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Хроматическое число слоек с запрещенными одноцветными арифметическими прогрессиями

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Аннотация

Для $h, n \geq 1$ и $e > 0$ рассматривается хроматическое число пространств вида $\mathbb{R}^n \times [0, e]^h$. Представлен обзор имеющихся результатов, рассмотрена задача о хроматическом числе нормированных пространств с запрещенными одноцветными арифметическими прогрессиями. Показано, что для любого n существует двуцветная раскраска пространства $\mathbb{R}^n$, при которой достаточно длинная арифметическая прогрессия содержит точки обоих цветов, и такая раскраска применима к пространствам вида $\mathbb{R}^n \times [0, e]^h$.

Ключевые слова: хроматическое число, задача Нельсона — Хадвигера

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On the chromatic number of slices without monochromatic unit arithmetic progressions

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Abstract

For $h, n \geq 1$ and $e > 0$ we consider a chromatic number of the spaces $\mathbb{R}^n \times [0, e]^h$ and general results in this problem. Also we consider the chromatic number of normed spaces with forbidden monochromatic arithmetic progressions. We show that for any n there exists a two-coloring of $\mathbb{R}^n$ such that all long unit arithmetic progressions contain points of both colors and this coloring covers spaces of the form $\mathbb{R}^n \times [0, e]^h$.

Keywords: chromatic number, Hadwiger -– Nelson problem

Bibliography: 21 titles.

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1. Introduction

The problem with which all the problems discussed in this article was first formulated in 1950 and is known as the Hadwiger–Nelson problem: what is the minimum number of colors required to color the plane such that no two points at a distance of 1 from each other have the same color? For an n-dimensional normed space $\mathbb{R}^n_N$, it is required to find its *chromatic number* $\chi(\mathbb{R}^n_N)$, defined as the smallest r for which there is a coloring of points $\mathbb{R}^n_N$ in r colors, i.e. an r*-coloring*, and with no two points of the same color unit distance apart. The chromatic number problem is one of the central problems of modern combinatorial geometry. This problem has been studied most actively for l_p-spaces $\mathbb{R}^n_p$. Recall that the l_p -norm of a point $x \in \mathbb{R}^n$ is defined by the equality $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$ for all real $p \geq 1$, but for $p = \infty$ by the equality $\|x\|_\infty = max_i|x_i|$. It is known that for all $1 \leq p \leq \infty$, the value of $\chi(\mathbb{R}^n_N)$ grows exponentially with the growth of n.

To generalize these problems, Erdős, together with his co-authors in [6, 7, 8] proposed prohibiting the monotony of more complex configurations. A subset $\mathfrak{M}' \subset \mathbb{R}^n$ is called an N*-isometric copy* of $\mathfrak{M}$ if there exists a bijection $f : \mathfrak{M} \to \mathfrak{M}'$ such that $\|x - y\|_N = \|f(x) - f(y)\|_N$ for all $x, y \in \mathfrak{M}$. For an n-dimensional normed space $\mathbb{R}^n_N$, and a subset $\mathfrak{M} \subset \mathbb{R}^n$, the *chromatic number* $\chi(\mathbb{R}^n_N, \mathfrak{M})$ is the smallest r such that there exists an r-coloring of $\mathbb{R}^n$ with no monochromatic N*-isometric copy* of $\mathfrak{M}$. In these terms, classical definition of chromatic number $\chi(\mathbb{R}^n_N) = \chi(\mathbb{R}^n_N, I)$, where I is a two-point set.

In our present note, we will mostly be interested in one-dimensional sets playing the role of $\mathfrak{M}$. Let us utilize the notation introduced in [15]: given a sequence of positive reals $\lambda_1, \dots, \lambda_k$, we call a set $\{0, \lambda_1, \lambda_1 + \lambda_2, \dots, \sum_{t=1}^k \lambda_t\} \subset \mathbb{R}$ a *baton* and denote it by $\mathscr{B}_k$. We consider the case $\lambda_1 = \dots = \lambda_k = 1$, i.e., the set $\mathscr{B}_k$ is just a unit arithmetic progression.

In 2016, in the work [12], A. Kanel-Belov, V. Voronov and D. Cherkashin progressed this topic by proposing the following interpretation of the problem. They suggested considering an intermediate case between a plane and a space, namely a layer between two planes of height e. It is obvious that with a sufficiently small value of e, a coloring of 7 colors without single-color dots at a unit distance is preserved. Thus, we will consider colorings of a set $\mathbb{R}^n_N \times [0, e]^h$, where natural $n, h \geq 1$ and real $e > 0$, in a finite number of colors with the forbidden distance 1 between monochromatic points. Further such sets are called *slices* and denoted by $Slice_N(n, h, e)$. We say that n is the *dimension* of a slice. These sets were introduced in the work [12] and were studied for 1-dimensional and 2-dimensional cases exclusively for Euclidean distance. In a recent paper [4], authors studied 3-dimensional slices, and in addition to real slices, they considered rational slices of the form $\mathbb{Q}^n_2 \times [0, e]^h$.

