ЧЕБЫШЕВСКИЙ СБОРНИК
Том 24. Выпуск 2.

УДК 517 DOI 10.22405/2226-8383-2023-24-2-38-62

О скорости сходимости средних Чезаро двойного ряда Фурье функций обобщенной ограниченной вариации

Р. К. Бера, Б. Л. Годадра

Бера Рамешбхай Каршанбхай — факультет математики, Бародский университет Махараджи Саяджирао (г. Гуджарат, Индия).
e-mail: rameshkbera8080@gmail.com
Годадра Бхикха Лила — факультет математики, Бародский университет Махараджи Саяджирао (г. Гуджарат, Индия).
e-mail: bhikhu_ghodadra@yahoo.com

Аннотация

В этой статье оценивается скорость сходимости средних Чезаро двойного ряда Фурье для 2π-периодической функции по каждой переменной и обобщенной ограниченной вариации. Полученный результат является обобщением результата С. М. Мажара для одного ряда Фурье и нашего более раннего результата для функции двух переменных.

Ключевые слова: двойной ряд Фурье, обобщенная ограниченная вариация, поточечная сходимость, скорость сходимости, среднее значение Чезаро.

Библиография: 20 названий.

Для цитирования:
Р. К. Бера, Б. Л. Годадра. О скорости сходимости средних Чезаро двойного ряда Фурье функций обобщенной ограниченной вариации // Чебышевский сборник, 2023, т. 24, вып. 2, с. 38–62.

CHEBYSHEVSKII SBORNIK
Vol. 24. No. 2.

UDC 517 DOI 10.22405/2226-8383-2023-24-2-38-62

On the rate of convergence of Cesàro means of double Fourier series of functions of generalized bounded variation

R. K. Bera, B. L. Ghodadra

Bera Rameshbhai Karshanbhai — department of mathematics, Maharaja Sayajirao University of Baroda (Gujarat, India).
email: rameshkbera8080@gmail.com
Ghodadra Bhikha Lila — department of mathematics, Maharaja Sayajirao University of Baroda (Gujarat, India).
email: bhikhu_ghodadra@yahoo.com

Abstract

In this paper, the rate of convergence of Cesàro means of the double Fourier series of a 2π-periodic function in each variable and of generalized bounded variation, is estimated. The result obtained is a generalization of a result of S. M. Mazhar for a single Fourier series and of our earlier result for a function of two variables.

Keywords: double Fourier series, generalized bounded variation, pointwise convergence, rate of convergence, Cesàro mean.

Bibliography: 20 titles.

For citation:
R. K. Bera, B. L. Ghodadra, 2023, "On the rate of convergence of Cesàro means of double Fourier series of functions of generalized bounded variation" , *Chebyshevskii sbornik*, vol. 24, no. 2, pp. 38–62.

1. Introduction

The Dirichlet-Jordan theorem (see [11] or [17, p. 57]) asserts that the Fourier series of a 2π-periodic function f of bounded variation on $[-\pi, \pi]$ converges at each point and the convergence is uniform on closed intervals of continuity of f. Bojanić [4], and Bojanić and Mazhar [6] have quantified this result by estimating the rate of convergence of the Fourier series and of Cesáro means of the Fourier series at each point, respectively. Also, Bojanić and Waterman [5], and Mazhar [13] have generalize the results of Bojanić [4], and Bojanić and Mazhar [6], respectively, for functions of generalized bounded variation. Hardy [10] proved the extension of the Dirichlet-Jordan theorem from single to double Fourier series. Similar to Bojanič [4], and Bojanić and Waterman [5], Morièz [14] and, Bera and Ghodadra [7] have quantified the result of Hardy, by estimating the rate of convergence of double Fourier series of functions of bounded variation and of generalized bounded varition, respecctively. Here we shall give an estimate of the rate of convergence of Cesàro means of the double Fourier series of a function f, 2π-periodic in each variable and of generalized bounded variation. Our result of this paper is a generalization of a result of Mazhar [13] for a single Fourier series and of our earlier result [7] for a double Fourier series.

2. Single Fourier Series

Here we shall recall certain results for pointwise convergence and rate of convergence of a single Fourier series. We need the following definitions.

Definition 1. *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function, which is Lebesgue integrable over $\mathbb{T} := [-\pi, \pi)$. The Fourier series of f, denoted by $S(f, x)$, is defined by*

$$S(f, x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) e^{-inu} du, \quad n \in \mathbb{Z}.$$

The n^{th} symmetric partial sum of the Fourier series of f, denoted by $S_n(f, x)$, is defined as

$$S_n(f, x) = \sum_{j=-n}^{n} c_j e^{ijx}, \; n = 0, 1, 2, \ldots.$$

Definition 2. *The (ordinary) oscillation of a function* $h : [a, b] \to \mathbb{C}$ *over a subinterval* J *of* $[a, b]$ *is defined as*

$$\operatorname{osc}_1(h, J) = \sup\{|h(t) - h(t')| : t, t' \in J\}.$$

In the sequel, we will distinguish the subintervals of the non-negative half of the one-dimensional torus $\bar{\mathbb{T}} = [-\pi, \pi] : I_{j,m} = [\theta_{j,m}, \theta_{j+1,m}]$, where $\theta_{j,m} = \frac{j\pi}{m+1}$, for $j = 0, 1, 2, \ldots m; m \in \mathbb{N} \cup \{0\}$.

Definition 3. *Let* f *be a real-valued function defined on an interval* $[a, b]$ *and* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *be a non-decreasing sequence of positive numbers such that* $\sum \frac{1}{\lambda_n}$ *diverges. Then the function* f *is said to be of* Λ*-bounded variation* ($f \in \Lambda BV$) *on* $[a, b]$ *if there exists a positive constant* M *such that*

$$\sum_{k=1}^{n} \frac{|f(a_k) - f(b_k)|}{\lambda_k} \leqslant M$$

for every choice $\{I_k\}$ *of non-overlapping intervals with* $I_k = [a_k, b_k] \subset [a, b]$, $k = 1, \ldots, n$. *If* $f \in \Lambda BV\,[a, b]$, *the* Λ*-variation of* f *is defined by*

$$V_\Lambda(f, [a, b]) = \sup \sum_{k=1}^{n} \frac{|f(a_k) - f(b_k)|}{\lambda_k},$$

where the supremum is extended over all sequences $\{I_k\}$ *as above.*

Note that for $\Lambda = \{1\}$, ΛBV=BV, the set of all functions of bounded variation on $[a, b]$. Also, note that if f is of Λ-bounded variation, then $f(x + 0)$ and $f(x - 0)$ exist at every point x of $[a, b]$ (see, e.g., [16, Theorem 4]). We define, for $x \in [a, b]$,

$$s(f, x) = \frac{1}{2}\{f(x + 0) + f(x - 0)\} \tag{1}$$

and

$$\phi_x(t) = f(x + t) + f(x - t) - 2f(x),\ t \in [a, b]. \tag{2}$$

Jordan [11] proved that if f is a 2π-periodic function of bounded variation on $[-\pi, \pi]$, then its Fourier series converges to $s(f, x)$ at each point of x. This result was quantified by Bojanič [4] by estimating the rate of convergence of the Fourier series of f at x by proving the following theorem.

