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# Методы нахождения оптимальных смешанных стратегий в матричных играх с коррелированными случайными выигрышами

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**Аннотация**

Рассматривается игра с природой при известных вероятностях состояний. Предлагается принцип оптимальности для принятия решений для игр с природой, основанный на оценках эффективности и риска. В отличие от традиционного подхода к определению смешанной стратегии в теории игр, в данной работе рассматривается возможность корреляционной зависимости случайных значений выигрышей для начальных альтернатив. Предлагаются два варианта реализации двухкритериального подхода к определению принципа оптимальности. Первый вариант — минимизировать дисперсию как оценку риска с более низким порогом математического ожидания выигрыша. Второй вариант — максимизировать математическое ожидание выигрыша с верхним порогом дисперсии. Получены аналитические решения обеих задач. Рассмотрено применение полученных результатов на примере процесса инвестирования на фондовом рынке. Инвестор, как правило, формирует портфель не сразу, а в виде последовательного процесса приобретения того или иного финансового актива. В этом случае смешанная стратегия может быть реализована в ее имманентном смысле, т.е. покупки осуществляются случайным образом с распределением, определяемым ранее найденным оптимальным решением. Если этот процесс достаточно длительный, то структура портфеля будет примерно соответствовать типу смешанной стратегии. Такой подход использования игры с природой с учетом корреляционной зависимости случайного выигрыша чистых стратегий может быть применен и к задачам принятия решений в других областях управления рисками.

*Ключевые слова:* управление риском, принцип оптимальности, двухкритериальный подход, математическое ожидание, стандартное отклонение.

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# Methods for Determining Optimal Mixed Strategies in Matrix Games with Correlated Random Payoffs

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**Abstract**

A game with nature for known state probabilities is considered. An optimality principle is proposed for decision-making for games with nature, based on efficiency and risk estimates. In contrast to the traditional approach to the definition of a mixed strategy in game theory, this paper considers the possibility of correlation dependence of random payoff values for initial alternatives. Two variants of the implementation of the two-criteria approach to the definition of the optimality principle are suggested. The first option is to minimize the variance as a risk estimate with a lower threshold on the mathematical expectation of the payoff. The second option is to maximize the mathematical expectation of the payoff with an upper threshold on the variance. Analytical solutions of both problems are obtained. The application of the obtained results on the example of the process of investing in the stock market is considered. An investor, as a rule, does not form a portfolio all at once, but as a sequential process of purchasing one or another financial asset. In this case, the mixed strategy can be implemented in its immanent sense, i.e. purchases are made randomly with a distribution determined by the previously found optimal solution. If this process is long enough, then the portfolio structure will approximately correspond to the type of mixed strategy. This approach of using the game with nature, taking into account the correlation dependence of random payoff of pure strategies, can also be applied to decision-making problems in other areas of risk management.

*Keywords:* risk management, optimality principle, two-criteria approach, mathematical expectation, standard deviation.

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## 1. Introduction

Decision theory describes and explains the behavior of a complex system consisting of human and information resources. In this case, the decision maker makes an informed choice between several options, each of which is considered achievable. This selection is based on available information. The result of a combination of the preferences of the decision maker and various decision options is the identification of a subjective decision that best meets the decision criteria [1, 2] [1, 2].

When modeling decision-making processes, the game-theoretic approach is widely used [3–13]. Control processes in complex systems are characterized by incomplete information about the state of the system and the environment. If one participant is explicitly distinguished, then a game with nature can be used as a mathematical model for making decisions in such situations. One of the players is a person or an institution acting as a decision-maker. The other player is nature and can affect the outcome of the game to various degrees. Nature constitutes a set of conditions affecting the results of taken decisions. To resolve a game of nature, it is required to apply certain decision-making criteria indicating the choice of the optimal decision which is to be made under the conditions of uncertainty concerning the future states of nature. In Wald's criterion [14], based on the loss function, the optimal decision corresponds to the lowest value of the maximum loss. On the other hand, when using the effectiveness function, the decision that maximizes the lowest value of the effectiveness function is the optimal one. In the case of Hurwicz's criterion [15], the parameter $\alpha$ is adopted to determine the coefficient of pessimism (expectations as to the realization of a given state of nature) about the possible future states of nature. This criterion determines the optimal decision, which maximizes the average value of the lowest and highest decision efficiency function with the weights $\alpha$ and $1-\alpha$ respectively. For a specific loss function, Hurwicz's criterion determines the optimal decision which minimizes the average value of the highest and the lowest loss function with the weights $\alpha$ and $1-\alpha$ respectively. Savage's criterion [16], which is based on Wald's criterion, refers to the minimum regret function (alternative loss function) resulting from wrong decisions for particular states of nature. The minimum regret function is formulated based on the decision effectiveness function or the loss function. According to Savage's criterion, it is first necessary to find the relative loss matrix (regret matrix). A loss is defined as the difference between the largest win possible in a particular state of nature, and the win corresponding to the decision currently under investigation.

When building a model and setting an optimization problem, the question arises about the availability of information concerning the states of nature. The definition of the concept of optimality or, as is sometimes said, the principle of optimality, depends on this. In this paper, it is assumed that the decision maker has information about the probabilities of the states of nature, i.e. the case of probabilistic uncertainty is considered (or, as it is fashionable to say, we are talking about risk management).

