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# EXTREMAL FORMS AND RIGIDITY IN ARITHMETIC GEOMETRY AND IN DYNAMICS

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**Abstract**

Ryshkov S. S. in his papers has investigated extremal forms and extremal lattices. Extremal forms and lattices are connected with hard or rigid (by M. Gromov and other) objects in mathematics. In their work with colleagues S. S. Ryshkov came also to the other hard (or rigid) objects, for instance, to rigidly connected chain.

Rigid and soft methods and results already evident in the study of the classical problems in number theory. Let us dwell briefly on the interpretation in terms of hard and soft methods of binary and ternary Goldbach problems. Since the binary (respectively ternary) Goldbach problems in their present formulation there are about equalities of the type $2n = p_1 + p_2$ (respectively $2n+1 = p_1+p_2+p_3$), where $n$ is a natural number greater than 1 (respectively $n$ is a natural number greater than 2), $p_1, p_2, p_3$ prime numbers, then these are hard (rigid) problems; the results of their studies are also hard.

However, the methods of their study include both rigid methods — the exact formula of the method of Hardy — Littlewood — Ramanujan and a combination of hard and soft methods under the investigation by the Vinogradov‘s method of trigonometric sums.

A number of problems of analytic number theory allow dynamic interpretation. We note in this regard that on connection of methods of analytic number theory and the theory of dynamical systems paid attention and has developed such analogies A. G. Postnikov.

The purpose of the paper is not to provide any sort of comprehensive introduction to rigidity in arithmetic and dynamics. Rather, we attempt to convey elementary methods, results and some main ideas of the theory, with a survey of some new results. We do not explore an exhaustive list of possible topics, nor do we go into details in proofs.

After giving an elementary number theoretic, algebraic and algebraic geometry introduction to rigid non-Archimedean spaces in the framework of local one dimensional complete regular rings, modules over rings, trees and formal schemes follow to I. R. Shafarevich, J.-P. Serre, J. Tate, D. Mumford, we review some novel results and methods on rigidity.

These include (but not exhaust) methods and results by H. Furstenberg, G. A. Margulis, G. D. Mostow, R. Zimmer, J. Bourgain, A. Furman, A. Lindenstrauss, S. Mozes, J. James, T. Koberda, K. Lindsey, C. Silva, P. Speh, A. Ioana, K. Kedlaya, J. Tuitman, and other.

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*Keywords:* rigid analytic space; Bruhat-Tits tree; formal scheme; rigid action; cocycle superrigidity; uniformly rigid ergodic action; superrigid action;

*Bibliography:* 51 titles.

# ЭКСТРЕМАЛЬНЫЕ ФОРМЫ И ЖЕСТКОСТЬ В АРИФМЕТИЧЕСКОЙ ГЕОМЕТРИИ И В ДИНАМИКЕ

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**Аннотация**

С. С. Рышков в своих работах исследовал экстремальные формы и экстремальные решетки. Экстремальные формы и экстремальные решетки связаны с жесткими (в смысле М. Громова и других) математическими объектами. В своих работах, а также в работах с коллегами С. С. Рышков пришел и к другим жестким объектам.

Жесткие и мягкие задачи, методы и результаты проявляются уже при исследовании классических проблем теории чисел. Остановимся очень кратко на интерпретации с точки зрения жестких и мягких методов бинарной и тернарной проблем Гольдбаха, проблем гольдбахова типа и методов их исследования. Так как в бинарной (соответственно, тернарной) проблемах Гольдбаха в их современной постановке речь идет о равенствах типа $2n = p_1 + p_2$ (соответственно $2n + 1 = p_1 + p_2 + p_3$), где $n$ — натуральное число, большее 1 (соответственно $n$ больше 2), $p_1, p_2, p_3$ — простые числа, то в своей постановке это жесткие проблемы; результаты их исследования также являются жесткими.

Однако методы их исследования включают как жесткие методы — точная формула метода Харди — Литтлвуда — Рамануджана (Х-Л-Р), получаемая методами комплексного анализа, так и сочетание жестких и мягких (soft) методов исследования главного члена в форме Х-Л-Р и остаточного члена методом тригонометрических сумм Виноградова.

Ряд задач аналитической теории чисел допускают динамическую интерпретацию. Отметим в связи с этим, что на связи методов аналитической теории чисел и теории динамических систем обращал внимание и развивал такие аналогии А. Г. Постников.

Целью предлагаемой работы не является исчерпывающее введение в жесткость в арифметике и в динамике. Скорее мы сделали попытку представить элементарные методы, результаты и некоторые основные идеи в этой области, вместе с обзором ряда новых результатов. Мы не даем исчерпывающего обзора возможных тем, а также не входим в детали доказательств.

После представления элементарного теоретико-числового, алгебраического и алгебро-геометрического введения в жесткие неархимедовы пространства на основе локальных одномерных полных регулярных колец, деревьев и формальных схем по И. Р. Шафаревичу, Ж.-П. Серру, Дж. Тэйту, Д. Мамфорду, мы даем обзор некоторых новых результатов и методов в направлении жесткости.

Изложение включает (но не исчерпывает) результаты и методы H. Furstenberg, G. A. Margulis, G. D. Mostow, R. Zimmer, J. Bourgain, A. Furman, A. Lindenstrauss, S. Mozes, J. James, T. Koberda, K. Lindsey, C. Silva, P. Speh, A. Ioana, K. Kedlaya, J. Tuitman, и других.

Я признателен В. М. Бухштаберу за полезные замечания в процессе обсуждения моего доклада.

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*Ключевые слова:* жесткое аналитическое пространство; дерево Брюа — Титса; формальная схема; жесткое действие; коцикленная супержесткость; равномерно жесткое эргодическое действие; супержесткое действие;

*Библиография:* 51 название.

# 1. Introduction

Ryshkov S. S. in his papers has investigated extremal forms and extremal lattices. Extremal forms and lattices are connected with hard or rigid (by M. Gromov and other) objects in mathematics. In their work with colleagues S. S. Ryshkov came also to the other hard (or rigid) objects, for instance, to rigidly connected chain.

Rigid and soft methods and results already evident in the study of the classical problems in number theory. Let us dwell briefly on the interpretation in terms of hard and soft methods of binary and ternary Goldbach problems[2, 3, 4, 5, 6]. Since the binary (respectively ternary) Goldbach problems in their present formulation there are about equalities of the type $2n = p_1 + p_2$ (respectively $2n + 1 = p_1 + p_2 + p_3$), where $n$ is a natural number greater than 1 (respectively $n$ is a natural number greater than 2) $p_1, p_2, p_3$ prime numbers, then these are hard (rigid) problems; the results of their studies are also hard. However, the methods of their study include both rigid methods — the exact formula of the method of Hardy — Littlewood — Ramanujan and a combination of hard and soft methods under the investigation by the Vinogradov‘s method of trigonometric sums.

A number of problems of analytic number theory allow dynamic interpretation. We note in this regard that on connection of methods of analytic number theory and the theory of dynamical systems paid attention and has developed such analogies A. G. Postnikov [45]. An interesting approach to rigid and soft models is proposed by V. Arnold [8]. Special considerations need for application of the approach to problems of number theory and algebra of our paper.

The purpose of the paper is not to provide any sort of comprehensive introduction to rigidity in arithmetic and dynamics. Rather, we attempt to convey elementary methods, results and some main ideas of the theory, with a survey of some new results. We do not explore an exhaustive list of possible topics, nor do we go into details in proofs.