Theorem 1. *If* f *is a* 2π*-periodic function and is of bounded variation on* $[-\pi, \pi]$, *then for all* x *and* n *we have*

$$|S_n(f, x) - s(f, x)| \leqslant \frac{3}{n} \sum_{k=1}^{n} V\left(\phi_x, \left[0, \frac{\pi}{k}\right]\right).$$

R. Bojanić and D. Waterman [5] have generalized this result for the larger class ΛBV, where $\Lambda = \{n^\gamma\}$, $0 \leqslant \gamma < 1$, and denoted that class by γBV and the corresponding variation by $V_\gamma(f, [a, b])$. Their result (including their Lemmas 1 and 2) is as follows.

Theorem 2. *Let* $f \in \gamma BV(\bar{\mathbb{T}})$, $0 \leqslant \gamma < 1$, *and let* $V_\gamma(\phi_x, u)$ *denote the generalized variation of* ϕ_x *on* $[0, u]$. *Then*

$$|S_n(f, x) - s(f, x)| \leqslant 2 \sum_{k=0}^{n} \frac{1}{k+1} \operatorname{osc}_1(\phi_x, I_{k,n}) \leqslant \frac{2(2-\gamma)}{(n+1)^{1-\gamma}} \sum_{k=1}^{n} \frac{1}{k^\gamma} V_\gamma\left(\phi_x, \frac{\pi}{k}\right),$$

where $s(f, x)$ *and* $\phi_x(t)$ *are as in* (1) *and* (2), *respectively.*

In order to obtain a result for Cesàro means, we first recall the following definition and properties, which can be found in ([17, pp. 94–95], [13, Theorem 1], or [6]).

Definition 4. *Let $K_n^\alpha(t)$ denote the (C,α) kernel and $\sigma_n^\alpha(f,x)$ the (C,α) mean of $S(f,x)$ for $-1<\alpha\leqslant 0$. Then*

$$K_n^\alpha(t)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}D_\nu(t),$$
$$\sigma_n^\alpha(f,x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)K_n^\alpha(t)dt,$$

and

$$\sigma_n^\alpha(f,x)-s(f,x)=\frac{1}{\pi}\int_0^\pi\phi_x(t)K_n^\alpha(t)dt, \tag{3}$$

where $D_\nu(t)$ and A_n^α are defined as

$$D_\nu(t)=\frac{1}{2}\sum_{j=-\nu}^{\nu}e^{ijt}=\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

and

$$A_0^\alpha=1,\quad A_n^\alpha=\binom{n+\alpha}{n}=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)},\ n\in\mathbb{N}.$$

Some properties of $K_n^\alpha(t)$ are as follows:

$$\frac{2}{\pi}\int_0^\pi K_n^\alpha(t)dt=1, \tag{4}$$

$$|K_n^\alpha(t)|\leqslant n+\frac{1}{2},\ 0<t<\pi, \tag{5}$$

$$\int_{\theta_{k,n}}^{\theta_{k+1,n}}|K_n^\alpha(t)|dt\leqslant\frac{C_1}{k^{1+\alpha}},\ k=1,2,\ldots,n,\ |\alpha|<1, \tag{6}$$

and

$$\left|\int_t^\pi K_n^\alpha(u)du\right|\leqslant\frac{C_2}{(nt)^{1+\alpha}}, \tag{7}$$

where C_1 and C_2 are constants.

Mazhar [13] generalize the result of Bojaniȁ and Waterman [5] by proving the following more general theorem (including their Lemmas 1 and 2).

Theorem 3. *Let $f\in\gamma BV(\bar{\mathbb{T}})$, $0\leqslant\gamma<1$ and let $V_\gamma(\phi_x,u)$ denote the generalized variation of ϕ_x on $[0,u]$. Then for $\alpha>\gamma-1$, $-1<\alpha\leqslant 0$,*

$$|\sigma_n^\alpha(f,x)-s(f,x)|\leqslant C_\alpha\sum_{k=0}^{n}\frac{1}{(k+1)^{1+\alpha}}\operatorname{osc}_1(\phi_x,I_{k,n})\leqslant\frac{(2+\alpha-\gamma)C_\alpha}{(n+1)^{\alpha-\gamma+1}}\sum_{k=1}^{n}\frac{1}{k^{\gamma-\alpha}}V_\gamma\left(\phi_x,\frac{\pi}{k}\right), \tag{8}$$

where $C_\alpha=\left(1+\frac{2^{1+\alpha}}{\pi}C_1+\frac{2^{1+\alpha}}{\pi^{2+\alpha}}C_2\right)$, and $s(f,x)$ and $\phi_x(t)$ are as in (1) and (2), respectively.

3. Double Fourier Series

In this section, we shall recall certain results for pointwise convergence and rate of convergence of a Double Fourier series. We need the following definitions.

Definition 5. *Let $f:\mathbb{R}^2\to\mathbb{C}$ be a function, 2π-periodic in each variable and integrable over $\mathbb{T}^2$. The double Fourier series of f is defined by*

$$S(f,x,y)=\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}c_{jk}e^{i(jx+ky)}, \tag{9}$$

where

$$c_{jk}=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)e^{-i(ju+kv)}dudv,\ j,k\in\mathbb{Z}. \tag{10}$$

We consider the double sequence of symmetric rectangular partial sums

$$S_{m,n}(f,x,y)=\sum_{j=-m}^{m}\sum_{k=-n}^{n}c_{jk}e^{i(jx+ky)},\ m,n=0,1,2,\dots. \tag{11}$$

The Cesàro (C,α,β)-means of the double Fourier series (9) *for $-1<\alpha,\beta\leqslant 0$ are defined (see, e.g., [9, p. 106]) by*

$$\sigma_{m,n}^{\alpha,\beta}(f,x,y)=\frac{1}{A_m^\alpha}\frac{1}{A_n^\beta}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}A_{m-\mu}^{\alpha-1}A_{n-\nu}^{\beta-1}S_{\mu,\nu}(f,x,y),\quad m,n=0,1,2,\dots.$$

Definition 6. *A function f defined on a rectangle $R:=[a,b]\times[c,d]$ is said to be of bounded variation in the sense of Vitali, in symbol, $f\in\mathrm{BV_V}(R)$, if*

$$\sup_{\mathcal{P}_1\times\mathcal{P}_2}\sum_{j=1}^{m}\sum_{k=1}^{n}|f(x_j,y_k)-f(x_{j-1},y_k)-f(x_j,y_{k-1})+f(x_{j-1},y_{k-1})|<\infty, \tag{12}$$

where the supremum is extended over all partitions

$$\mathcal{P}_1: a=x_0<x_1<\dots<x_m=b\ \text{ and }\ \mathcal{P}_2: c=y_0<y_1<\dots<y_n=d$$

of $[a,b]$ and $[c,d]$ respectively. The supremum in (12) *denoted by $V(f,[a,b],[c,d])$ is called the total variation of f over R.*

If a function $f\in\mathrm{BV_V}(R)$ is such that the marginal functions $f(\cdot,c)$ and $f(a,\cdot)$ are of bounded variation over the intervals $[a,b]$ and $[c,d]$ respectively, then f is said to be of bounded variation in the sense of Hardy and Krause, in symbols, $f\in\mathrm{BV_H}(R)$.