A large number of works are devoted to the application of mathematical methods in risk-based decision making (see, for example, [17–23]). In the paper [20], a two-criteria approach "efficiency – risk" was proposed to determine the principle of optimality when making decisions in stochastic conditions. The mathematical expectation of the gain was used as an efficiency assessment, and the VAR function was used as a risk assessment. As it is known, the VAR function and variance are the most widely used quantities as a risk assessment (see, for example, [24–26]).

The paper [21] outlined the two-criteria approach "efficiency-risk"to the definition of the principle of optimality in decision-making under stochastic conditions, using the mathematical expectation of the payoff as an efficiency estimate and the standard deviation as a risk estimate. Note that if, under known probabilities of states of nature, we are talking about maximizing the mathematical expectation of the payoff, then using a mixed strategy does not make sense. The situation is different with the two-criterion approach, namely, the optimal mixed strategy, generally speaking, gives a greater gain than any pure strategy.

The main difference of this paper from the traditional approach to the definition of a mixed strategy in game theory is that it takes into account the possibility of correlation dependence of random payoff values of the original alternatives (pure strategies). It should be noted that it is in the two-criteria approach that taking into account correlation becomes essential. Usually, in games with nature, either the mathematical expectation of the payoff or the risk according to Savage is considered as a criterion. In this case, the possible correlation of random payoffs under different pure strategies does not play any role. If there are two criteria, one of which is the standard deviation, taking into account the correlation significantly affects the formulation of the problem and the method of its solution.

Here we consider two problems: the first is to minimize the variance as a risk criterion with a lower threshold on the mathematical expectation of the payoff; the second is to maximize the mathematical expectation of the payoff with an upper threshold on the variance. Analytical and algorithmic results will be obtained concerning the solution of these problems taking into account the correlation of random payoffs of each pair of pure strategies. These results are illustrated by the example of the investment process using real statistical data.

## 2. Determining the Optimal Mixed Strategy with a Restriction on the Mathematical Expectation of the Payoff

So, we consider the situation when the decision maker can choose one of the strategies (alternatives) $i = 1, \ldots, n$, with a known set of possible options for the states of the environment (nature) $j = 1, \ldots, m$. The gain from the $i$-th decision in the $j$-th state of the environment is $a_{ij}$. The payoff matrix from the implementation of possible solutions is $A = \|a_{ij}\|$. The probabilities of states of nature $q_j$ will be considered known. The decision maker needs to choose the strategy that will lead, if possible, to a greater gain, but at the same time, possible losses due to the ambiguity of the outcome will be as small as possible.

As an estimation of the effectiveness of a pure strategy $i$ we take the mathematical expectation of a payoff $\overline{a}_i = \sum_{j=1}^{m} a_{ij} q_j$ , and as a risk estimate - the standard deviation $\sigma_i =$ $= \left(\sum_{j=1}^{m} (a_{ij} - \overline{a}_i)^2 q_j\right)^{0.5}$.

When using a mixed strategy, value $\overline{a}_i$ is a conditional mathematical expectation of payoff under realization of the pure strategy $i$. We denote by $p_i$ the probability of choosing the pure strategy $i$. Then the mathematical expectation of payoff when using the strategy $p = (p_1, \ldots, p_n)$ is $\sum_{i=1}^{n} \overline{a}_i p_i$.

Let $\sigma_{ik}$ be the covariance moments of random values of payoff for pure strategies $i$ and $k$, which are determined by the formula $\sigma_{ik} = \sum_{j=1}^{m} (a_{ij} - \overline{a}_i)(a_{kj} - \overline{a}_k) q_j$.

We denote the covariance matrix $D = \|\sigma_{ik}\|$. As known, the covariance matrix is always non-negative definite. In what follows, we will assume a little more, namely, that it is positive definite.

The standard deviation of the random value of payoff for the strategy $p = (p_1, \ldots, p_n)$ in the case of correlation is determined, obviously, by the formula $\sigma = \left(\sum_{i=1}^{n} \sum_{k=1}^{n} \sigma_{ik} p_i p_k\right)^{0.5}$ or in the matrix-vector form $\sigma = \langle p, Dp \rangle^{0.5}$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product of vectors.

It is convenient to present all the data in the form of Table 1.

The first $m$ columns of the table are the initial data imported from external sources, and the last $n + 1$ columns are the calculated data.

We introduce $n$-dimensional vectors $\overline{a} = (\overline{a}_1, \ldots, \overline{a}_n)$ and $e = (1, \ldots, 1)$.

Let us formulate a problem for the minimum variance under a lower bound on the mathematical expectation of the payoff:

$$\underset{p \in P}{min} \langle p, Dp \rangle , \quad P = \{p | \langle \overline{a}, p \rangle \geqslant a_0, \quad \langle p, e \rangle = 1, \ p \geqslant 0\}. \tag{1}$$

The set $P$ is non-empty, closed, bounded if the threshold value $a_0$ is not greater than the maximum of the values $\overline{a}_i$. Hence, for

$$a_0 \leqslant \underset{i=1,\ldots,n}{max} \overline{a}_i, \tag{2}$$

problem (1) has a solution.