After giving an elementary number theoretic, algebraic and algebraic geometry introduction to rigid non-Archimedean spaces in the framework of local one dimensional complete regular rings, modules over rings, trees and formal schemes follow to I. R. Shafarevich, J.-P. Serre, J. Tate, D. Mumford, we review some novel results and methods on rigidity.

These include (but not exhaust) methods and results by S. S. Ryshkov [1], H. Furstenberg, G. A. Margulis, G. D. Mostow, M. Gromov, R. Zimmer, J. Bourgain, A. Furman, A. Lindenstrauss, S. Mozes, J. James, T. Koberda, K. Lindsey, C. Silva, P. Speh, A. Ioana, K. Kedlaya, J. Tuitman, and other.

## 2. Quadratic modules over integers and sums of squares

Here we consider the partial case of quadratic modules [21] over integer numbers when the quadratic form is the sum of $n$ squares. Let $\Lambda$ be a lattice [21] in $n$-dimensional real euclidean space that is defined by congruences. Davenport, Mordell, Cassels and others used the lattices and Minkowski‘s convex body theorem for proving results about existence of nontrivial solutions of some Diophantine equations.

We will give examples below. Recall the case of positive quadratic forms.

Let $\tau$ be a complex number, $Im\ \tau > 0, q = \exp \pi i \tau$,

$$\theta_3(\tau) = \sum_{m=-\infty}^{\infty} q^{m^2}$$

the Jacobi function. Denote by $\mathbb{Z}^n$ the $d$-dimensional integer lattice.

Let $r_n(m)$ be the number of ways of writing $m$ as a sum $f(x_1, \ldots, x_n) = f$ of $n$ squares. Put $\Theta_{\mathbb{Z}^n} = \theta_3(\tau)^n$.

## 2.1. Sums of two squares

Let $p \equiv 1 \pmod 4$. In the case there is the integer $l$ such that $l^2 + 1 \equiv 0 \pmod p$. The lattice $\Lambda$ of pairs $(a, b)$ of integer numbers is defined by congruences $a \equiv lb \pmod p$ and has determinant $d(\Lambda) \leqslant p$. From this and Minkowski‘s convex body theorem follow that every prime $p \equiv 1 \pmod 4$ is the sum of two squares.

Let $\chi$ be the nontrivial Dirichlet character mod 4, integer $m > 0$. There is the well known

Proposition 1. *The number of integer solutions of the equation* $x_1^2 + x_2^2 = m$ *is equal* $4\sum_{d|m} \chi(d)$.

*In the framework of the function* $\Theta_{\mathbb{Z}^n}$ *we have*

$$\Theta_{\mathbb{Z}^2} = \sum_{m=0}^{\infty} r_2(m) q^m.$$

## 2.2. Sums of three squares

In the case and in the case $n = 4$ it is possible to use quaternions (hermitions) but for simplicity we will formulate the well known result by $\Theta_{\mathbf{Z}^3}$ and $r_3(m)$.

Proposition 2.

$$\Theta_{\mathbb{Z}^3} = \sum_{m=0}^{\infty} r_3(m) q^m.$$

## 2.3. Sums of four squares

The quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$ represents all positive numbers (Lagrange). The number of solutions of the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = m$, where $m$ is a positive integer is given by Jacobi.

Proposition 3. *The number of integer solutions of the equation*

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = m$$

*is equal*

$$8 \sum_{d|m} d$$

*if* $m = 2k + 1$, *and is equal*

$$24 \sum_{d|m} d$$

*if* $m = 2k$.

*In the framework of the function* $\Theta_{\mathbb{Z}^n}$ *we have*

$$\Theta_{\mathbb{Z}^4} = \sum_{m=0}^{\infty} r_4(m) q^m.$$

### 2.4. Sums of squares greater than four

Recall elements of Hardy-Littelwood-Kloosterman method in the case. This is valid also in the previous case $n = 4$. Consider a function of complex variable $u, |u| < 1$

$$\vartheta(f, u) = \sum_{x_1 \cdots x_n = -\infty}^{\infty} u^{f(x_1 \cdots x_n)}$$

Then the number $r_n(m)$ of ways of writing $m$ as a sum of $n$ squares by Cauchy's integral formula is given as

$$r_n(m) = \frac{1}{2\pi i} \oint_{\Gamma} \vartheta(f, u) u^{-m-1} du$$

where $\Gamma$ is the circle $|u| = \exp(-\frac{1}{m})$. We omit here the very important step of the dividing $\Gamma$ into Farey-arcs.

## 3. Elements of history of rigidity

The history of rigidity is reflected in papers by A. Selberg, E. Calabi, E. Vesentini, A. Weil, H. Furstenberg, G. Mostow, G. A. Margulis and their colleagues [22, 23, 24, 25, 26, 27]. There are interesting surveys by D. Fisher [28] and R. Spatzier [29]. Let $G$ be a finitely generated group, $D$ a topological group, and $h : G \to D$ a homomorphism. Follow to [28] recall

DEFINITION 1. *Given a homomorphism* $h : G \to D$, *it is said that* $h$ *is locally rigid if any other homomorphism* $h^{‘}$ *which is close to* $h$ *is conjugate to* $h$ *by a small element of* $D$.

Recall follow to [23, 24] in framework of [29] the Local Rigidity Theorem.

THEOREM 1. *Cocompact discrete subgroups* $H$ *in semisimple Lie groups without compact nor* $SL(2, \mathbf{R})$ *nor* $SL(2, \mathbf{C})$ *local factors is deformation rigid.*

The notion of uniform rigidity was introduced as a topological version of rigidity by S. Glasner and D. Maon [30].

# 4. Rigid non-Archimedean spaces and Formal Groups

At first we formulate very briefly some elementary (and probably well known) results on connections among local one dimensional complete regular rings, trees and formal schemes. We follow to [9, 10, 52].

Let $A$ be a local one dimensional complete regular ring with maximal ideal $\pi$, $K$ its field of fractions with the multiplicative group $K^*$, $V$ a two dimensional vector space over $K$, $M$ a module of the rank 2 over $A$ (a two-dimensional lattice in the space $V$). Denote by $S(M)$ the symmetric algebra of the module $M$. The main example is the case of the ring $A = \mathbb{Z}_p$ of integer $p-$ adic numbers, $K = \mathbb{Q}_p$ the field of $p-$ adic numbers, $\pi = p$ the prime number, $M$ a module of the $rank_{\mathbb{Z}_p} M = 2$ over $\mathbb{Z}_p$.

Definition 2. *Let $K$ be a locally compact non-Archimedean field, $A$ its valuation ring, $\mathfrak{m}$ the maximal ideal of $A$. A free module of rank $n$ over $A$ is called a lattice in $K^n$.*

Two modules $M$ and $M^‘$ of the rank 2 over $A$ are called similar if $M^‘ = xM$, $x \in K^*$. Denote by $\mathcal{T}$ the set of classes of similar modules.

Definition 3. *Let $X$ be the graph whose vertices are equivalence classes $[M]$ of similar modules $M$ of the rank 2 over $A$ in $V$ , where two vertices $x$ and $y$ are joint by an edge if $x = [M]$ and $y = [M^‘]$ with $M^‘ \subset M$, $M^‘ \not\subset \pi M$, $M/M^‘ \simeq A/\pi A$.*

Two modules are called adjacent if the length $l(M/M^‘) = 1$, i.e. $M/M^‘ \simeq A/\pi A$.