Definition 7. *The rectangular oscillation of a function $f:[a,b]\times[c,d]\to\mathbb{C}$ over a subrectangle $J\times K$ of $[a,b]\times[c,d]$ is defined as*

$$\mathrm{osc}_2(f,J,K)=\sup\{|f(u,v)-f(u',v)-f(u,v')+f(u',v')|: u,u'\in J, v,v'\in K\}.$$

We also recall that the modulus of continuity of a function f on $\mathbb{T}^2$ is defined by

$$\omega(f,\delta_1,\delta_2):=\sup_{|u-u'|\leqslant\delta_1,\ |v-v'|\leqslant\delta_2}|f(u,v)-f(u',v)-f(u,v')+f(u',v')|,$$

the partial moduli of continuity of f are defined by

$$\omega_x(f,\delta):=\sup_{|u-u'|\leqslant\delta_1,\ v\in\mathbb{T}}|f(u,v)-f(u',v)|,$$

and

$$\omega_y(f,\delta) := \sup_{|v-v'|\leqslant\delta_2,\ u\in\mathbb{T}} |f(u,v)-f(u,v')|,$$

and also

$$\omega_i(f,\delta_i) = \sup_{|h_i|\leqslant\delta_i} \{|f(x+h_i)-f(x)| \ : \ x\in\mathbb{T}^n\}, \quad i=1,2,\ldots,n.$$

Definition 8. *Let $f(x+0,y+0) := \lim_{s\to0^+,t\to0^+} f(x+s,y+t)$ be the limiting value of f as (x,y) is approached along any path lying north and east of (x,y). The other three quadrant limits $f(x-0,y+0)$,$f(x+0,y-0)$ and $f(x-0,y-0)$ can be defined analogously.*

Hardy [10] proved that if f is function of bounded variation (in the sense of Hardy and Krause) on $\bar{\mathbb{T}}^2$, 2π-periodic in each variable, then its Fourier series (9) converges to $s(f,x,y)$ at each point of (x,y).

Morièz [14] quantified Hardy's result by estimating the rate of convergence of double Fourier series of f at (x,y) by proving following theorems.

Theorem 4 ([14, Theorem 2]). *If f is a bounded, measurable function on $\bar{\mathbb{T}}^2$, 2π-periodic in each variable, such that the four limits $f(x\pm0,y\pm0)$ exist at a certain point (x,y), and the four limit functions $f(x\pm0,\cdot)$ and $f(\cdot,y\pm0)$ exist, then for any $m,n\geqslant0$ we have*

$$\begin{aligned}|S_{m,n}(f,x,y)-s(f,x,y)| \leqslant \left(1+\frac{1}{\pi}\right)^2 \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{1}{(j+1)(k+1)} \operatorname{osc}_2(\phi_{xy},I_{j,m},I_{k,n})\\ + \left(1+\frac{1}{\pi}\right)\sum_{j=0}^{m}\frac{1}{(j+1)}\operatorname{osc}_1(\phi_{xy}(\cdot,0),I_{j,m})\\ + \left(1+\frac{1}{\pi}\right)\sum_{k=0}^{n}\frac{1}{(k+1)}\operatorname{osc}_1(\phi_{xy}(0,\cdot),I_{k,n}),\end{aligned}$$

where

$$s(f,x,y) = \frac{1}{4}[f(x+0,y+0)+f(x-0,y+0)+f(x+0,y-0)+f(x-0,y-0)] \tag{13}$$

and

$$\phi_{xy}(u,v) = \begin{cases} f(x+u,y+v)+f(x-u,y+v)+f(x+u,y-v) & \\ +f(x-u,y-v)-4s(f,x,y), & \text{if } u,v>0;\\ f(x+0,y+v)+f(x-0,y+v)+f(x+0,y-v) & \\ +f(x-0,y-v)-4s(f,x,y), & \text{if } u=0 \text{ and } v>0;\\ f(x+u,y+0)+f(x-u,y+0)+f(x+u,y-0) & \\ +f(x-u,y-0)-4s(f,x,y), & \text{if } u>0 \text{ and } v=0;\\ 0, & \text{if } u=v=0.\end{cases} \tag{14}$$

Theorem 5 ([14, Theorem 3]). *If $f(x,y)$ is 2π-periodic in each variable and of bounded variation over $\bar{\mathbb{T}}^2$ in the sense of Hardy and Krause, then for all $m,n\geqslant0$, we have*

$$\begin{aligned}|S_{m,n}(f,x,y)-s(f,x,y)| \leqslant \frac{4\left(1+\frac{1}{\pi}\right)^2}{(m+1)(n+1)}\sum_{j=1}^{m}\sum_{k=1}^{n} V\left(\phi_{xy},\left[0,\frac{\pi}{j}\right],\left[0,\frac{\pi}{k}\right]\right)\\ + \frac{2\left(1+\frac{1}{\pi}\right)}{m+1}\sum_{j=1}^{m} V\left(\phi_{xy}(\cdot,0),\left[0,\frac{\pi}{j}\right]\right)\\ + \frac{2\left(1+\frac{1}{\pi}\right)}{n+1}\sum_{k=1}^{n} V\left(\phi_{xy}(0,\cdot),\left[0,\frac{\pi}{k}\right]\right),\end{aligned}$$

where $s(f,x,y)$ *and* $\phi_{xy}(u,v)$ *are as in* (13) *and* (14), *respectively.*

In [20], Zhizhiashvili have proved the following theorem for function of several variables.