Let us find the left boundary $a^*$ of the range of values $a_0$, at which the first constraint in problem (1) becomes significant. To do this, consider an auxiliary problem of quadratic programming:

Таблица 1: Model Data.

| | $q_1$ | $q_2$ | $\cdots$ | $q_m$ | $\overline{a}_i$ | 1 | 2 | $\cdots$ | $n$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | $a_{11}$ | $a_{12}$ | $\cdots$ | $a_{1m}$ | $\overline{a}_1$ | $\sigma_{11}$ | $\sigma_{12}$ | $\cdots$ | $\sigma_{1n}$ |
| 2 | $a_{21}$ | $a_{22}$ | $\cdots$ | $a_{2m}$ | $\overline{a}_2$ | $\sigma_{21}$ | $\sigma_{22}$ | $\cdots$ | $\sigma_{2n}$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $n$ | $a_{n1}$ | $a_{n2}$ | $\cdots$ | $a_{nm}$ | $\overline{a}_n$ | $\sigma_{n1}$ | $\sigma_{n2}$ | $\cdots$ | $\sigma_{nn}$ |

$$d_0 = \min_{p \in P_0} \langle p, Dp \rangle \ , \quad P_0 = \{p \mid \langle p, e \rangle = 1, \ p \geqslant 0\}. \tag{3}$$

Problem (3) has a unique solution $p^*$. Obviously, $a^* = \langle \overline{a}, p^* \rangle$. Denote by $\hat{D}$ an arbitrary square submatrix of the matrix $D$ of dimension $k \times k$, obtained by deleting rows and columns with the same numbers, $I_1$ - the set of not deleted row and column numbers, $I_2$ - the set of deleted row and column numbers, $\hat{D}^+$ - additional submatrix obtained from $D$ by deleting rows with numbers from $I_1$ and columns with numbers from $I_2$, $\hat{e}$ – part of the vector $e$ of dimension $k$, $\hat{e}^+$ – part of the vector $e$ of dimension $n-k$, $\hat{a}$ – part of the vector $\overline{a}$ with components from $I_1$. The following lemma gives a formula for finding $a^*$.

Lemma 1. *There is a unique matrix $\hat{D}$ such that $\hat{D}^+\hat{p} - \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \hat{e}^+ \geqslant 0$, where*

$$\hat{p} = \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \hat{D}^{-1}\hat{e}. \tag{4}$$

*Wherein*

$$a^* = \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \left\langle \hat{a}, \hat{D}^{-1}\hat{e} \right\rangle. \tag{5}$$

Proof. Compose the Lagrange function $L_0(p, \mu) = 0.5 \langle p, Dp \rangle + \mu(1 - \langle p, e \rangle)$.

The Karush-Kuhn-Tucker (KKT) extremum conditions for problem (3): $\frac{\partial L_0(p,\mu)}{\partial p_i} = 0$, $i \in I$, $\frac{\partial L_0(p,\mu)}{\partial p_i} \geqslant 0$, $i \notin I$, where $I$ – the set of indices corresponding to nonzero $p_i$. For problem (2) these conditions are necessary and sufficient, and since the solution of problem (3) $p^*$ is unique, they are satisfied only for the given vector. For nonzero components of the vector $p^*$, the first part of the KKT conditions gives the system of equations: $\hat{D}\hat{p} - \mu\hat{e} = 0$. The square submatrices of the positive-definite matrix $D$ are also positive-definite and hence non-degenerate. Therefore $\hat{p} = \mu\hat{D}^{-1}\hat{e}$ and from the restriction we have $\mu \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle = 1$. The matrix $\hat{D}^{-1}$ is also positive definite, so $\mu = \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1}$ and $\hat{p} = \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \hat{D}^{-1}\hat{e}$, i.e. we get (4). The second part of the KKT conditions leads to the inequality $\hat{D}^+\hat{p} - \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \hat{e}^+ \geqslant 0$.

Multiply the vector (4) by the vector $\hat{a}$:
$\langle \hat{a}, \hat{p} \rangle = \left\langle \hat{a}, \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \hat{D}^{-1}\hat{e} \right\rangle = \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \left\langle \hat{a}, \hat{D}^{-1}\hat{e} \right\rangle$.

Hence $a^* = \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \left\langle \hat{a}, \hat{D}^{-1}\hat{e} \right\rangle$, i.e. we get (5). The lemma is proven. □

Note that for this case, the KKT conditions are necessary and sufficient. Therefore, if $\hat{p} \geqslant 0$ and the rest of the KKT conditions are satisfied, namely, the nonnegativity of the derivatives of the Lagrange function with respect to $p_i$ with numbers corresponding to zero components, then the vector $\tilde{p}$, padded with zeros in the corresponding places, is a solution to problem (3).

Thus, the method for solving the problem (3) is reduced to enumerating the square submatrices of the matrix $D$, solving the systems of equations based on them using the obtained formulas, and checking the conditions for non-negativity of the components of the obtained vectors and the corresponding derivatives of the Lagrange function. Moreover, since the conditions of the KKT are necessary and sufficient, the enumeration stops as soon as a vector satisfying them is found.