Theorem 2. *The graph $X$ is a homogeneous or a regular tree. We will denote the tree by $\mathcal{T}$.*

By $\partial\mathcal{T}$ denote the set of ends of $\mathcal{T}$ and by $\mathbb{P}^1(A)$ denote the one-dimensional projective space over $A$.

Theorem 3. $\partial\mathcal{T} \simeq \mathbb{P}^1(A)$.

Recall that a group $G$ acts on a set $X$ if there is a map $G \times X \to X, (g; x) \mapsto gx$ such that the following are true: (i) For $e$ the identity of $G, ex = x$; (ii) For $h; g \in G, x \in X, h(gx) = (hg)x$. On the space $V$ act the projective linear group $PGL_2(K)$ and its subgroups. This action extends to the action on the tree $\mathcal{T}$.

Theorem 4. *Let a group $G$ acts on a tree $\mathcal{T}$ without fixed points and without inversions. Then $G$ is the free group.*

Let a group $G \subset PGL_2(K)$ acts on $\mathcal{T}$ discretely and freely. Follow to Mumford it is possible to construct a subtree $\mathcal{T}_G$ of $\mathcal{T}$.

Theorem 5. *If the group $G$ has finite number of generators then $\mathcal{T}_G/G$ is finite.*

For the above mentioned symmetric algebra $S(M)$ of the module $M$ define the corresponding scheme $\mathbb{P}(M)$ by the formula $\mathbb{P}(M) = Proj\ S(M)$. For each module $M \hookrightarrow V$ there is the birational isomorphism $\mathbb{P}(M) = \mathbb{P}^1(A) \otimes_A K \xrightarrow{\varphi_M} \mathbb{P}^1_K$. Now let $\mathcal{S}$ be a finite subtree of $\mathcal{T}$. It is possible to construct many formal schemes from these data. We indicate here the formal scheme $\mathcal{P}$ that is the formal completion $(\mathbb{P}(\mathcal{S})_0)$ of the scheme $\mathbb{P}(\mathcal{S})$ along its closed fibre $\mathbb{P}(\mathcal{S})_0$ only. Recall that the generic fiber of $\mathbb{P}^1(A)$ is the one-dimensional projective space $\mathbb{P}^1_K$ over $K$.

# 5. Formal groups and formal stacks

Here we present results on two-dimensional commutative formal groups and on formal stacks

## 5.1. On two-dimensional commutative formal groups

Let $F$ be a commutative formal group low of $n$ variables over commutative ring $R$ with unit. In the case $n = 1$, following to the known results by M. Lazard, there is only one $1-$ bud of the form $x + y + \alpha xy$.

Proposition 4. *Let $n = 2$, $A = \mathbb{Z}_p[\alpha, \beta]$ be the ring of polynomials with integer p-adic coefficients from $\alpha, \beta$. $1-$ buds are*

$$F(x,y) = \begin{cases} x_1 + y_1 + \alpha x_1 y_1 \\ x_2 + y_2 + \beta x_2 y_2, \end{cases}$$

$$F_a(x,y) = \begin{cases} x_1 + y_1 + \alpha x_1 y_1 \\ x_2 + y_2 + \beta x_1 y_1, \end{cases}$$

$$F_b(x,y) = \begin{cases} x_1 + y_1 + \alpha x_2 y_2 \\ x_2 + y_2 + \beta x_2 y_2, \end{cases}$$

$$F_c(x,y) = \begin{cases} x_1 + y_1 + \alpha(x_1 + x_2)(y_1 + y_2) \\ x_2 + y_2 + \beta(x_1 + x_2)(y_1 + y_2), \end{cases}$$

Remark 1. *$1-$ buds given in Proposition 1 are also two-dimensional formal group lows, whose coefficients under terms of degrees $\geqslant 3$ are zeros.*

Remark 2. *These group lows define classes of group lows. In particular, the class $F_a$ contains under values of parameters $\alpha = 0, \beta = -1$, the Witt group, that corresponds to prime number $p = 2$.*

### 5.2. Formal stacks

Let now the ring $R$ is the field $k$. Recall, that formal $k-$ scheme is formal $k-$ functor, that is the limit of directed inductive system of finite $k-$ schemes, and a formal group is a group object in the category of formal $k-$ schemes. The notion of a stack, as one of category theory variants of moduli space is defined by P. Deligne and D. Mumford.

Proposition 5. *There exist formal stacks, that are categories that are bundled on formal groupoids and that satisfy axioms of decent theory.*

## 6. Uniformly rigid and measurable weak mixing

Authors of the paper [14] investigate properties of uniformly rigid transformations and analyze the compatibility of uniform rigidity and measurable weak mixing along with some of their asymptotic convergence properties.

This interesting survey includes some resent results on genericity of rigid and multiply recurrent infinite measure preserving and nonsingular transformations by O. Ageev and C. Silva [31] and on measurable sensitivity by J. James, T. Koberda, K. Lendsey, C. Silva, P. Speh [32].

All spaces of the paper [14] are considered simultaneously as topological spaces and as measure spaces. Presented results concern either the measurable dynamics on the spaces or the interplay between the measurable and topological dynamics.

After some introductory section, second section of the paper[14] considers functional analytic properties of uniform rigidity that is similar to the properties of rigidity. Authors formulate and prove

Theorem 6. *(Theorem 1.) Every totally ergodic finite measure-preserving transformation on a Lebesgue space has a representation that is not uniformly rigid, except in the case where the space consists of a single atom.*

The proof of the theorem connects with results of authors of the paper [14] that uniform rigidity and weak mixing are mutually exclusive notions on a Cantor set, and follows from the Jewett-Krieger Theorem by [33].

Third section concerns with uniform rigidity and measurable weak mixing.

Author motivation for this topic is that a (nontrivial) measure-preserving weakly mixing transformation that is uniformly rigid would yield an example of a measurable sensitive transformation that is not strongly measurably sensitive. For a subset $Y$ of a metric space $X$ and a measurable transformation of $X$ authors of the paper [14] define the notion of uniformly rigid transformation on $Y$ and prove Theorem 3.4 that is reminiscent of Egorov Theorem by P. Halmos [34]. In forth section authors present asymptotic convergence behavior. Let $X$ be a compact metric space and let $T$ be a finite measure-preserving ergodic transformation. Authors prove:

PROPOSITION 6. *If $T$ is uniformly rigid, then the uniform rigidity sequence has zero density.*

The aim of section five is to study group action and generalized uniform rigidity. Let $G$ be a countable group endowed with the discrete topology acting faithfully on a finite measure space by measure-preserving transformations. Following authors of the paper [14] the action of $G$ is uniformly rigid if there exists a sequence $\{g_i\}$ of group elements that leaves every compact $K \subset G$, denoted $g_i \to \infty$, such that $d(x, g_i \cdot x) \to 0$ uniformly. The main result of the section is Theorem 5.3:

THEOREM 7. *Let $X$ admit a weakly mixing group action and a uniformly rigid action by nontrivial subgroups of a fixed group of automorphisms $G$. Then there exists a $G-$action on $X$ that is simultaneously weakly mixing and uniformly rigid.*

Authors formulate several interesting questions that arise under investigations of weak mixing and uniform rigidity.

Some results and methods that are connected with topics of this and next section are considered in the paper [51].

# 7. Actions of groups and semigroups

Furstenberg and Berent investigate the action of abelian semigroups on the torus $\mathbf{T}^d$ for $d = 1$ and $d > 1$ respectively. The authors of the paper [12] extend to the noncommutative case some results of Furstenberg and Berent. Author‘s results answer problems raising by H. Furstenberg [35] and by Y. Guivarc'h [private communication to authors of the paper [12]].