THEOREM 6. (a) *If* $f \in C(\mathbb{T}^n)$ *and*

$$\omega_i(f,\delta_i) = O\left\{\left(\log\frac{1}{\delta_i}\right)^{-n-\epsilon}\right\} \ (\delta_i \to 0,\ i=1,2,\ldots,n),\ \ \epsilon > 0,$$

then the Fourier series of f *is uniformly convergent in the sense of Pringsheim.*

(b) *If*

$$\omega_i(f,\delta_i) = o\left\{\left(\log\frac{1}{\delta_i}\right)^{-n}\right\} \ (\delta_i \to 0,\ i=1,2,\ldots,n),$$

then the Fourier series of the function f *is uniformly convergent in the sense of Pringsheim.*

Morièz [15] have also proved the similar type of result for function of two variables, which is as follows

THEOREM 7 ([15, Corollary 1.2]). *If* f *is continuous on* $\mathbb{T}^2$,

$$\omega(f,\delta_1,\delta_2) = o\left\{\left(\log\frac{1}{\delta_1}\right)^{-1}\left(\log\frac{1}{\delta_1}\right)^{-1}\right\} \ (\delta_1,\delta_2 \to 0),$$

$$\omega_x(f,\delta) = o\left\{\left(\log\frac{1}{\delta}\right)^{-1}\right\} \ (\delta \to 0),\ \ \text{and}\ \ \omega_y(f;\delta) = o\left\{\left(\log\frac{1}{\delta}\right)^{-1}\right\} \ (\delta \to 0),$$

then the Fourier series of the function f *is uniformly convergent in the sense of Pringsheim.*

In [9], D'yachenko have constructed a continuos function of $2m$ variables ($m \in \mathbb{N}$) with modulus of continuity

$$\omega_i(f,\delta_i) = O\left(\left(\log\left(\frac{1}{\delta_i}\right)\right)^{-m}\right) \tag{15}$$

and its Fourier series is divergent almost everywhere in the Pringsheim sense based on example of Bakhbukh and Nikishin [8]. Similar results for λ-divergent Fourier series are also constructed by Bakhvalov (see [1],[2]).

DEFINITION 9. *Let* f *be a measurable function defined on the rectangle* $[a,b]\times[c,d]$ *and* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *and* $\Lambda' = \{\lambda'_n\}_{n=1}^{\infty}$ *be non-decreasing sequences of positive numbers such that* $\lambda_n, \lambda'_n \to \infty$ *and* $\sum \frac{1}{\lambda_n}$, $\sum \frac{1}{\lambda'_n}$ *diverges. Then the function* f *is said to be of* (Λ,Λ')*-bounded variation* ($f \in (\Lambda,\Lambda')BV$) *on* $[a,b]\times[c,d]$, *if*

(1) $f(\cdot,c) \in \Lambda BV[a,b]$ *and* $f(a,\cdot) \in \Lambda' BV[c,d]$, *and*

(2) *if* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are the sets of finite collections of non-overlapping intervals* $I_j = [a_j,b_j]$, $j=1,2,\ldots,m$, *and* $J_k = [c_k,d_k]$, $k=1,2,\ldots,n$, *in* $[a,b]$ *and* $[c,d]$ *respectively, and* $f(I_j\times J_k) = f(a_j,c_k) - f(a_j,d_k) - f(b_j,c_k) + f(b_j,d_k)$, *then*

$$\sup_{\mathcal{I}_1,\mathcal{I}_2}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{|f(I_j\times J_k)|}{\lambda_j\lambda'_k} < \infty. \tag{16}$$

We denote the supremum in (16) *by* $V_{(\Lambda,\Lambda')}(f,[a,b],[c,d])$.

Here we shall consider the class $(\Lambda, \Lambda')\,BV$, where $\Lambda = \{n^\gamma\}$ and $\Lambda' = \{n^\delta\}$, for $\gamma, \delta \geqslant 0$, $\gamma + \delta \leqslant 1$; denote this class by $(\gamma, \delta)BV$ and the corresponding variations by $V_\gamma(f(\cdot, c), [a, b])$, $V_\delta(f(a, \cdot), [c, d])$ and $V_{\gamma\delta}(f, [a, b], [c, d])$ respectively. The present authors have proved (see [7, Theorem 7]) that if $f(x, y) \in (\gamma, \delta)BV(\bar{\mathbb{T}}^2)$, then all the four limits $f(x \pm 0, y \pm 0)$ exist at every point (x, y). They have also generalized Theorem 5 of Morič́z and proved the following (see [7, Theorem 8]).

Theorem 8. *Let $f \in (\gamma, \delta)BV(\bar{\mathbb{T}}^2)$, $\gamma, \delta \geqslant 0$, $\gamma + \delta \leqslant 1$, and let $V_\gamma(\phi_{xy}(\cdot, 0), u)$, $V_\delta(\phi_{xy}(0, \cdot), v)$, and $V_{\gamma\delta}(\phi_{xy}, u, v)$ denote the generalized variation of ϕ_{xy} on $[0, u]$, $[0, v]$ and $[0, u] \times [0, v]$, respectively. Then*

$$
\begin{aligned}
|S_{m,n}(f, x, y) - s(f, x, y)| \leqslant & \frac{\left(1 + \frac{1}{\pi}\right)^2 (2 - \gamma)(2 - \delta)}{(m + 1)^{1-\gamma}(n + 1)^{1-\delta}} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{1}{j^\gamma k^\delta} V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right) \\
& + \frac{\left(1 + \frac{1}{\pi}\right)(2 - \gamma)}{(m + 1)^{1-\gamma}} \sum_{j=1}^{m} \frac{1}{j^\gamma} V_\gamma\left(\phi_{xy}(\cdot, 0), \frac{\pi}{j}\right) \\
& + \frac{\left(1 + \frac{1}{\pi}\right)(2 - \delta)}{(n + 1)^{1-\delta}} \sum_{k=1}^{n} \frac{1}{k^\delta} V_\delta\left(\phi_{xy}(0, \cdot), \frac{\pi}{k}\right),
\end{aligned}
$$

where $s(f, x, y)$ and $\phi_{xy}(u, v)$ are as in (13) *and* (14)*, respectively.*

We note that if the four quadrant limits $f(x \pm 0, y \pm 0)$ exist at each point (x, y), then in view of (10) and (11), we have the representation

$$
\sigma_{m,n}^{\alpha,\beta}(f, x, y) - s(f, x, y) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \phi_{x,y}(u, v) K_m^\alpha(u) K_n^\beta(v)\,du dv. \tag{17}
$$

Zhizhiashvili (see [18, Theorem A] or [19, p. 233]) redicovered this result with the supplement that if f is continuous on a rectangle R, then its Fourier series (9) converges to $f(x, y)$ uniformly on any rectangle R_1 inside R. He also proved that Hardy's result remains valid if convergence is replaced by (C, α, β)-summability, where $\alpha, \beta > -1$ are fixed real numbers. Bakhvalov [3] generalized the Zhizhiashvli's theorem for larger class of several variable function (see [3, Theorem 1]). In particular, Bakhvalov proved the following theorem (see [3, Corollary 1]).

Theorem 9. *Let α, $\beta \in \left(-\frac{1}{2}, 0\right)$ and $\gamma = \alpha + 1$, $\delta = \beta + 1$. Then, for any function $f \in (\gamma, \delta)BV(\mathbb{T}^2)$, its Fourier series is (C, α, β)-summable to $s(f, x, y)$ and the summability is uniform on any compact set in the neighborhood of which the function is continuous.*

4. Main Results

The main results of this paper are as follows.