In what follows, we will assume that all $\overline{a}_i$ are distinct. We will need this purely technical assumption to formulate a theorem on the method for solving problem (1). It allows us to exclude trivial cases when the optimal solution is a pure strategy. But this assumption is quite natural and does not violate the generality of the consideration.

The following theorem substantiates a method for finding optimal truly mixed (containing at least two nonzero components) strategies.

Theorem 1. *If*

$$a^*{<}a_0{<}\max_{i=1,\ldots,n}\overline{a}_i,$$

*all* $\overline{a}_i$ *are distinct, matrix* $D{=}\|\sigma_{ik}\|$ *is positive definite, then problem (1) has a unique solution* $p^0$ *and true mixed optimal strategy can be represented as*

$$\tilde{p}^0{=}\tilde{D}^{-1}\left(\lambda^0\tilde{a}{+}\mu^0\tilde{e}\right), \tag{6}$$

$$\begin{gathered}\lambda^0{=}\frac{\max\left\{a_0\left\langle\tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle-\left\langle\tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle,0\right\}}{\left\langle\tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle\left\langle\tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle-\left\langle\tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle^2},\\ \mu^0{=}\frac{\left\langle\tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle-a_0\left\langle\tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle}{\left\langle\tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle\left\langle\tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle-\left\langle\tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle^2},\end{gathered} \tag{7}$$

$\tilde{D}$ *is some (unique) square submatrix of the matrix* $D$ *obtained by deleting rows and columns with the same numbers,* $\tilde{p}^0$ *is a vector of nonzero components of the vector* $p^0$*,* $\tilde{a}$ *is a vector of the part of the components of the vector* $\overline{a}$*,* $\tilde{e}$ *is a vector from a part of the components of the vector* $e$*, obtained by deleting the components with numbers corresponding to the zero components of the vector* $p^0$*.*

Proof. If condition (2) is satisfied, the set $P$ is not empty, closed, and bounded; therefore, the convex programming problem (1) has a solution, and it is unique, because the objective function is strictly convex. The KKT conditions for it are necessary and sufficient (in a problem with linear constraints, the Slater regularity condition is not required). The Lagrange function has the form $L_1(p,\lambda,\mu){=}0.5\langle p,Dp\rangle{+}\lambda\left(a_0{-}\langle\overline{a},p\rangle\right){+}\mu\left(1{-}\langle p,e\rangle\right)$, $\lambda\geqslant 0$. Let $I$ be the set of indices corresponding to non-zero $p_i$. The KKT extremum conditions for problem (1) have the form $\frac{\partial L_1(p,\lambda,\mu)}{\partial p_i}{=}\,0,\ i\in I,\ \frac{\partial L_1(p,\lambda,\mu)}{\partial p_i}\geqslant 0,\ i\notin I.$

For the non-zero components of the vector $p$, we have the system of equations: $\tilde{D}\tilde{p}{-}\lambda\tilde{a}{-}\mu\tilde{e}{=}\,0$, where $\tilde{D}$ is a square submatrix of the matrix $D$ obtained by deleting rows and columns with numbers corresponding to the zero components of the vector $p$, $\tilde{p}$ is a vector of non-zero components of the vector $p$, $\tilde{a}$ is a vector from the part of the components of the vector $\overline{a}$, $\tilde{e}$ is a vector from the part of the components of the vector $e$, obtained by deleting the components with numbers corresponding to the zero components of the vector $p$.

Suppose first that $\lambda{>}0$, then the first constraint in (1) is active. As mentioned above, the square submatrices of the positive-definite matrix $D$ are also positive-definite and, therefore, non-degenerate. Therefore, we have $\tilde{p}{=}\tilde{D}^{-1}(\lambda\tilde{a}{+}\mu\tilde{e})$. We substitute this expression into the constraints of problem (1): $\left\langle\tilde{a},\tilde{D}^{-1}(\lambda\tilde{a}{+}\mu\tilde{e})\right\rangle{=}a_0,\left\langle\tilde{D}^{-1}(\lambda\tilde{a}{+}\mu\tilde{e}),\tilde{e}\right\rangle{=}\,1$. We transform the first equality to the form $\lambda\left\langle\tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle{+}\mu\left\langle\tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle{=}a_0$. From the second equality, we express $\mu{=}\,(1{-}\lambda\left\langle\tilde{e},\tilde{D}^{-1}\tilde{a}\right\rangle)\left\langle\tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle^{-1}$ and substitute into the first:
$\left\langle\tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle{+}(1{-}\lambda\left\langle\tilde{e},\tilde{D}^{-1}\tilde{a}\right\rangle)\left\langle\tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle^{-1}\left\langle\tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle{=}a_0.$

Thus, taking into account the fact that the matrix $\tilde{D}^{-1}$ is symmetric, we obtain an expression for $\lambda$:

$$\lambda=\frac{a_0-\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle^{-1}\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle}{\left\langle \tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle-\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle^{-1}\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle^{2}}, \tag{8}$$

or, after transformation $\lambda=\frac{a_0\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle-\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle}{\left\langle \tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle-\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle^{2}}$. Let us show that the denominator is positive, i.e. there is an inequality

$$\left\langle \tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle-\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle^{2}>0. \tag{9}$$