In the next section, all the latest results on slices studies will be described in more detail. Section 3 is devoted to the problem of the chromatic number of normed spaces with forbidden monochromatic set $\mathscr{B}_k$, and we will consider this problem on slices and discussing further issues.

2. The chromatic numbers of rational and real slices

Obviously for any positive e the next inequalities are true:

$$\chi(\mathbb{R}^n_N) \leq \chi(Slice_N(n, h, e)) \leq \chi(\mathbb{R}^{n+h}_N).$$

Currently, the best known bounds on the plane are $5 \leqslant \chi(\mathbb{R}^2_2) \leqslant 7$. The lower bound is a relatively recent breakthrough by de Grey [5] (reproved quickly after by Exoo and Ismailescu [9]). The upper bound here is classical. See also Soifer's account of the history of this problem in [20]. As for the growing dimension case, currently the best asymptotic lower and upper bounds belong to Raigorodskii [18] and Larman and Rogers [16, 17] respectively: $(1.239+o(1))^n \leqslant \chi(\mathbb{R}^n_2) \leqslant (3+o(1))^n$ as $n \to \infty$. It is obvious that the chromatic number of $Slice_N(n, h, e)$ is finite. So by the de Bruijn–Erdos theorem it is achieved on a finite subgraph.

As stated earlier, the study of slices colorings started in the work [12] for 1-dimensional and 2-dimensional Euclidean slices. The following main result.

Theorem 1. *For an arbitrary $e > 0$, the following inequality holds:*

$$6 \leq \chi(\mathbb{R}^2 \times [0, e]^2).$$

The following lower and upper bounds obtained in the works[2, 13] are known. Statement. Let $0 < e \leq \sqrt{\frac{3}{4h}}$. Then

$$\chi(\mathbb{R} \times [0, e]^h) = 3.$$

Let $\sqrt{\frac{3}{4h}} < e < \sqrt{\frac{8}{9h}}$. Then

$$\chi(\mathbb{R} \times [0, e]^h) = 4.$$

Consider the intermediate case between a plane and a 3-dimensional space $\mathbb{R}^2 \times [0, e]$ (slice with height e). This set allows correct coloring in 7 colors. However, the lower bound is less trivial than for the plane:

$$5 \leq \chi(\mathbb{R}^2 \times [0, e]) \leq 7.$$

In [12] it is proved that the upper bound remains the same with increasing dimension due to the fact that the coloring of the plane in 7 colors does not contain distances belonging to a certain interval, and also that the lower bound can be improved with the value $h = 2$.

Theorem 2. *Let $h \in \mathbb{Z}$, $e < e_o(h)$ is positive. Then*

$$\chi(\mathbb{R}^2 \times [0, e]^h) \leq 7.$$

For the lower estimates given, it is sufficient to consider the coloring of a bounded area whose diameter does not depend on the choice of valuea e.

In the work [21] authors considered 3-dimensional real and 2-dimensional rational slices. They proved the next theorem.

Theorem 3. *There is $e_0 > 0$, such that for an arbitrary positive $e > e_0$ holds*

$$10 \leq \chi(Slice(3, 6, e)) \leq 15.$$

The upper bound follows from the proof of the upper bound $\chi(\mathbb{R}^3) \leq 15$, which was obtained independently by Coulson [3] and Radoicic, Toth [19].

In the case of rational slices, in [21] authors showed that the chromatic number of 2-dimensional rational slice is at most 4:

$$\chi(\mathbb{Q}^2 \times [0, e]^2_{\mathbb{Q}}) = 4$$

where $[0, e]^2_{\mathbb{Q}}$ is the set of rational numbers from $[0, e]$ and $e > 0$. In [12] authors considered 1-dimensional slices and showed that

$$\chi(\mathbb{Q} \times [0, e]^3_{\mathbb{Q}}) = 3.$$

3. The chromatic number of normed spaces with forbidden one-color arithmetic progressions

In a recent series of papers [10, 11, 15], the chromatic numbers $\chi(\mathbb{R}^n_\infty, \mathfrak{M})$ of the n-dimensional Chebyshev spaces $\mathbb{R}^n_\infty$ were studied. In particular, it was proven in [15] that

$$\chi(\mathbb{R}^n_\infty, \mathscr{B}_k) \geqslant \left(\frac{k+1}{k}\right)^n \tag{1}$$

for all $k, n \in \mathbb{N}$. This inequality shows that, unlike the Euclidean case, for any given k, every two-coloring of $\mathbb{R}^n$ contains a monochromatic ℓ_∞-isometric copy of $\mathscr{B}_k$ whenever the dimension n is large enough in terms of k.