Let $\nu$ be a probability measure on $SL_d(\mathbf{Z})$ satisfying the moment condition

$$\mathbf{E}_\nu(\| g \|^\varepsilon) < \infty$$

for some $\varepsilon$. The authors of the paper [12] show that if the group generated by the support of $\nu$ is large enough, in particular if this group is Zariski dense in $SL_d$, for any irrational $x \in \mathbf{T}^d$ the probability measures $\nu^{*n} * \delta_x$ tend to the uniform measure on $\mathbf{T}^d$. If in addition $x$ is Diophantine generic, authors show this convergence is exponentially fast.

This interesting survey includes resent results on rigidity theory by M. Einsiedler, E. Lindenstrauss [36] and by G.A. Margulis [37], convolution of measures, on $\nu-$stiff action, on Fourier coefficients of measures and on notions of coarse dimension.

Let the action of semigroup $\Gamma$ on $\mathbf{T}^d$ satisfy the following three conditions: ($\Gamma$−0) $\Gamma < SL_d(\mathbf{R})$, ($\Gamma$−1) $\Gamma$ acts strongly irreducibly on $\mathbf{R}^d$, ($\Gamma$−2) $\Gamma$ contains a proximal element: there is some $g \in \Gamma$ with a dominant eigenvalue which is a simple root of its characteristic polynomial.

In Section 1 authors formulate main result of the paper.

Theorem 8. *Let $\Gamma < SL_d(\mathbf{R})$ satisfy ($\Gamma-1$) and ($\Gamma-2$) above, and let $\nu$ be a probability measure supported on a set of generators of $\Gamma$ satisfying*

$$\sum_{g\in\Gamma} \nu(g) \parallel g \parallel^{\epsilon} < \infty$$

*for some $\epsilon > 0$. Then for any $0 < \lambda < \lambda_1(\nu)$ there is a constant $C = C(\nu, \lambda)$ so that if for a point $x \in \mathbf{T}^d$ the measure $\mu_n = \nu^{*n} * \delta_x$ satisfies that for some $a \in \mathbf{Z}^d \setminus \{0\}$ $\mid \hat{\mu}_n(a) \mid > t > 0$, with $n > C \cdot\ log(\frac{2\|a\|}{t})$, then $x$ admits a rational approximation $p/q$ for $p \in \mathbf{Z}^d$ and $q \in \mathbf{Z}_+$ satisfying $\parallel x - \frac{p}{q} \parallel < \exp^{-\lambda n}$ and $\mid q \mid < (\frac{2\|a\|}{t})^C$.*

Authors of [12] denote the theorem as Theorem A.

Section 2 is devoted to the deduction of two corollaries from Theorem A. Let in the corollaries $\Gamma$ and $\nu$ be as in theorem A.

Corollary 1. *Let $x \in \mathbf{T}^d \setminus (\mathbf{Q}/\mathbf{Z})^d$. Then the measures $\mu_n = \nu^{*n} * \delta_x$ converge to the Haar measure $m$ on $\mathbf{T}^d$ in weak-* topology.*

This is authors [12] Corollary B. Next corollery is the authors [12] Corollary C:

Corollary 2. *Let $x \in \mathbf{T}^d$ and $\mu_n = \nu^{*n} * \delta_x$. Then there are $c_1, c_2$ depending only on $\nu$ so that the following holds: (1) Assume $x$ is Diophantine generic in the sense that for some $M$ and $Q$ $\parallel x - \frac{p}{q} \parallel > q^{-M}$ for all integers $q \geqslant Q$ and $p \in \mathbf{Z}^d$. Then for $n > c_1 \log Q$ $\max_{b\in\mathbf{Z}^d, \|b\|<B} \mid \hat{\mu}_n(b) \mid < B \exp^{-c_2 n/M}$. (2) Assume $x \notin \mathbf{Q}/\mathbf{Z})^d$. Then there is a sequence $n_i \to \infty$ along which $\max_{b\in\mathbf{Z}^d, 0<\|b\|<\exp^{c_2 n_i}} \mid \hat{\mu}_n(b) \mid < \exp^{-c_2 n_i}$.*

Section 3 gives the deduction of authors' solution of Furstenberg problem from the authors [12] Proposition 3.1:

Proposition 7. *Let $\Gamma$ and $\nu$ be as in theorem A, $0 < \lambda < \lambda_1(\nu)$. Then for some constant $C$ depending on $\nu, \lambda$ the following holds: for any probability measure $\mu_0$ on $\mathbf{Z}^d$, if $\mu_n = \nu^{*n} * \mu_0$ has a nontrivial Fourier coefficient $a \in \mathbf{Z}^d \setminus \{0\}$ $\mid \hat{\mu}_n(a) \mid > t$, with $n > C \cdot\ log(\frac{2\|a\|}{t})$, then $\mu_0(W_{Q,\exp^{-\lambda n}}) > (\frac{t}{2})^C$ where $Q = (\frac{2\|a\|}{t})^C$.*

Theorem A follows from Proposition 3.1.

Section 4 is devoted to random matrix products. It includes estimates of the metric on $\mathbf{P}^{d-1}$ and random walks. In Section 5 two notions of coarse dimension are discussed. Section 6 describes the structure of the set of $t-$large Fourier coefficients. The last Section "Granulated measures"gives the prove of Proposition 3.1.

The results of the paper[12] will be of use to specialists interested in Diophantine approximation, measure theory and algebraic dynamics.

# 8. Rigid cohomology

At first we review the necessary results on the connection and on the Gauss-Manin connection. We follow the ideas and results by Grothendieck, Griffiths, Manin, Katz, Deligne and others.

## 8.1. Connection

Let $S/k$ be the smooth scheme over field $k$, $U$ an element of open covering of $S$, $\mathcal{O}_S$ the structure sheaf on $S$, $\Gamma(U, \mathcal{O}_S)$ the sections of $\mathcal{O}_S$ on $U$. Let $\Omega^1_{S/k}$ be the sheaf of germs of 1−dimension differentials, $\mathcal{F}$ a coherent sheaf on $S$. The *connection* on the sheaf $\mathcal{F}$ is the sheaf homomorphism

$$\nabla : \mathcal{F} \to \Omega^1_{S/k} \otimes \mathcal{F},$$

such that, if $f \in \Gamma(U, \mathcal{O}_S),\ g \in \Gamma(U, \mathcal{F})$ then

$$\nabla(fg) = f\nabla(g) + df \otimes g.$$

There is the dual definition. Let $\mathcal{F}$ be the locally free sheaf, $\Theta^1_{S/k}$ the dual to sheaf $\Omega^1_{S/k}$, $\partial \in \Gamma(U, \Theta^1_{S/k})$. The *connection* is the homomorphism

$$\rho : \Theta^1_{S/k} \to End_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}),$$

$$\rho(\partial)(fg) = \partial(f)g + f\rho(\partial).$$

## 8.2. Integration of connection

Let $\Omega^i_{S/k}$ be the sheaf of germs of $i$−differentials,

$$\nabla^i(\alpha \otimes f) = d\alpha \otimes f + (-1)^i \alpha \wedge \nabla(f).$$

Then $\nabla,\ \nabla^i$ define the sequence of homomorphisms:

$$\mathcal{F} \to \Omega^1_{S/k} \otimes \mathcal{F} \to \Omega^2_{S/k} \otimes \mathcal{F} \to \cdots,. \tag{1}$$

The map

$$K = \nabla \circ \nabla^1 : \mathcal{F} \to \Omega^2_{S/k} \otimes \mathcal{F}$$

is called the curvature of the connection $\nabla$.