Theorem 10. *If f is a bounded, measurable function on $\mathbb{T}^2$, 2π-periodic in each variable, such that the four limits $f(x \pm 0, y \pm 0)$ exist at a certain point (x, y), and the four limit functions $f(x \pm 0, \cdot)$ and $f(\cdot, y \pm 0)$ exist, then for any $m, n \geqslant 0$ and $-1 < \alpha, \beta \leqslant 0$, we have*

$$
\begin{aligned}
\left|\sigma_{m,n}^{\alpha,\beta}(f, x, y) - s(f, x, y)\right| \leqslant & C_\alpha C_\beta \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{1}{(j + 1)^{1+\alpha}(k + 1)^{1+\beta}} \operatorname{osc}_2(\phi_{xy}, I_{j,m}, I_{k,n}) \\
& + C_\alpha \sum_{j=0}^{m} \frac{1}{(j + 1)^{1+\alpha}} \operatorname{osc}_1(\phi_{xy}(\cdot, 0), I_{j,m}) \\
& + C_\beta \sum_{k=0}^{n} \frac{1}{(k + 1)^{1+\beta}} \operatorname{osc}_1(\phi_{xy}(0, \cdot), I_{k,n}),
\end{aligned} \tag{18}
$$

where constants C_α *and* C_β *are as in Theorem* 3.

Our second result, which is a particular case of Theorem 10, reads as follows.

THEOREM 11. *Let* $f \in (\gamma,\delta)BV(\bar{\mathbb{T}}^2)$, $\gamma,\delta \geqslant 0$, $\gamma+\delta \leqslant 1$, *and let* $V_\gamma(\phi_{xy}(\cdot,0),u)$, $V_\delta(\phi_{xy}(0,\cdot),v)$, *and* $V_{\gamma\delta}(\phi_{xy},u,v)$ *denote the generalized variation of* ϕ_{xy} *on* $[0,u]$, $[0,v]$ *and* $[0,u]\times[0,v]$, *respectively. Then for* $\alpha > \gamma - 1$, $\beta > \delta - 1$, *and* $-1 < \alpha,\beta \leqslant 0$,

$$\begin{aligned}\left|\sigma^{\alpha,\beta}_{m,n}(f,x,y) - s(f,x,y)\right| \leqslant & \frac{(2+\alpha-\gamma)(2+\beta-\delta)C_\alpha C_\beta}{(m+1)^{\alpha-\gamma+1}(n+1)^{\beta-\delta+1}} \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{1}{j^{\gamma-\alpha}k^{\delta-\beta}} V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right) \\ & + \frac{(2+\alpha-\gamma)C_\alpha}{(m+1)^{\alpha-\gamma+1}} \sum_{j=1}^{m} \frac{1}{j^{\gamma-\alpha}} V_\gamma\left(\phi_{xy}(\cdot,0), \frac{\pi}{j}\right) \\ & + \frac{(2+\beta-\delta)C_\beta}{(n+1)^{\beta-\delta+1}} \sum_{k=1}^{n} \frac{1}{k^{\delta-\beta}} V_\delta\left(\phi_{xy}(0,\cdot), \frac{\pi}{k}\right), \end{aligned} \tag{19}$$

where constants C_α *and* C_β *are as in Theorem* 3.

In particular, taking $\gamma=\delta=0$ in Theorem 11, we get the following corollary.

COROLLARY 1. *Let* $f \in BV_H(\bar{\mathbb{T}}^2)$, *and let* $V(\phi_{xy}(\cdot,0),u)$, $V(\phi_{xy}(0,\cdot),v)$, *and* $V(\phi_{xy},u,v)$ *denote the variation of* ϕ_{xy} *on* $[0,u]$, $[0,v]$ *and* $[0,u]\times[0,v]$, *respectively. Then for* $-1<\alpha,\beta\leqslant 0$, *we have*

$$\begin{aligned}\left|\sigma^{\alpha,\beta}_{m,n}(f,x,y) - s(f,x,y)\right| \leqslant & \frac{(2+\alpha)(2+\beta)C_\alpha C_\beta}{(m+1)^{\alpha+1}(n+1)^{\beta+1}} \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{1}{j^{-\alpha}k^{-\beta}} V\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right) \\ & + \frac{(2+\alpha)C_\alpha}{(m+1)^{\alpha+1}} \sum_{j=1}^{m} \frac{1}{j^{-\alpha}} V\left(\phi_{xy}(\cdot,0), \frac{\pi}{j}\right) \\ & + \frac{(2+\beta)C_\beta}{(n+1)^{\beta+1}} \sum_{k=1}^{n} \frac{1}{k^{-\beta}} V\left(\phi_{xy}(0,\cdot), \frac{\pi}{k}\right), \end{aligned} \tag{20}$$

where constants C_α *and* C_β *are as in Theorem* 3.

Also, we can derive the following corollary from Theorem 10.

COROLLARY 2. *Let* $\alpha,\beta \in (-1,0)$.

(i) *If*

$$\omega(f,\delta_1,\delta_2) = o\left(\delta_1^{-\alpha}\delta_2^{-\beta}\right) \quad (\delta_1,\delta_2 \to 0),$$
$$\omega_x(f;\delta_1) = o\left(\delta_1^{-\alpha}\right) \quad (\delta_1 \to 0), \quad \text{and} \quad \omega_y(f;\delta_2) = o\left(\delta^{-\beta}\right) \quad (\delta_2 \to 0),$$

then the Fourier series of the function f *is uniformly* (C,α,β)*- summable in the sense of Pringsheim.*

(ii) *If* $f \in C(\mathbb{T}^2)$ *and*

$$\omega_x(f,\delta_1) = O\left(\delta_1^{-2\alpha+\epsilon}\right) \quad \text{and} \quad \omega_y(f;\delta_2) = O\left(\delta_2^{-2\beta+\epsilon}\right) \quad (i=1,2), \quad \epsilon > 0,$$

then the Fourier series of the function f *is uniformly* (C,α,β)*- summable in the sense of Pringsheim.*

(iii) *If*

$$\omega_x(f,\delta_1) = o\left(\delta_1^{-2\alpha}\right) \ \ \textit{and} \ \ \omega_y(f;\delta_2) = o\left(\delta_2^{-2\beta}\right) \ \ (\delta_i \to 0, \ i=1,2),$$

then the Fourier series of the function f is uniformly (C,α,β)- summable in the sense of Pringsheim.

(iv) *there exists a continuous function on $\mathbb{T}^2$ satisfying*

$$\omega_x(f,\delta_1) = O\left(\delta_1^{-\alpha}\right) \ \ (\delta_1 \to 0) \ \ \textit{and} \ \ \omega_y(f,\delta_2) = O\left(\delta_2^{-\beta}\right) \ \ (\delta_2 \to 0), \tag{21}$$

and its (C,α,β)-mean of Fourier series diverges almost everywhere in restricted sense.