Indeed, since $\tilde{D}^{-1}$ is a positive definite matrix, there exists a nondegenerate matrix $B$ such that $\tilde{D}^{-1}=B^TB$. Substituting this decomposition of the matrix into the left-hand side of the inequality (9), we have $\langle \tilde{a},B^TB\tilde{a}\rangle\langle \tilde{e},B^TB\tilde{e}\rangle-\langle \tilde{e},B^TB\tilde{a}\rangle^2=\langle B\tilde{a},B\tilde{a}\rangle\langle B\tilde{e},B\tilde{e}\rangle-\langle B\tilde{e},B\tilde{a}\rangle^2$. We apply the Cauchy-Bunyakovsky inequality: $\langle x,y\rangle^2\leqslant\|x\|^2\cdot\|y\|^2$, setting $x=B\tilde{a}$, $y=B\tilde{e}$. In the Cauchy-Bunyakovsky inequality equality holds only if the vectors $x$ and $y$ are collinear. But the vectors $B\tilde{a}$ and $B\tilde{e}$ cannot be collinear, since otherwise, when they are multiplied by the matrix $B^{-1}$, the vectors $\tilde{a}$ and $\tilde{e}$ are also collinear. This contradicts the condition of the theorem, since by assumption, all $\overline{a}_i$ are distinct, and all components of the vector $e$ are equal to ones. Therefore, if these vectors have at least two components, (9) holds.

The numerator in (8) is non-negative, because otherwise for the submatrix $\tilde{D}$ the threshold value $a_0$ is less than the mathematical expectation of the payoff corresponding to the minimum of the variance (it follows from the lemma, see formula (5)).

Substituting $\lambda$ into the expression for $\mu$ we have

$$\mu=\left(1-\frac{a_0-\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle^{-1}\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle}{\left\langle \tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle-\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle^{-1}\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle^{2}}\left\langle \tilde{e},\tilde{D}^{-1}\tilde{a}\right\rangle\right)\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle^{-1}=$$

$$=\frac{1}{\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle}-\frac{a_0\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle-\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle^{-1}\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle^{2}}{\left\langle \tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle-\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle^{2}}=\frac{\left\langle \tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle-a_0\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle}{\left\langle \tilde{a},\tilde{D}^{-1}\tilde{a}\right\rangle\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle-\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle^{2}}.$$

If $\tilde{p}\geqslant 0$ and the rest of the KKT conditions are satisfied, namely, the non-negativity of the derivatives of the Lagrange function with respect to $p_i$ with numbers corresponding to zero components, then the vector $\tilde{p}$, padded with zeros at the appropriate places, is a solution to problem (1).

Let now $\lambda=0$, then we have $\tilde{D}\tilde{p}-\mu\tilde{e}=0$. Combining both cases, we obtain formulas (6), (7). The theorem has been proven. □

*Note*: If formula (8) gives $\lambda<0$, i.e. numerator $a_0-\left\langle \tilde{e},\tilde{D}^{-1}\tilde{e}\right\rangle^{-1}\left\langle \tilde{a},\tilde{D}^{-1}\tilde{e}\right\rangle<0$, then this means that for the given submatrix $\tilde{D}$ the first constraint of problem (1) for a given $a_0$ cannot be active and the case $\lambda=0$ takes place. The algorithm for finding a solution to problem (1) includes enumeration of sets of nonzero components $I$. Since for the convex programming problem (2) the KKT optimality conditions are also sufficient, if a solution satisfies them, then the enumeration process ends.

## 3. Determining the Optimal Mixed Strategy with a Restriction on the Payoff Variance

The problem for the maximum of mathematical expectation of payoff under an upper bound on the standard deviation has the form:

$$\underset{p\in P}{max}\left\langle \overline{a},p\right\rangle,\ P=\{p|\langle p,Dp\rangle^{0.5}\leqslant\sigma_0,\ \langle p,e\rangle=1,\ p\geqslant 0\}. \tag{10}$$

The set $P$ is not empty if the threshold value $\sigma_0$ is not less than the minimum value of the standard deviation on the set $P_0=\{p\mid\ \langle p,e\rangle=1,\ \ p\geqslant 0\}$. To find this value, you need to solve an auxiliary quadratic programming problem (2). Substituting this vector $\hat{p} = \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \hat{D}^{-1}\hat{e}$ into the objective function (2), we obtain the value $d_0 = \left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1}$.

In what follows, we will assume again that all $\overline{a}_i$ are different. This does not violate the generality of the consideration, since if two pure strategies have the same mathematical expectation of payoff and the standard deviations are also equal, then such strategies are equivalent within the framework of this approach and one of them can be excluded. If one of these strategies has a larger standard deviation than the other, then, within the framework of this approach, such a pure strategy cannot be included in the optimal mixed strategy with a nonzero weight.

The following theorem substantiates a method for finding the optimal truly mixed (containing at least two nonzero components) strategies.