The *cochain complex*

$$(K^\bullet, d) = \{K^0 \xrightarrow{d} K^1 \xrightarrow{d} K^2 \xrightarrow{d} \cdots\}$$

is the sequence of abelian groups and differentials $d : K^p \to K^{p+1}$ with the condition $d \circ d = 0$.

Let $\mathcal{A}$ be an abelian category, $\mathcal{K}(\mathcal{A})$ the category of complexes over $\mathcal{A}$. Furthermore, there are various full subcategories of $\mathcal{K}(\mathcal{A})$ whose respective objects are the complexes which are bounded below, bounded above, bounded in both sides.

A connection is *integrable* if (1) is a complex.

PROPOSITION 8. *The statements a), b), c) are equivalent:*
*a) the connection $\nabla$ is integrable;*
*b) $K = \nabla \circ \nabla^1 = 0$;*
*c) $\rho$ is the Lie-algebra homomorphism of sheaves of Lie algebras.*

More generally, let $X, S$ be smooth schemas over $k, f : X \to S$ a smooth morphism,

$$\Omega^\bullet_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \cdots ,$$

the de Rham complex of relative differentials. There is the exact sequence

$$0 \to f^*(\Omega^\bullet_S) \to \Omega^\bullet_X \to \Omega^\bullet_{X/S} \to 0.$$

In the case there is an integrable connection, the Gauss-Manin connection.

By an abelian sheaf we mean a sheaf of abelian groups. Let $R^0 f_*$ be the functor from the category of complexes of abelian sheaves on $X$ to the category of abelian sheaves on $S$. Denote by $\mathcal{H}^i_{DR}(X/S)$ the sheaf of de Rham cohomologies such that

$$\mathcal{H}^i_{DR}(X/S) = R^i f_* \Omega^\bullet_{X/S}.$$

Recall that

$$\mathcal{H}^i = \mathcal{H}^i_{DR} = R^i f_* \Omega^\bullet_{X/S}$$

is called the Gauss-Manin bundle.

Here $R^i f_*, i > 0$, are the hyperderved functor of $R^0 f_*$.

For each $i \geqslant 0, \mathcal{H}^i_{DR}(X/S)$ is a locally free coherent algebraic sheaf on $S$, whose fiber at each point $s \in S$ is the $\mathbf{C}-$vector space $H^i(X_s, \mathbf{C})$ and has the Gauss-Manin connection. $\mathcal{H}^i_{DR}(X/S)$ has the main interpretation as the Picard-Fuchs equations and $H^i(X_s, \mathbf{C})$ in the interpretation is the local system of germs of solutions of the equations.

There is the canonical filtration of $\Omega^\bullet_{X/S}$ by locally free subsheaves

$$\Omega^\bullet_{X/S} = F^0(\Omega^\bullet_{X/S}) \supset F^1(\Omega^\bullet_{X/S}) \supset \cdots$$

given by

$$F^i(\Omega^n_X) := Im(f^*(\Omega^i_S) \otimes \Omega^{n-i}_X \to \Omega^n_X).$$

Let

$$gr^i = F^i/F^{i+1},\ i = 0, 1, \dots$$

Example 1. *Let $S = Spec\, B, X = Spec\, A$ be schemas over $k$ of algebraic dimensions $dim\ B = 1$ (for instance, $B = k[x]$, ) $dim\ A = 2$,*

$$\mathcal{H}^r_{DR}(X/S) = H^r_{DR}(A/B).$$

*Let $\Omega^1_B = Bdt,\ r = 1$,*

$$0 \to gr^1 \to F^0/F^2 \to gr^0 \to 0$$

*where $F^0 = \Omega^\bullet_A$. The exect sequence has the form*

$$0 \to \Omega^1_B \otimes A \oplus \Omega^2_A \to \Omega^\bullet_A \to A \oplus \Omega^1_{A/B} \to 0.$$

Consider the simplest case of elliptic curve

$$y^2 = x^3 + t.$$

In the case

$$B = k[t, t^{-1}], A = B[x, y]/(y^2 - x^3 + t),$$

and $\omega = \frac{dx}{y} \in \Omega^1_A$. Finally, we obtain a partial case of Fuchs equation:

$$\frac{d\omega}{dt} + \frac{1}{6t}\omega = 0.$$

### 8.3. Rigidity

Let $p$ be a prime, $n$ a positive integer, and $\mathbf{F}_q$ the finite field with $q = p^n$ elements. Let $\mathbf{Q}_q$ denote the unique unramified extension of degree $n$ of the field of $p$-adic numbers. Let $U$ be an open dense subscheme of the projective space $\mathbf{P}^1_{\mathbf{Q}_q}$ with nonempty complement $Z$. Let $V$ be the rigid analytic subspace of $\mathbf{P}^1_{\mathbf{Q}_q}$ which is the complement of the union of the open disks of radius 1 around the points of $Z$. A Frobenius structure on $\mathcal{E}$ with respect to $\sigma$ is an isomorphism $\mathcal{F} : \sigma^*\mathcal{E} \simeq \mathcal{E}$ of vector bundles with connection defined on some strict neighborhood of $V$.

A meromorphic connection on $\mathbf{P}^1$ over a $p$-adic field admits a Frobenius structure defined over a suitable rigid analytic subspace. Authors of the paper[11] give an effective convergence bound for this Frobenius structure by studying the effect of changing the Frobenius lift. They also give an example indicating that their bound is optimal.

The techniques used are computational. This is a good place to see the interplay between matrix representation of a Frobenius structure and a Gauss-Manin connection.

The theory of rigid $p$-adic cohomology are developed by Berthelot [38] and others. Rigid cohomology in some sense extends crystalline cohomology. Review of some novel results and applications of crystalline cohomology is given in paper [52].

## 9. Superrigidity

The notion of property (T) for locally compact groups was defined by D. Kazhdan [39] and the notion of relative property (T) for inclusion of countable groups $\Gamma_0 \subset \Gamma$ was defined by G. Margulis [40].

The concept of superrigidity was introduced by G. D. Mostow [41] and by G. A. Margulis [42] in the context of studying the structure of lattices in rank one and higher rank Lie groups respectively. The first result on orbit equivalent (OE) superrigid actions was obtained by A. Furman [43], who combined the cocycle superrigidity by R. Zimmer [44] with ideas from geometric group theory to show that the actions $SL_n(\mathbf{Z}) \to T^n (n \geqslant 3)$ are OE superrigid. The deformable actions of rigid groups are OE superrigid by S. Popa [45].

The paper [13] presents a new class of orbit equivalent superrigid actions. The main result of the paper [13] is the Theorem A on orbit equivalence (OE) superrigidity. As a consequence of Theorem A the author can constructs uncountable many non-OE profinite actions for the arithmetic groups $SL_n(\mathbf{Z})(n \geqslant 3)$, as well as for their finite subgroups, and for the groups that are semi direct products of groups $SL_m(\mathbf{Z})$ and $\mathbf{Z}^m(m \geqslant 2)$. The author deduces Theorem A as a consequence of the Theorem B on cocycle superrigidity.