Замечание 2. *Our Theorem* 10 is more general than the Theorem 4 of Morìcz (except for exact constants). Our Theorem 11 is a two-dimensional analogue of Theorem 3 and in a particular case, we provide a quantitative version of Theorem 9. Also, setting $\alpha=\beta=0$ in Theorem 11, we get our earlier result Theorem 8 (except for exact constants). In Corollary 1, we provide a quantitative version of Zhizhiashvili's result (see [18, Theorem A] or [19, p. 233]) for (C,α,β)-summability, for $\alpha,\beta>-1$. Our Corollary 2 is more general than the Theorem 7 and also for the case of function of two variable in Theorem 6.

5. Proofs

We need the partial summation formulas for single and double sequences, which are as follows.

Lemma 1. *Consider $n\in\mathbb{N}$. For $j=0,1,\ldots,n$, let a_j and b_j be real numbers. Let $B_j=\sum\limits_{k=j}^{n} b_k$ for $j=0,1,2,\ldots,n$, and $B_{n+1}=0$. Then*

$$\sum_{j=1}^{n} a_j b_j = \sum_{j=1}^{n} \left(a_j - a_{j-1}\right) B_j + a_0 B_1.$$

Lemma 2 ([12, Proposition 7.37]). *Consider $(m,n)\in\mathbb{N}^2$. For $j=0,1,\ldots,m$ and $k=0,1,\ldots,n$, let $a_{j,k}$ and $b_{j,k}$ be real numbers, and let $B_{m,n}=\sum\limits_{j=0}^{m}\sum\limits_{k=0}^{n} b_{j,k}$. Then*

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_{j,k}b_{j,k} = a_{m,n}B_{m,n} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\left(a_{j,k}-a_{j+1,k}-a_{j,k+1}+a_{j+1,k+1}\right)B_{j,k}$$
$$+\sum_{j=0}^{m-1}\left(a_{j,n}-a_{j+1,n}\right)B_{j,n} + \sum_{k=0}^{n-1}\left(a_{m,k}-a_{m,k+1}\right)B_{m,k}. \tag{22}$$

Also, if we assume that $B_{j,k}=\sum\limits_{j'=j}^{m}\sum\limits_{k'=k}^{n} b_{j',k'}$ and $B_{m+1,n+1}=B_{j,n+1}=B_{m+1,k}=0$, for $j=0,1,\ldots,m$, $k=0,1,\ldots,n$, then

$$\sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}b_{j,k} = \sum_{j=1}^{m}\sum_{k=1}^{n}\left(a_{j,k}-a_{j,k-1}-a_{j-1,k}+a_{j-1,k-1}\right)B_{j,k}$$
$$+\sum_{j=1}^{m}(a_{j,0}-a_{j-1,0})B_{j,1} + \sum_{k=1}^{n}(a_{0,k}-a_{0,k-1})B_{1,k} + a_{0,0}B_{1,1}. \tag{23}$$

The proof of our Theorem 10 is similar to that of a result of Moriĉz [14, Theorem 2] and the proof of Theorem 11 is similar to that of our earlier result [7, Theorem 8].

Proof of Theorem 10. Let $m, n \in \mathbb{N}$ be fixed. We start with the representation (17) of the difference of $\sigma_{m,n}^{\alpha,\beta}(f,x,y)$ and $s(f,x,y)$. By writing ϕ instead of ϕ_{xy}, in view of (4), it is clear that

$$\begin{aligned}
&\sigma_{m,n}^{\alpha,\beta}(f,x,y) - s(f,x,y)\\
&\quad= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \{\phi(u,v)-\phi(u,0)-\phi(0,v)\}K_m^\alpha(u)K_n^\beta(v)dudv\\
&\quad\quad+\frac{1}{2\pi}\int_0^\pi \phi(u,0)K_m^\alpha(u)du + \frac{1}{2\pi}\int_0^\pi \phi(0,v)K_n^\beta(v)dv\\
&\quad= \mathrm{A}_{mn} + \mathrm{B}_m + \mathrm{C}_n, \text{ say.}
\end{aligned}\tag{24}$$

Defining $g(u,v) = \phi(u,v) - \phi(u,0) - \phi(0,v)$, we decompose the double integral defining A_{mn} as

$$\begin{aligned}
\pi^2\mathrm{A}_{mn} &= \int_{I_{0,m}}\int_{I_{0,n}} g(u,v)K_m^\alpha(u)K_n^\beta(v)dudv\\
&+\sum_{j=1}^m\int_{I_{j,m}}\int_{I_{0,n}}\{g(u,v)-g(\theta_{j,m},v)\}K_m^\alpha(u)K_n^\beta(v)dudv\\
&+\sum_{j=1}^m\int_{I_{j,m}}\int_{I_{0,n}} g(\theta_{j,m},v)K_m^\alpha(u)K_n^\beta(v)dudv\\
&+\sum_{k=1}^n\int_{I_{0,m}}\int_{I_{k,n}}\{g(u,v)-g(u,\theta_{k,n})\}K_m^\alpha(u)K_n^\beta(v)dudv\\
&+\sum_{k=1}^n\int_{I_{0,m}}\int_{I_{k,n}} g(u,\theta_{k,n})K_m^\alpha(u)K_n^\beta(v)dudv\\
&+\sum_{j=1}^m\sum_{k=1}^n\int_{I_{j,m}}\int_{I_{k,n}}\{g(u,v)-g(u,\theta_{k,n})-g(\theta_{j,m},v)+g(\theta_{j,m},\theta_{k,n})\}K_m^\alpha(u)K_n^\beta(v)dudv\\
&+\sum_{j=1}^m\sum_{k=1}^n\int_{I_{j,m}}\int_{I_{k,n}}\{g(\theta_{j,m},v)-g(\theta_{j,m},\theta_{k,n})\}K_m^\alpha(u)K_n^\beta(v)dudv\\
&+\sum_{j=1}^m\sum_{k=1}^n\int_{I_{j,m}}\int_{I_{k,n}}\{g(u,\theta_{k,n})-g(\theta_{j,m},\theta_{k,n})\}K_m^\alpha(u)K_n^\beta(v)dudv\\
&+\sum_{j=1}^m\sum_{k=1}^n\int_{I_{j,m}}\int_{I_{k,n}} g(\theta_{j,m},\theta_{k,n})K_m^\alpha(u)K_n^\beta(v)dudv\\
&= \mathrm{A}_1 + \mathrm{A}_2 + \cdots + \mathrm{A}_9, \text{ say.}
\end{aligned}\tag{25}$$