However, in [14] authors showed that this is not the case in the "opposite"situation, when k is sufficiently large in terms of n. For arbitrary $n \in \mathbb{N}$, they construct a two-coloring of $\mathbb{R}^n$ with the maximum metric satisfying the following. For any finite set $\mathscr{B}_k$ with diameter greater than 5^n such that the distance between any two consecutive points of $\mathscr{B}_k$ does not exceed one, no isometric copy of $\mathscr{B}_k$ is monochromatic.

Theorem 4. *For any $1 \leq p \leq \infty$ and any natural n, there exist sufficiently large $k = k(p, n)$, such that $\chi(\mathbb{R}^n_p, \mathscr{B}_k) = 2$.*

Proof. The proof of this theorem is constructive.

For the case $p = \infty$, the proof is carried out using an explicit, but technically complex two-color coloring consisting of identical layers located on top of each other - "snakes"[14], the colors of which we alternate. It is shown that this coloring of the space $\mathbb{R}^n$ does not contain one-color l_∞-isometric copies of progressions $\mathscr{B}_k$ for $k \geq 5^n$.

Suppose now that $1 \leq p \leq \infty$. It is known that the unit ball of the l_p norm in this case is strictly convex. This means that every l_p - isometric copy of the set $\mathscr{B}_k$ lies on some straight line. As a consequence, it is an arithmetic progression in the space $\mathbb{R}^n_\infty$, the length of the link (and hence the diameter) which can be controlled in terms of n and p. Here we use the fact that l_p- and l_∞-norms on $\mathbb{R}^n$ are 'equivalent' to each other, i.e. for some positive $c = c(n, p)$ and $C = C(n, p)$ it is true that $c\|x\|_\infty \leqslant \|x\|_p \leqslant C\|x\|_\infty$ for all $x \in \mathbb{R}^n$. So, from the absence in some coloring of the space $\mathbb{R}^n$ with the norm l_∞ of sufficiently long one-color arithmetic progressions, it really follows from this that there are no one-color l_p-isometric copies of the sets $\mathscr{B}_k$ for all sufficiently large values of k.

Finally, we consider the case $p = 1$. In a sense, this situation is diametrically opposite to the previous one, since the unit ball of the l_1-norm is a convex centrally symmetric polyhedron (more precisely, a hyperoctahedron or a cross-polytope). It is known that every such polyhedron with f pairs of opposite faces is the central section of an f-dimensional hypercube by some hyperplane. This means that the $\mathbb{R}^m_1$ space can be isometrically embedded in $\mathbb{R}^m_\infty$ for $m = 2^{n-1}$. Therefore, to construct the desired two-coloring of $\mathbb{R}^n_1$, it is sufficient to consider such a coloring of $\mathbb{R}^m_\infty$, and then simply induce it to the corresponding subspace.

Corollary 1. *For any normed space $\mathbb{R}^n_N$ there exists a real $\delta = \delta(\mathbb{R}^n_N)$ such that the following holds. There exists a two-coloring of $\mathbb{R}^n$ with no monochromatic collinear N-isometric copies of all batons $\mathscr{B}_k$ such that $\max_t \lambda_t \leqslant 1$ and $\sum_{t=1}^k \lambda_t \geqslant \delta$. In particular, all sufficiently long unit arithmetic progressions in $\mathbb{R}^n_N$ contain points of both colors under this coloring.*

The following theorem is a corollary of Theorem 4.

Theorem 5. *For any $1 \leq p \leq \infty$ and any natural $h, n \geq 1$ and $e > 0$, there is $k = k(n, h, e)$ such that $\chi(Slice_p(n, h, e), \mathscr{B}_k) = 2$.*

Доказательство. Since $Slice_p(n, h, e) \subset \mathbb{R}^{n+h}_p$. If we want to color the slice $\mathbb{R}^n \times [0, e]^h$ so that there is no monochromatic $\mathscr{B}_k$ in it, then it is not difficult to see that this is an intermediate case between coloring of $\mathbb{R}^{n+h}$ and $\mathbb{R}^{n+h+1}$. Moreover, if e is smaller than the diameter of $\mathscr{B}_k$, then this task exactly coincides with the task of two-coloring $\mathbb{R}^{n+h}$ and the set $\mathbb{R}^n \times [0, e]^h$ is also two-colorable. But if e is large, then the $Slice(n, h, e)$ roughly coincides with the space $\mathbb{R}^{n+h+1}$. This is obvious, but it can be strictly proved through the De Bruijn—Erdős compactness theorem for hypergraphs.

□ Note that the question of the asymptotic behavior of the optimal constant $k = k(p, n)$ from the Theorem 4 is open and also raises the following problem accordingly. Problem. Find the smallest $k = k(n, h, e)$ such that $\chi(Slice(n, h, e), \mathscr{B}_k) = 2$.

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