Let $\Gamma \to X$ be a free ergodic measure-preserving profinite action (i.e., an inverse limit of actions $\Gamma \to X_n$ with $X_n$ finite) of a countable property (T) group $\Gamma$ (more generally, of a group $\Gamma$ which admits an infinite normal subgroup $\Gamma_0$ such that the inclusion $\Gamma_0 \subset \Gamma$ has relative property (T) and $\Gamma/\Gamma_0$ is finitely generated) on a standard probability space $X$. The author prove that if $\omega : \Gamma \times X \to \Lambda$ is a measurable cocycle with values in a countable group $\Lambda$, then $\omega$ is a cohomologous to a cocycle $\omega^{'}$ which factors through the map $\Gamma \times X \to \Gamma \times X_n$, for some $n$. As a corollary, he shows that any free ergodic measure-preserving action $\Lambda \to Y$ comes from a (virtual) conjugancy of actions.

# 10. Newton strata in the loop group of a reductive group

Let $G$ be a split connected reductive group over $\mathbb{F}_p$, let $T$ be a split maximal torus of $G$ and let $LG$ be the loop group of $G$ by Faltings [46].

Let $R$ be a $\mathbb{F}_q$-algebra and $K$ be the sub-group scheme of $LG$ with $K(R) = G(R[[z]])$. Let $\sigma$ be the Frobenius of $k$ over $\mathbb{F}_q$ and also of $k((z))$ over $\mathbb{F}_q((z))$. For algebraically closed $k$, the set of $\sigma$-conjugacy classes $[b] = \{g^{-1}b\sigma(g) | g \in G(k((z)))\}$ of elements $b \in LG(k)$ is classified by two invariants, the Kottwitz point $\kappa_G(b)$ and the Newton point $\nu$.

The author of the paper [47] proves the following two main results.

THEOREM 9. *Let $S$ be an integral locally noetherian scheme and let $b \in LG(S)$. Let $j \in J(\nu)$ be a break point of the Newton point $\nu$ of $b$ at the generic point of $S$. Let $U_j$ be the open subscheme of $S$ defined by the condition that a point $x$ of $S$ lies in $U_j$ if and only if $\mathrm{pr}_{(j)}(\nu_b(x)) = \mathrm{pr}_{(j)}(\nu)$. Then $U_j$ is an affine $S$-scheme.*

THEOREM 10. *Let $\mu_1 \preceq \mu_2 \in X_*(T)$ be dominant coweights. Let*

$$S_{\mu_1,\mu_2} = \bigcup_{\mu_1 \preceq \mu^{'} \preceq \mu_2} K z^{\mu^{'}} K.$$

*Let $[b]$ be a $\sigma$-conjugacy class with $\kappa_G(b) = \overline{\mu}_1 = \overline{\mu}_2$ as elements of $\pi_1(G)$ and with $\nu_b \preceq \mu_2$. Then the Newton stratum $N_b = [b] \cap S_{\mu_1,\mu_2}$ is non-empty and pure of* *codimension* $\langle \rho, \mu_2 - \nu_b \rangle + \frac{1}{2} def(b)$ *in* $S_{\mu_1,\mu_2}$. *The closure of* $N_b$ *is the union of all* $N_{b^{`}}$ *for* $[b^{`}]$ *with* $\kappa_G(b^{`}) = \overline{\mu_1}$ *and* $\nu_{b^{`}} < \nu_b$.

Here $\rho$ is the half-sum of the positive roots of $G$ and the defect $def(b)$ is defined as $\mathrm{rk}G - \mathrm{rk}_{\mathbf{F}_q} J_b$ where $J_b$ is the reductive group over $\mathbb{F}_q$ with $J_b(k((z))) = \{g \in LG(\overline{k}) | gb = b\sigma(g)\}$ for every field $k$ containing $\mathbb{F}_q$ and with algebraically closed $\overline{k}$.

The proof of Theorem 9 is based on a generalization of some results by Vasiu [48]. An interesting feature of E. Viehmann method in the prove of Theorem10 is the using of various results on the Newton stratification on loop groups as Theorem 9 and the dimension formula for affine Deligne-Lusztig varieties by G'ortz, Haines, Kottwitz, Reuman [49] together with results on lengths of chains of Newton points by Chai [50].

# 11. Conclusion

Rigid problems, methods and results in arithmetic algebraic geometry and in dynamics have presented. Diverse notions of rigidity and respective novel results are reviewed.

## REFERENCES


1. Ryshkov, S. S. 1970, "The polyhedron $u(m)$ and some extremal problems of the geomety of numbers" , *Soviet Math. Dokl.*, vol. 11, pp. 1240–1244. (Russian)

2. Hardy, G. H. & Littlewood, J. E. 1923, "Some problems of "Partitio Numerorum": V. A further contribution to the study of Goldbach's problem" , *Proc. London Math. Soc., Ser. 2*, vol. 22, pp. 46–56.

3. Vinogradov, I. M. 1980, "Metod trigonometricheskikh summ v teorii chisel" [The method of trigonometric sums in the theory of numbers] Second edition. *"Nauka", Moscow*, 144 pp. (Russian)

4. Vinogradov, I. M. 1976, "Osobye varianty metoda trigonometricheskikh summ" [Special variants of the method of trigonometric sums] *Izdat. "Nauka", Moscow*, 119 pp. (Russian)

5. Arkhipov, G. I., Karatsuba, A. A. & Chubarikov, V. N. 1987, "Teoriya kratnykh trigonometricheskikh summ" [Theory of multiple trigonometric sums] *"Nauka", Moscow*, 368 pp. (Russian)

6. Arkhipov, G. I. & Chubarikov, V. N. 2002, "On the exceptional set in a Goldbach-type binary problem" , *Dokl. Akad. Nauk*, vol. 387, no. 3, pp. 295–296. (Russian)

7. Postnikov, A. G. 2005, "Izbrannye trudy. Pod redaktsiej V. N. Chubarikova.", *Izdal. Fizmatlit, Moscow*, 512 p. (Russian)

8. Arnild, V. I. 2004, "«Gestkie» i «myagkie» matematicheskie modeli" ["Hard" and "soft" mathematical models] *Izdat. MTsNMO, Moscow.* (Russian)

9. Shapharevich, I. R. 1988, "Foundations of Algebraic Geometry", vol. 1, vol. 2, *Moscow: Nauka*, 351 p., 304 p. (Russian)

10. Serre, J.-P. 2003, "Trees.", *Berlin–Heidelberg–New York: Springer –Verlag.*

11. Kedlaya, K. & Tuitman, J. 2012, "Effective convergence bounds for Frobenius structures on connections", *Rend. Semin. Mat. Univ. Padova*, vol. 128, pp. 7–16.

12. Bourgain, J., Furman, A., Lindenstraussl, E. & Mozes, S. 2011, "Stationary measures and equidistribution for orbits of nonabelian semigroups on the torus", *J. Am. Math. Soc.*, vol. 1, pp. 231–280.

13. Ioana, A. 2011, "Cocycle superrigidity for profinite actions of property (T) groups", *Duke Math. J.* 2, pp. 337–367.

14. James, J., Koberda, T., Lindsey, K., Silva, C. & Speh, P. 2009, "On ergodic transformations that are both weakly mixing and uniformly rigid", *New York J. Math.* 15, pp. 393–403.

15. Gromov, M. 1986, "Soft and Hard Symplectic Geometry", *Proceedings of the International Congress of Mathematicians, Berkeley, California, USA.*, vol. I, pp. 81–98.

16. Hartshorne, R. 1977, "Algebraic Geometry," *Springer –Verlag, Berlin–Heidelberg–New York.*

17. Tate, J. 1971, "Rigid analytic spaces", *Invent. Math.*, vol. 12, pp. 257–289.

18. Mumford, D. 1972, "An analytic construction of degenerating curves over complete local rings", *Compositio Mathematica.*, vol. 24, pp. 129–174.