Theorem 2. *If $\sigma_0 > d_0^{0.5}$, all $\overline{a}_i$ are different, the matrix $D = \|\sigma_{ik}\|$ is positive definite, then the problem (10) has a solution $p^0$ and the truly mixed optimal strategy can be represented as*

$$\tilde{p}^0 = \lambda^{0^{-1}} \tilde{D}^{-1} \left( \tilde{a} - \mu^0 \tilde{e} \right), \tag{11}$$

$$\lambda^0 = \sqrt{\frac{\left\langle \tilde{a}, \tilde{D}^{-1}\tilde{a} \right\rangle \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle - \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle^2}{\sigma_0^2 \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle - 1}}, \qquad \mu^0 = \frac{\left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle - \lambda^0}{\left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle}, \tag{12}$$

*$\tilde{D}$ is some square submatrix of matrix $D$ obtained by deleting rows and columns with the same numbers, $\tilde{p}^0$ is a vector from nonzero components of the vector $p^0$, $\tilde{a}$ is a vector from a part of the components of the vector $\overline{a}$, $\tilde{e}$ is a vector from parts of the components of the vector $e$ obtained by deleting the components with numbers corresponding to the zero components of the vector $p^0$.*

Proof. For $\sigma_0 > d_0^{0.5}$ the set $P$ is not empty, closed and bounded; therefore, convex programming problem (10) has a solution and satisfies the Slater condition, and the KKT conditions for it are necessary and sufficient. In problem (10), to apply the extremum conditions, it is more convenient to square the first constraint. Then the Lagrange function has the form
$L_2(p, \lambda, \mu) = \langle \overline{a}, p \rangle + \frac{1}{2}\lambda \left( \sigma_0^2 - \langle p, Dp \rangle \right) + \langle \mu, 1 - \langle p, e \rangle \rangle, \ \ \lambda \geqslant 0.$

Let, as before, $I$ be the set of indices corresponding to nonzero $p_i$. The conditions KKT of extremum for the problem (10) have the form $\frac{\partial L_2(p,\lambda,\mu)}{\partial p_i} = 0$, $i \in I$, $\frac{\partial L_2(p,\lambda,\mu)}{\partial p_i} \leqslant 0$, $i \notin I$.

For nonzero components of the vector $p$, we have a system of equations: $\tilde{a} - \lambda \tilde{D}\tilde{p} - \mu \tilde{e} = 0$, where $\tilde{D}$ is a square submatrix of the matrix $D$ obtained by deleting rows and columns with numbers corresponding to the zero components of the vector $p$, $\tilde{p}$ is a vector of nonzero components of the vector $p$, $\tilde{a}$ is a vector from a part of the components of the vector $\overline{a}$, $\tilde{e}$ is a vector from a part of the components of the vector $e$ obtained by deleting the components with numbers corresponding to the zero components of the vector $p$.

If $\lambda = 0$, then we have $\tilde{a} - \mu\tilde{e} = 0$. But by virtue of the assumption of the theorem that all $\overline{a}_i$ are different, this equality is possible only for one index, so in this case, the optimality conditions can be satisfied only for the set $I$ containing one index. If the quadratic constraint in the problem (10) is not active, then $\lambda = 0$. Therefore, for a truly mixed optimal strategy with at least two components different from zero, the quadratic constraint in (10) must be active and $\lambda > 0$.

As mentioned above, the square submatrices of the positive definite matrix $D$ are also positive definite and, therefore, nondegenerate. Therefore, we have $\tilde{p} = \lambda^{-1}\tilde{D}^{-1}(\tilde{a} - \mu\tilde{e})$. We substitute this expression into the constraints of problem (10):
$\left\langle \tilde{D}^{-1}(\tilde{a} - \mu\tilde{e}), (\tilde{a} - \mu\tilde{e}) \right\rangle = \lambda^2\sigma_0^2, \quad \lambda^{-1} \left\langle \tilde{D}^{-1}(\tilde{a} - \mu\tilde{e}), \tilde{e} \right\rangle = 1.$

We transform the first equality to the form: $\left\langle \tilde{a}, \tilde{D}^{-1}\tilde{a} \right\rangle + \mu^2 \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle - 2\mu \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle = \lambda^2\sigma_0^2$.

From the second equality, we express $\mu$: $\mu = \left( \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle - \lambda \right) \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle^{-1}$, and substitute into

the first equality $\left\langle \tilde{a}, \tilde{D}^{-1}\tilde{a} \right\rangle \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle + \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle^2 - 2\lambda \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle + \lambda^2 -$
$-2\left( \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle^2 - \lambda \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle \right) = \lambda^2\sigma_0^2 \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle.$

In this expression, the coefficient at $\lambda$ is zero. Thus, we obtain a quadratic equation for $\lambda$: $\lambda^2 \left( \sigma_0^2 \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle - 1 \right) = \left\langle \tilde{a}, \tilde{D}^{-1}\tilde{a} \right\rangle \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle - \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle^2$. As it was shown the free term in the last equation is positive (see the inequality (9)). Let us show that the coefficient at $\lambda^2$ is also positive. To do this, we will use the form of solution of the problem (3) obtained above. If we solve a similar problem of minimizing the variance with the covariance matrix $\tilde{D}$, corresponding to the solution of the problem (10), we obtain the minimum value of the variance $\left\langle \tilde{D}^{-1}\tilde{e}, \tilde{e} \right\rangle^{-1}$. By the assumption of the theorem, $\sigma_0^2$ is greater than this value, i.e. $\sigma_0^2 > \left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle^{-1}$. Considering, that $\lambda > 0$, $\lambda$ is a solution with a plus sign in front of the radical.