To estimate A_1 and A_2, from (5) and by definition of $g(u,v)$, as $\phi(0,0)=0$, we have

$$\begin{aligned}
|\mathrm{A}_1| &\leqslant \int_{I_{0,m}}\int_{I_{0,n}} |g(u,v)||K_m^\alpha(u)||K_n^\beta(v)|dudv\\
&= \int_{I_{0,m}}\int_{I_{0,n}} |\phi(u,v)-\phi(u,0)-\phi(0,v)+\phi(0,0)||K_m^\alpha(u)||K_n^\beta(v)|dudv\\
&\leqslant \mathrm{osc}_2(\phi, I_{0,m}, I_{0,n})\int_{I_{0,m}}\int_{I_{0,n}}\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)dudv\\
&\leqslant \pi^2\mathrm{osc}_2(\phi, I_{0,m}, I_{0,n})
\end{aligned}\tag{26}$$

and using (6), we have

$$\begin{aligned}
|\mathrm{A}_2| &= \left|\sum_{j=1}^{m}\int_{I_{j,m}}\int_{I_{0,n}}\{g(u,v)-g(\theta_{j,m},v)\}K_m^{\alpha}(u)K_n^{\beta}(v)dudv\right| \\
&\leqslant \sum_{j=1}^{m}\int_{I_{j,m}}\int_{I_{0,n}}|\phi(u,v)-\phi(u,0)-\phi(\theta_{j,m},v)+\phi(\theta_{j,m},0)|\,|K_m^{\alpha}(u)|\left|K_n^{\beta}(v)\right|dudv \\
&\leqslant \sum_{j=1}^{m}\mathrm{osc}_2(\phi,I_{j,m},I_{0,n})\left(\int_{I_{j,m}}\frac{C_1}{j^{1+\alpha}}du\right)\left(\int_{I_{0,n}}\left(n+\frac{1}{2}\right)dv\right) \\
&\leqslant \pi C_1\sum_{j=1}^{m}\frac{1}{j^{1+\alpha}}\mathrm{osc}_2(\phi,I_{j,m},I_{0,n}) \\
&\leqslant \pi 2^{1+\alpha}C_1\sum_{j=0}^{m}\frac{1}{(j+1)^{1+\alpha}}\mathrm{osc}_2(\phi,I_{j,m},I_{0,n}). \qquad (27)
\end{aligned}$$

Similalry, we can get

$$|\mathrm{A}_4| \leqslant \pi 2^{1+\beta}C_1\sum_{k=0}^{n}\frac{1}{(k+1)^{1+\beta}}\mathrm{osc}_2(\phi,I_{0,m},I_{k,n}). \qquad (28)$$

Next, we estimate A_3. Put

$$R_{j,m}^{\alpha} = \int_{\theta_{j,m}}^{\pi}K_m^{\alpha}(u)du, \quad j=0,1,\ldots,m+1 \qquad (29)$$

and

$$R_{k,n}^{\beta} = \int_{\theta_{k,n}}^{\pi}K_n^{\beta}(v)dv, \quad k=0,1,\ldots,n+1. \qquad (30)$$

Then by (4) and (7), we have

$$|R_{j,m}^{\alpha}| \leqslant C_2\left(\frac{2}{\pi j}\right)^{1+\alpha}, \ j=1,2,\ldots,m; \ R_{0,m}^{\alpha}=\frac{\pi}{2}, \ R_{m+1,m}^{\alpha}=0 \qquad (31)$$

and similarly

$$|R_{k,n}^{\beta}| \leqslant C_2\left(\frac{2}{\pi k}\right)^{1+\beta}, \ k=1,2,\ldots,n; \ R_{0,n}^{\beta}=\frac{\pi}{2}, \ R_{n+1,n}^{\beta}=0. \qquad (32)$$

Now, by definition of A_3, we have

$$\begin{aligned}
\mathrm{A}_3 &= \sum_{j=1}^{m}\int_{I_{j,m}}\int_{I_{0,n}}g(\theta_{j,m},v)K_m^{\alpha}(u)K_n^{\beta}(v)dudv \\
&= \int_{I_{0,n}}\left\{\sum_{j=1}^{m}g(\theta_{j,m},v)\int_{I_{j,m}}K_m^{\alpha}(u)du\right\}K_n^{\beta}(v)dv \\
&= \int_{I_{0,n}}\left\{\sum_{j=1}^{m}g(\theta_{j,m},v)\left(R_{j,m}^{\alpha}-R_{j+1,m}^{\alpha}\right)\right\}K_n^{\beta}(v)dv.
\end{aligned}$$

Using the partial summation formula of Lemma 1 with $a_j = g(\theta_{j,m}, v)$ and $b_j = R^{\alpha}_{j,m} - R^{\alpha}_{j+1,m}$, for $j = 0, 1, \ldots, m$, we have

$$\begin{aligned} \mathrm{A}_3 &= \int_{I_{0,n}} \Bigg\{ \sum_{j=1}^{m} \left(g(\theta_{j,m}, v) - g(\theta_{j-1,m}, v)\right) \left(R^{\alpha}_{j,m} - R^{\alpha}_{m+1,m}\right) \\ &\qquad\qquad + g(\theta_{0,m}, v) \left(R^{\alpha}_{1,m} - R^{\alpha}_{m+1,m}\right) \Bigg\} K^{\beta}_{n}(v) dv \\ &= \int_{I_{0,n}} \left\{ \sum_{j=1}^{m} \left(\phi(\theta_{j,m}, v) - \phi(\theta_{j,m}, 0) - \phi(\theta_{j-1,m}, v) + \phi(\theta_{j-1,m}, 0)\right) R^{\alpha}_{j,m} \right\} K^{\beta}_{n}(v) dv, \end{aligned} \tag{33}$$

because $g(\theta_{j,m}, v) - g(\theta_{j-1,m}, v) = \phi(\theta_{j,m}, v) - \phi(\theta_{j,m}, 0) - \phi(\theta_{j-1,m}, v) + \phi(\theta_{j-1,m}, 0)$, $g(\theta_{0,m}, v) = g(0, v) = 0$ by definition of $g(u, v)$, and $R^{\alpha}_{m+1,m} = 0$ by (31).