19. Raynaud, M. 1974, "Géométrie analytique rigide d'aprés Tate, Kiehl,. . . ,table ronde a'analyse non-archimedienne", *Bull. Soc. math. France*, vol. 39–40, pp. 319–327.

20. Demazure, M. 1972, "Lectures on $p$-divisible groups", *Springer Verlag, Berlin*, LNM 302.

21. Serre, J.-P. 1973, "A Course of Arithmitic," *Springer –Verlag, Berlin–Heidelberg–New York.*

22. Selberg, A. 1960, "On discontinuous groups in higher-dimensional symmetric spaces" , *Contributions to function theory (Internat. Colloq. Function Theory), Bombay, Tata Institute of Fundamental Research.*, vol. 4, pp. 147–164.

23. Calabi, E. & Vesentini, E. 1960, "On compact locally symmetric Kähler manifolds" , *Ann. of Math.*, vol. 71, pp. 472–507.

24. Weil, A. 1960, "On discrete subgroups of Lie groups I" , *Ann. Math.*, vol. 72, pp. 369–384.

25. Furstenberg, H. 1967, "Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation" , *Math. Systems Theory.*, vol. 1, pp. 1–49.

26. Mostow, G. 1968, "Quasi-conformal mappings in $n$-space and the rigidity of the hyperbolic space forms" , *Publ. Math. IHES*, vol 34, pp. 53–104.

27. Margulis, G. 1974, "Discrete groups of motions of manifolds of nonpositive curvature" , *Proceedings of the International Congress of Mathematicians, Vancouver, Canada.*, vol. II, pp. 21–34.

28. Fisher, D. 2007, "Local rigidity of group actions: past, present, future" , *Recent Progress in Dynamics, MSRI Publications,* vol. 54, pp. 211–231.

29. Spatzier, R. 2004, "An invitation to rigidity theory" , *Modern dynamical systems and applications, Cambridge University Press, Cambridge,* pp. 45–97.

30. Glasner, S. & Maon, D. 1989, "Rigidity in topological dynamics" , *Ergodic Theory Dynam. Systems,* vol. 9, pp. 309–320.

31. Ageev, O. & Silva, C. 2002, "Genericity of rigid and multiply recurrent infinite measure-preserving and nonsingular transformations" , *Proceedings of the 16ty Summer Conference on General Topology and its applications (New York), Topology Proc.,* vol. 26, no. 2, pp. 357–365.

32. James, J., Koberda, T., Lendsey, K., Silva, C. & Speh, P. 2008, "Measurable sensitivity" , *Proc. Amer. Math. Soc.,* vol. 136, pp. 3549–3559.

33. Peterson, K. 1983, "Ergodic Theory. Cambridge Studies in Advanced Mathematics, 2" , *Cambridge University Press, Cambridge.*

34. Halmos, P. R. 1950, "Measure theory," *D. van Nostrand Co., Inc., New York,* xi+304 pp.

35. Furstenberg, H. 1998, "Stiffness of group actions" , *Tata Ins. Fund. Res. Stud. Math., Bombay,* vol. 14, pp. 105–117.

36. Einsiedler, M. & Lindenstrauss, E. 2003, "Rigidity properties of $\mathbf{Z}^d$-actions on tori and and solinoids" , *Electron. Res. Announc. Amer. Math. Soc.,* vol. 9, pp. 99–110.

37. Margulis, G. 2000, "Problems and conjectures in rigidity theory" , *Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI,* pp. 161–174.

38. Berthelot, P. 1986, "Géométrie rigide et cohomologie des variétés algébriques de caractéristique $p$" , *Mémoires de la Société Mathématique de France, Nouvelle Série,* vol. 23, pp. 7–32.

39. Kazhdan, K. 1967, "On the connection of the dual space of a group with the structure of the closed subgroups" , *Funct. Anal. and its Appl.,* vol. 1, pp. 63-65.

40. Margulis, G. 1982, "Finitely additive invariant measures on Euclidian spaces" , *Ergodic Theory Dynam. Systems,* vol. 2, pp. 383–396.

41. Mostow, G. 1973, "Strong rigidity of locally symmetric spaces," *Princeton University Press, Princeton, N.J., Annals of Mathematics Studies*, no. 78.

42. Margulis, G. 1991, "Discrete subgroups of semisimple Lie groups," *Ergeb. Math. Grenzgeb. 17, Springer-Verlag, Berlin.*

43. Furman, A. 1999, "Gromov‘s measure equivalence and rigidity of higher rank lattices" , *Ann. of Math.*, vol. 2, pp. 1059–1081, 1083–1108.

44. Zimmer, R. 1984. "Ergodic theory and semisimple groups" , Monographs in Mathematics, vol. 81, *Birkhuser Verlag, Basel.*

45. Popa, S. 2007, "Deformation and rigidity for group actions and von Newmann algebras" , *Proceedings of the International Congress of Mathematicians, Eur. Math. Soc., Zürich.*, vol. I, pp. 445–477.

46. Faltings, G. 2003, "Algebraic loop group and moduli spaces of bundles" , *Journ. Eur. Math. Soc. (JEMS)*, vol. 5, pp. 41–68.

47. Viehmann, E. 2013, "Newton strata in the loop group of a reductive group" , *Am. J. Math.*, vol. 135, No. 2, pp. 499–518.

48. Vasiu, A. 2006, "Crystalline boundedness principle" , *Ann. Sci. École Norm. Sup.*, vol. (4) 39, no. 2, pp. 245–300.

49. G'ortz, U., Haines, T., Kottwitz, R. & Reuman, D. 2006, "Dimensions of some affine Deligne-Lusztig varieties" , *Ann. Sci. École Norm. Sup.*, vol. (4) 39, pp. 467–511.

50. Chai, C. 2003, "Newton polygons as lattice points" , *Journ. Amer. Math. Soc.*, vol. 13, pp.209–241.

51. Glazunov, N. M. 2014, " On norm maps and "universal norms"of formal groups over integer rings of local fields" , *Continuous and Distributed Systems. Theory and Applications. Springer*, pp. 73–80.

52. Glazunov, N. M., 2014, "Crystalline cohomology and their applications" , *Algebra and Number Theory: Modern Problems and Application: XII International Conference. Tula, RFFI*, pp. 52–54.

## СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Рышков С. С. Полиэдр $\mu(m)$ и некоторые экстремальные задачи геометрии чисел // Доклады АН СССР. 1970. Т. 194, № 3. С. 514–517.

2. Hardy G. H., Littlewood J. E. Some problems of "Partitio Numerorum" : V. A further contribution to the study of Goldbach‘s problem // Proc. London Math. Soc., 1923. (2), 22, P. 46–56.

3. Виноградов И. М. Метод тригонометрических сумм в теории чисел. – М.: Наука. 1980.

4. Виноградов И. М. Особые варианты метода тригонометрических сумм. — М.: Наука. 1976. 119 с.

5. Архипов Г. И., Карацуба А. А., Чубариков В. Н. Теория кратных тригонометрических сумм. — М.: Наука. 1987. 368 с.

6. Архипов Г. И., Чубариков В. Н. О мощности исключительного множества в бинарной аддитивной проблеме гольдбахова типа // Доклады РАН. 2002. Т. 387, № 3. С. 295–296.

7. Постников А. Г. Избранные труды. Под редакцией В. Н. Чубарикова. М.: Физматлит. 2005. 512 с.