If $\tilde{p} \geqslant 0$ and the rest of the KKT conditions are satisfied, namely, the non-positiveness of the derivatives of the Lagrange function with respect to $p_i$ with numbers corresponding to zero components, then the vector $\tilde{p}$, padded with zeros in the corresponding places, is a solution to the problem (10).

As a result, we obtain formulas (11) and (12). Q.E.D. □

## 4. Calculation Examples for Stock Investment Problems

Let us consider the application of the obtained results on the example of the process of investing in the stock market. Usually, a mixed strategy is interpreted as a vector of shares of financial instruments in a portfolio. Without excluding such an interpretation, we will offer a slightly different point of view. An investor, as a rule, does not form a portfolio all at once, but as a sequential process of purchasing one or another financial asset. In this case, the mixed strategy can be implemented in its immanent sense, i.e. purchases are made randomly with a distribution determined by the previously found optimal solution. If this process is long enough, then the portfolio structure will approximately correspond to the type of mixed strategy. Within the framework of this model, as a game with nature, when applied to the stock market, short sales are unacceptable, because the solution is mixed strategies, the components of which, in principle, cannot be negative.

We will conduct a technical analysis and find the optimal investment strategy using real data on stock quotes of Russian companies for the period from 02/01/2021 to 05/01/2021. This period was chosen because the later data period characterizes the fall of market indices and is associated not so much with economic as with political reasons.

Three relatively successful companies were selected, namely VTB Bank (VTBR), Gazprom (SAGP), Sberbank of Russia (SBER). Based on data on daily closing prices, the daily value of company returns, average returns, variance, and covariance for a given period were calculated (data taken from the site of FINAM Investment Company [27]).

Strategy 1 – investment in shares of VTB Bank, strategy 2 – investment in shares of Gazprom, strategy 3 – investment in shares of Sberbank of Russia. In this case, the average values of returns are equal to $\overline{a}_1 = 0.00548$ (0.548%), $\overline{a}_2 = 0.00127$ (0.127%), $\overline{a}_3 = 0.002$ (0.2%), the covariance matrix has the form $D = \begin{pmatrix} 0.00034 & 0.00010 & 0.000095 \\ 0.00010 & 0.00016 & 0.000094 \\ 0.000095 & 0.000094 & 0.00017 \end{pmatrix}$.

At first, we will solve the problem (1) for the minimum variance with a constraint on the mathematical expectation of the payoff. According to the condition of Theorem 1, we calculate the left and right ends of the interval

$$\left\langle \hat{D}^{-1}\hat{e}, \hat{e} \right\rangle^{-1} \left\langle \hat{a}, \hat{D}^{-1}\hat{e} \right\rangle < a_0 < \underset{i=1,\ldots,n}{max}\ \overline{a}_i.$$

The solution of problem (3) gives a full-size portfolio
$p = (0.11532,\ 0.44388,\ 0.44079)$, therefore, for the initial matrix $D$ and the initial vector of expected payoffs $\overline{a} = (0.00548,\ 0.00127,\ 0.002)$ we have
$D^{-1} = \begin{pmatrix} 3815.458 & -1597.97754 & -1296.22457 \\ -1597.97754 & 9840.62394 & -4696.6592 \\ -1296.22457 & -4696.6592 & 9514.1783 \end{pmatrix}$, $\langle e, D^{-1}e \rangle = 7988.538$,
$\langle \overline{a}, D^{-1}e \rangle = 16.60911$. Then we get $0.00208 < a_0 < 0.00548$.

Let us solve the problem (1) with the threshold value of the mathematical expectation of the payoff $a_0 = 0.003$. For clarity, we present a detailed procedure for solving this problem using formulas (6), (7).

Let us take $I = \{1,\ 2,\ 3\}$, i.e. we use the original vector of expected payoffs $\overline{a} = (0.00548, 0.00127, 0.002)$ and the original covariance matrix $D$, then we get $\langle \overline{a}, D^{-1}\overline{a} \rangle = 0.093993$ . By formulas (7) we have $\lambda = 0.01549$, $\mu = 0.00009$. Using formula (6), we have $p$= (0.33775, 0.24212, 0.42013).

Let us now solve the problem (1) with the threshold value of the mathematical expectation of the payoff $a_0$= 0.0045. Let's take $I = \{1,\ 2,\ 3\}$, then similarly by formulas (7) we have $\lambda = 0.04071$, $\mu = 0.00004$. Using formula (6), we have $p$= (0.70007, −0.08654, 0.38648). The non-negativity condition $p \geqslant 0$ is not satisfied in this case, which means that this vector $p$ is not a solution. Since $p_2$ is negative in this case, we can assume that the optimal mixed strategy contains a second zero component.

So let us take $I = \{1,\ 3\}$ , then $\tilde{a}$= (0.00548, 0.002),
$\tilde{D} = \begin{pmatrix} 0.00034 & 0.000095 \\ 0.000095 & 0.00017 \end{pmatrix}$, $\tilde{D}^{-1} = \begin{pmatrix} 3555.96955 & -2058.89531 \\ -2058.89531 & 7272.59202 \end{pmatrix}$,
$\left\langle \tilde{a}, \tilde{D}^{-1}\tilde{a} \right\rangle$=0.090743 , $\left\langle \tilde{e}, \tilde{D}^{-1}\tilde{e} \right\rangle$=6710.771, $\left\langle \tilde{e}, \tilde{D}^{-1}\tilde{a} \right\rangle$=18.64708 and by formulas (7) we obtain $\lambda = 0.04422$, $\mu = 0.00003$. Using formula (6), we have a vector of nonzero components $\tilde{p}$= (0.71830, 0.28170).

