbb{N}}$.

The *concatenation* of the finite words $\mathbf{u} = \mathbf{u}[1] \cdots \mathbf{u}[n]$, $|\mathbf{u}| = n$ and $\mathbf{w} = \mathbf{w}[1] \cdots \mathbf{w}[m]$, $|\mathbf{w}| = m$ is the word

$$\mathbf{s} = \mathbf{uw} = \mathbf{u} = \mathbf{u}[1] \cdots \mathbf{u}[n]\mathbf{w}[1] \cdots \mathbf{w}[m], \quad |\mathbf{s}| = |\mathbf{u}| + |\mathbf{w}| = n + m.$$

Let $\mathbf{u}$ and $\mathbf{w}$ be two words. If there are words $\mathbf{s}$ and $\mathbf{v}$ such that $\mathbf{w} = \mathbf{suv}$, then the word $\mathbf{u}$ is called a *factor* of the word $\mathbf{w}$. The set of all factors of $\mathbf{w}$ is denoted by $\mathcal{L}_{\mathbf{w}}$ and it is called *language generated* by $\mathbf{w}$. If $\mathbf{s} = \varepsilon$, then $\mathbf{u}$ is called a *prefix* of the word $\mathbf{w}$, if $\mathbf{v} = \varepsilon$, it is named a *suffix*. The factor $\mathbf{w}[i]\mathbf{w}[i+1] \cdots \mathbf{w}[j]$ where $i \leqslant j$ is denoted by $\mathbf{w}[i \cdots j]$.

This work investigates concatenating words from a complexity perspective. There are numerous approaches to quantify the complexity of a word over a finite alphabet. The complexity function will be used as our primary measure of complexity. This function was first studied by Hedlund and Morse in 1938 [4].

Definition 1.1. The *factor complexity* of a finite or infinite word $\mathbf{w}$ is the function $k \mapsto \mathrm{p}_{\mathbf{w}}(k)$, which, for each integer $k$, give the number $\mathrm{p}_{\mathbf{w}}(k)$ of distinct factors of length $n$ in that word.

It is clear that the factor complexity is between zero and $(\#\Sigma)^k$. If $\mathrm{p}_{\mathbf{w}}(k) = (\#\Sigma)^k$, then the word $\mathbf{w}$ is said to have *full factor complexity*. This kind of words are called *disjunctive word* [4].

It is also easy to see that the factor complexity of any infinite word is a non-decreasing function, and the complexity of a finite word first increases, then decreases to zero.

The arithmetic complexity of an infinite word is the function that counts the number of words of a specific length composed of letters in arithmetic progression (and not only consecutive). In fact, it's a generalization of the complexity function. This concept was introduced by Avgustinovich and Frid in [1].

Definition 1.2. Let $\mathbf{w} = (a_n)_{n \geqslant 0} \in \Sigma^{\mathbb{N}}$. The *arithmetic closure* of $\mathbf{w}$ is the set

$$A(\mathbf{w}) = \{a_i a_{i+d} a_{i+2d} \cdots a_{i+kd} \mid d \geqslant 1, k \geqslant 0\}.$$

The *arithmetic complexity* of $\mathbf{w}$ is the function $a_{\mathbf{w}}$ mapping $n$ to the number $\mathrm{a}_{\mathbf{w}}(n)$ of words with length $n$ in $A(\mathbf{w})$.

If $\mathrm{a}_{\mathbf{w}}(k) = (\#\Sigma)^k$, then the word $\mathbf{w}$ is said to have *full arithmetic complexity*. The following statement immediately follows from the definition:

Proposition 1.3. Let $\mathbf{w} \in \Sigma^{\mathbb{N}}$ and $\#\Sigma = k$. Then for all $n \in \mathbb{N}$ we have

$$1 \leqslant \mathrm{p}_{\mathbf{w}}(n) \leqslant \mathrm{a}_{\mathbf{w}}(n) \leqslant k^n.$$

### 1.2. Ergodic theory

In mathematics, a sequence $(s_n)_{n\geqslant 0}$ of real numbers is said to be *equidistributed* or *uniformly distributed* on a non-degenerate interval $[a,b]$, if the proportion of terms that fall into a sub-interval is proportional to the length of this interval, i.e., if for any sub-interval $[c,d]$ of $[a,b]$ we have

$$\lim_{n\to\infty}\frac{\#\left(\{s_1,\ldots,s_n\}\cap[c,d]\right)}{n}=\frac{d-c}{b-a}.$$

The theory of uniform distribution modulo 1 deals with the distribution behavior of sequences of real numbers.