8. Арнольд В. И. «Жесткие» и «мягкие» математические модели . – М.: Издательство МЦНМО. 2004. 32 с.

9. И. Р. Шафаревич Основания алгебраической геометрии. Т. 1, Т. 2. – М.: Наука. 1988. 351 с., 304 с.

10. Serre J.-P. Trees. – Berlin–Heidelberg–New York: Springer –Verlag. 2003.

11. K. Kedlaya, J. Tuitman Effective convergence bounds for Frobenius structures on connections // Rend. Semin. Mat. Univ. Padova. 2012. Vol. 128. P. 7–16.

12. J. Bourgain, A. Furman, E. Lindenstraussl, S. Mozes Stationary measures and equidistribution for orbits of nonabelian semigroups on the torus // J. Am. Math. Soc. 2011. Vol. 1. P. 231–280.

13. A. Ioana Cocycle superrigidity for profinite actions of property (T) groups // Duke Math. J. 2011. Vol. 2. P. 337–367.

14. J. James, T. Koberda, K. Lindsey, C. Silva, P. Speh On ergodic transformations that are both weakly mixing and uniformly rigid // New York J. Math. 2009. Vol. 15. P. 393–403.

15. M. Gromov Soft and Hard Symplectic Geometry // Proceedings of the International Congress of Mathematicians, Berkeley, California, USA. 1986. Vol. I. P. 81–98.

16. R. Hartshorne Algebraic Geometry. Springer –Verlag, Berlin–Heidelberg–New York. 1977.

17. J. Tate Rigid analytic spaces // Invent. Math. 1971. Vol. 12. P. 257–289.

18. D. Mumford An analytic construction of degenerating curves over complete local rings // Compositio Mathematica. 1972. Vol. 24. P. 129–174.

19. M. Raynaud Géométrie analytique rigide d'aprés Tate, Kiehl,...,table ronde a'analyse non-archimedienne // Bull. Soc. math. France. 1974. Vol. 39–40. P. 319–327.

20. M. Demazure Lectures on p - divisible groups, LNM 302. Springer Verlag, Berlin. 1972.

21. J.-P. Serre A Course of Arithmitic. Springer –Verlag, Berlin–Heidelberg–New York. 1973.

22. A. Selberg On discontinuous groups in higher-dimensional symmetric spaces // Contributions to function theory (Internat. Colloq. Function Theory, Bombay, Tata Institute of Fundamental Research. 1960. Vol. 4. P. 147–164.

23. E. Calabi, E. Vesentini On compact locally symmetric Kähler manifolds // Ann. of Math. 1960. Vol. 71. P. 472–507.

24. A. Weil On discrete subgroups of Lie groups I // Ann. Math. 1960. Vol. 72. P. 369–384.

25. H. Furstenberg Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation // Math. Systems Theory. 1967. Vol. 1. P. 1–49.

26. G. Mostow Quasi-conformal mappings in $n$-space and the rigidity of the hyperbolic space forms // Publ. Math. IHES 1968. Vol. 34. P. 53–104.

27. G. Margulis Discrete groups of motions of manifolds of nonpositive curvature // Proceedings of the International Congress of Mathematicians, Vancouver, Canada. 1974. Vol. II. P. 21–34.

28. D. Fisher Local rigidity of group actions: past, present, future // Recent Progress in Dynamics, MSRI Publications. 2007. Vol. 54. P. 211–231.

29. R. Spatzier An invitation to rigidity theory // Modern dynamical systems and applications, Cambridge University Press, Cambridge. 2004. P. 45–97.

30. S. Glasner, D. Maon Rigidity in topological dynamics // Ergodic Theory Dynam. Systems. 1989. Vol. 9. P. 309–320.

31. O. Ageev, C. Silva Genericity of rigid and multiply recurrent infinite measure-preserving and nonsingular transformations // Proceedings of the 16ty Summer Conference on General Topology and its applications (New York), Topology Proc. 2002. Vol. 26. No. 2. P. 357–365.

32. J. James, T. Koberda, K. Lendsey, C. Silva, P. Speh Measurable sensitivity // Proc. Amer. Math. Soc. 2008. Vol. 136. P. 3549–3559.

33. K. Peterson Ergodic Theory. Cambridge Studies in Advanced Mathematics, 2. Cambridge University Press, Cambridge. 1983.

34. P. Halmos Measure theory. D. van Nostrand Co., New York. 1950.

35. H. Furstenberg Stiffness of group actions // Tata Ins. Fund. Res. Stud. Math., Bombay. 1998. Vol. 14. P. 105-117.

36. M. Einsiedler, E. Lindenstrauss Rigidity properties of $\mathbf{Z}^d$-actions on tori and and solinoids // Electron. Res. Announc. Amer. Math. Soc. 2003. Vol. 9. P. 99–110.

37. G. Margulis Problems and conjectures in rigidity theory // Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI. 2000. P. 161–174.

38. P. Berthelot Géométrie rigide et cohomologie des variétés algébriques de caractéristique $p$ // Mémoires de la Société Mathématique de France, Nouvelle Série. 1986. Vol. 23. P. 7–32.

39. K. Kazhdan On the connection of the dual space of a group with the structure of the closed subgroups // Funct. Anal. and its Appl. 1967. Vol. 1. P. 63-65.

40. G. Margulis Finitely additive invariant measures on Euclidian spaces // Ergodic Theory Dynam. Systems. 1982. Vol. 2. P. 383–396.

41. G. Mostow Strong rigidity of locally symmetric spaces. Princeton University Press, Princeton, N.J., Annals of Mathematics Studies, No. 78. 1973.

42. G. Margulis Discrete subgroups of semisimple Lie groups. Ergeb. Math. Grenzgeb. 17, Springer-Verlag, Berlin. 1991.

43. A. Furman Gromov's measure equivalence and rigidity of higher rank lattices // Ann. of Math. 1999. Vol. 2. P. 1059–1081, 1083–1108.

44. R. Zimmer Ergodic theory and semisimple groups. Monographs in Mathematics, 81, Birkhuser Verlag, Basel. 1984.

45. S. Popa Deformation and rigidity for group actions and von Newmann algebras // Proceedings of the International Congress of Mathematicians, Eur. Math. Soc., Zürich. 2007. Vol. I. P. 445–477.

46. G. Faltings Algebraic loop group and moduli spaces of bundles // Journ. Eur. Math. Soc. (JEMS) 2003. Vol. 5. P. 41–68.

47. E. Viehmann Newton strata in the loop group of a reductive group // Am. J. Math. 2013. Vol. 135, No. 2. P. 499–518.

48. A. Vasiu Crystalline boundedness principle // Ann. Sci. École Norm. Sup. 2006. Vol. (4) 39, no. 2. P. 245–300.

49. U. G'ortz, T. Haines, R. Kottwitz, D. Reuman Dimensions of some affine Deligne-Lusztig varieties // Ann. Sci. École Norm. Sup. 2006. Vol. (4) 39. P. 467–511.

50. C. Chai Newton polygons as lattice points // Journ. Amer. Math. Soc. 2003. Vol. 13. P. 209–241.

51. Glazunov N. M. On norm maps and "universal norms" of formal groups over integer rings of local fields // Continuous and Distributed Systems. Theory and Applications. Springer. 2014. P. 73–80.

52. Glazunov N. M. Crystalline cohomology and their applications // Algebra and Number Theory: Modern Problems and Application: XII International Conference. Tula, RFFI. 2014. P. 52–54.

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