Let us check the fulfillment of the KKT condition for the crossed-out number $i$= 2. The derivative of the Lagrange function with respect to $p_2$ is $\frac{\partial L_1(p,\lambda,\mu)}{\partial p_2} = \sum_{k=1}^{3} \sigma_{2k} p_k - \lambda \overline{a}_2 - \mu$. When substituting the vector (0.71830, 0, 0.28170) and the Lagrange multipliers $\lambda$= 0.04422 and $\mu$= 0.00003, it is equal $\frac{\partial L_1(p,\lambda,\mu)}{\partial p_2}$= 0.00002. This means that all KKT conditions are satisfied and the optimal solution to problem (1) has the form $p^0$= (0.71830, 0, 0.28170).

Now let us solve problem (10) for the maximum mathematical expectation of the payoff with a restriction on the variance. For the original matrix $D$, the solution of the problem (3) is the strategy $p = (0.11532,\ 0.44388,\ 0.44079)$ and the corresponding minimum value of the objective function is $d_0 = 0.00013$.

Let us solve the problem (10) at the threshold value of the standard deviation $\sigma_0 = 0.014$ (or the variance $\sigma_0^2 = 0.0002$).

Take $I = \{1,\ 2,\ 3\}$, i.e. we will use the original vector of expected returns
$\overline{a} = (0.00548,\ 0.00127,\ 0.002)$ and the original covariance matrix $D$, then we get by formulas (11), (12) $\lambda = 28.1906, \mu = -0.00145$, $p = (0.62479,\ -0.01826,\ 0.39346)$. The non-negativity condition $p \geqslant 0$ is not satisfied in this case, which means that this vector $p$ is not a solution. Since $p_2$ is negative in this case, it can be assumed that the optimal mixed strategy contains the second zero component.

Therefore, we take $I = \{1,\ 3\}$, then $\tilde{a} = (0.00548,\ 0.002)$,
$\tilde{D} = \begin{pmatrix} 0.00034 & 0.000095 \\ 0.000095 & 0.00017 \end{pmatrix}$, and we get by (12) $\lambda = 27.63183$, $\mu = -0.00134$. Using (11), we have the vector of nonzero components $\tilde{p}$= (0.62840, 0.37161).

Let us check the fulfillment of the KKT condition for the crossed-out number $i = 2$. The derivative of the Lagrange function with respect to $p_2$ is $\frac{\partial L_2(p,\lambda,\mu)}{\partial p_2} = \overline{a}_2 - \lambda \sum_{k=1}^{3} \sigma_{2k} p_k - \mu$. When substituting the vector (0.62840, 0, 0.37161) and Lagrange multipliers $\lambda = 27.63183$ and $\mu = -0.00134$, we have $\frac{\partial L_2(p,\lambda,\mu)}{\partial p_2} = -0.00009$.

This means that all the KKT conditions are satisfied and the optimal solution of the problem (10) has the form $p^0 = (0.62840,\ 0,\ 0.37161)$.

In [21], an example of investing in shares of Russian companies for the period from 10/01/2019 to 12/31/2019 was considered. Analysis of statistical data showed that the values of the covariance of the returns of the companies under consideration were an order of magnitude less than the values of their variances. So covariances practically did not affect the calculation results, and it was legitimate to assume that they could be neglected.

In the above example with data on stock quotes of Russian companies for the period from 02/01/2021 to 05/01/2021 the covariances and variances have approximately the same order, and the covariances are positive. It can be assumed that this is due to recovery growth after the peak of the pandemic.

If we neglect the covariances in this example, then we have the following results.

Having solved problem (10) with covariances equal to zero and the same threshold value of the standard deviation, we obtain a solution of the problem (10) $p = (0.75124,\ 0,\ 0.24876)$. As you can see, the structure of the strategy has not changed, but the values of the first and third components differ significantly from these values of the vector $p^0 = (0.62840,\ 0,\ 0.37161)$.

Thus, the idea of mixed strategy calculations without neglecting the covariance of random payoffs of different pure strategies in games with nature is founded. Of course, this idea is not new in portfolio analysis, but games with nature can be models for other management tasks.

## 5. Conclusion

The purpose of this work is to develop a new approach in game theory, specifically in games with nature, related to the consideration of the correlation of random payoffs for each pair of pure strategies. The obtained theoretical results, in our opinion, can find applications in various decision-making problems. The considered example of stock investment is an illustration of the practical application of the results obtained. At the same time, we note that in general theoretical terms, we are talking about finding an optimal mixed strategy for which the condition of non-negativity of the components is mandatory (which, by the way, significantly complicates the search for a solution). Therefore, when applying this approach to stock investing, short selling is excluded. However, for the stock markets, restrictions on short sales, up to their complete ban, are not so rare.

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