Definition 1.4. A sequence $(a_n)_{n\geqslant 0}$ of real numbers is said to be *equidistributed modulo* 1 or *uniformly distributed modulo* 1 if the sequence of fractional parts of $(a_n)_{n\geqslant 0}$ is equidistributed in the interval $[0,1]$.

Theorem 1.5. (Weyl's criterion[2]) A sequence $(a_n)_{n\geqslant 0}$ is equidistributed modulo 1 if and only if for all non-zero integers $N$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathrm{e}^{2\pi \mathrm{i} N a_j}=0.$$

Lemma 1.6. [3] The fractional part of the sequence $(\log(n!))_{n\geqslant 0}$ is dense in $[0,1]$.

Theorem 1.7. [3] If $k$ is any positive integer having $m$ digits, there exists a positive integer $n$ such that the first $m$ digits of $n!$ constitute the integer $k$.

Also we can state this result for any arbitrary base. Let $d_m\ldots d_n$ be a word over $\Sigma_b$ with $d_m\neq 0$. There exists $n$ such that the base$_b$ expansion of $n!$ starts with $d_m\ldots d_n$.

## 2. Main result

### 2.1. Statement of problem

The infinite word $\mathfrak{F}=\mathfrak{F}_b:=({}_bf_n)_{n\geqslant 0}$ is defined by concatenating non-negative base-$b\geqslant 2$ representation of the recursive $n!$.

by concatenating base-10 representation of the recursive $n!$:

$$\mathfrak{F}:=(f_n)_{n\geqslant 0}=1\ 1\ 2\ 6\ 24\ 120\ 720\ 5040\ 40320\cdots.$$

*What is the factor complexity of the $\mathfrak{F}$, i.e. $\mathrm{p}_{\mathfrak{F}}(k)$ ? What about arithmetic complexity, i.e., $\mathrm{a}_{\mathfrak{F}}(k)$?*

In fact, this problem can be easily generalized for any natural bases.

Theorem 2.1. **(i)** The factor complexity of the infinite word $\mathfrak{F}_b$ is full.

**(ii)** The arithmetic complexity of the infinite word $\mathfrak{F}_b$ is full.

**Proof. (i)** Let the alphabet for base-$b$ is $\Sigma_b=\{0,\cdots,b-1\}$. Then $\mathfrak{F}_b\in\Sigma_b^{\mathbb{N}}$. Now we want to find

$$k\mapsto\#\{f_i\cdots f_{i+k-1}|i\geqslant 0\}.$$

By Lemma 1.2 and Theorem 1.2, we claim that $({}_bf_n)_{n\geqslant 0}$ is equidistributed modulo 1. Then we have a same result such Theorem 1.2, but for an arbitrary bases, i.e., there exists an $n$ such that the $b$-expansion of $n!$ begins with these digits. On the other hand, each word $\mathbf{T}\in\Sigma_b^+$ will appearance at least one position in $\mathfrak{F}_b$, i.e., $\mathfrak{F}_b[i\cdots j]=\mathbf{T}$, because there exist $\mathbf{S}\in\Sigma_b^+$ such that it begins by $\mathbf{T}$.

Hence, $\mathcal{L}_{\mathfrak{F}_b}=\Sigma_b^+$, and $\mathfrak{F}_b$ is full factor complexity, i.e.,

$$\mathrm{p}_{\mathfrak{F}_b}(k)=(\#(\Sigma_b))^k=b^k.$$

**(ii)** In the previous part we show that $\mathrm{p}_{\mathfrak{F}_b}(k) = b^k$. Now by Proposition 1.1, we can say that for all $k \in \mathbb{N}$

$$b^k \leqslant \mathrm{a}_{\mathfrak{F}_b}(k) \leqslant b^k.$$

This inequality is true for all natural numbers $k$, this implies that $\mathrm{a}_{\mathfrak{F}_b}(k) = b^k$.

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