From (5), (31), and (33), we conclude that

$$\begin{aligned} |\mathrm{A}_3| &\leqslant \int_{I_{0,n}} \left\{ \sum_{j=1}^{m} |\phi(\theta_{j,m}, v) - \phi(\theta_{j,m}, 0) - \phi(\theta_{j-1,m}, v) + \phi(\theta_{j-1,m}, 0)| |R^{\alpha}_{j,m}| \right\} |K^{\beta}_{n}(v)| dv \\ &\leqslant \frac{C_2 2^{1+\alpha}}{\pi^{1+\alpha}} \sum_{j=1}^{m} \frac{1}{j^{1+\alpha}} \mathrm{osc}_2(\phi, I_{j-1,m}, I_{0,n}) \int_{I_{0,n}} \left(n + \frac{1}{2}\right) dv \\ &\leqslant \frac{C_2 2^{1+\alpha}}{\pi^{\alpha}} \sum_{j=0}^{m-1} \frac{1}{(j+1)^{1+\alpha}} \mathrm{osc}_2(\phi, I_{j,m}, I_{0,n}). \end{aligned} \tag{34}$$

Analogously, now using (32) instead of (31), we can see that

$$|\mathrm{A}_5| \leqslant \frac{C_2 2^{1+\beta}}{\pi^{\beta}} \sum_{k=0}^{n-1} \frac{1}{(k+1)^{1+\beta}} \mathrm{osc}_2(\phi, I_{0,m}, I_{k,n}). \tag{35}$$

Next we estimate A_6. By definition of $g(u, v)$, we have

$$\begin{aligned} & g(u, v) - g(u, \theta_{k,n}) - g(\theta_{j,m}, v) + g(\theta_{j,m}, \theta_{k,n}) \\ &\qquad = \phi(u, v) - \phi(u, \theta_{k,n}) - \phi(\theta_{j,m}, v) + \phi(\theta_{j,m}, \theta_{k,n}), \end{aligned} \tag{36}$$

and hence by definition of A_6 and (6), we have

$$\begin{aligned} |\mathrm{A}_6| &= \left| \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{I_{j,m}} \int_{I_{k,n}} \{g(u, v) - g(u, \theta_{k,n}) - g(\theta_{j,m}, v) + g(\theta_{j,m}, \theta_{k,n})\} K^{\alpha}_{m}(u) K^{\beta}_{n}(v) du dv \right| \\ &\leqslant \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{I_{j,m}} \int_{I_{k,n}} |\phi(u, v) - \phi(u, \theta_{k,n}) - \phi(\theta_{j,m}, v) + \phi(\theta_{j,m}, \theta_{k,n})| \, |K^{\alpha}_{m}(u)| \left|K^{\beta}_{n}(v)\right| du dv \\ &\leqslant \sum_{j=1}^{m} \sum_{k=1}^{n} \mathrm{osc}_2(\phi, I_{j,m}, I_{k,n}) \int_{I_{j,m}} \int_{I_{k,n}} |K^{\alpha}_{m}(u)| \left|K^{\beta}_{n}(v)\right| du dv \\ &\leqslant C_1^2 \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{1}{j^{1+\alpha} k^{1+\beta}} \mathrm{osc}_2(\phi, I_{j,m}, I_{k,n}) \\ &\leqslant 2^{2+\alpha+\beta} C_1^2 \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{1}{(j+1)^{1+\alpha} (k+1)^{1+\beta}} \mathrm{osc}_2(\phi, I_{j,m}, I_{k,n}). \end{aligned} \tag{37}$$

To estimate A_7, using notation (29) and the partial summation formula of Lemma 1 with $a_j = g(\theta_{j,m}, v) - g(\theta_{j,m}, \theta_{k,n})$ and $b_j = R^\alpha_{j,m} - R^\alpha_{j+1,m}$ for $j = 0, 1, \dots, m$, we obtain

$$\begin{aligned}\mathrm{A}_7 &= \sum_{j=1}^{m}\sum_{k=1}^{n}\int_{I_{j,m}}\int_{I_{k,n}}\{g(\theta_{j,m}, v) - g(\theta_{j,m}, \theta_{k,n})\}K^\alpha_m(u)K^\beta_n(v)dudv \\ &= \sum_{k=1}^{n}\int_{I_{k,n}}\left\{\sum_{j=1}^{m}(g(\theta_{j,m}, v) - g(\theta_{j,m}, \theta_{k,n}))(R^\alpha_{j,m} - R^\alpha_{j+1,m})\right\}K^\beta_n(v)dv \\ &= \sum_{k=1}^{n}\int_{I_{k,n}}\left\{\sum_{j=1}^{m}([g(\theta_{j,m}, v) - g(\theta_{j,m}, \theta_{k,n})] - [g(\theta_{j-1,m}, v) - g(\theta_{j-1,m}, \theta_{k,n})])(R^\alpha_{j,m} - R^\alpha_{m+1,m})\right. \\ &\qquad\qquad \left. + (g(\theta_{0,m}, v) - g(\theta_{0,m}, \theta_{k,n}))(R^\alpha_{1,m} - R^\alpha_{m+1,m})\right\}K^\beta_n(v)dv \\ &= \sum_{k=1}^{n}\int_{I_{k,n}}\left\{\sum_{j=1}^{m}(\phi(\theta_{j,m}, v) - \phi(\theta_{j,m}, \theta_{k,n}) - \phi(\theta_{j-1,m}, v) + \phi(\theta_{j-1,m}, \theta_{k,n}))R^\alpha_{j,m}\right\}K^\beta_n(v)dv,\end{aligned}$$

because $g(\theta_{0,m}, v) = g(\theta_{0,m}, \theta_{k,n}) = 0$ by definition of $g(u, v)$, $R^\alpha_{m+1,m} = 0$ by (31), and by (36) with $u = \theta_{j-1,m}$. Now, in view of (6) and (31), it follows that

$$\begin{aligned}|\mathrm{A}_7| &\leqslant \sum_{k=1}^{n}\int_{I_{k,n}}\left\{\sum_{j=1}^{m}|\phi(\theta_{j,m}, v) - \phi(\theta_{j,m}, \theta_{k,n}) - \phi(\theta_{j-1,m}, v) + \phi(\theta_{j-1,m}, \theta_{k,n})||R^\alpha_{j,m}|\right\}|K^\beta_n(v)|dv \\ &\leqslant \frac{C_1C_2 2^{1+\alpha}}{\pi^{1+\alpha}}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{j^{1+\alpha}k^{1+\beta}}\mathrm{osc}_2(\phi, I_{j-1,m}, I_{k,n}) \\ &\leqslant \frac{C_1C_2 2^{2+\alpha+\beta}}{\pi^{1+\alpha}}\sum_{j=0}^{m-1}\sum_{k=0}^{n}\frac{1}{(j+1)^{1+\alpha}(k+1)^{1+\beta}}\mathrm{osc}_2(\phi, I_{j,m}, I_{k,n}). \end{aligned} \tag{38}$$

Similarly, using (32) instead of (31), we can estimate

$$\begin{aligned}|\mathrm{A}_8| &\leqslant \frac{C_1C_2 2^{1+\beta}}{\pi^{1+\beta}}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{j^{1+\alpha}k^{1+\beta}}\mathrm{osc}_2(\phi, I_{j,m}, I_{k-1,n}) \\ &\leqslant \frac{C_1C_2 2^{2+\alpha+\beta}}{\pi^{1+\beta}}\sum_{j=0}^{m}\sum_{k=0}^{n-1}\frac{1}{(j+1)^{1+\alpha}(k+1)^{1+\beta}}\mathrm{osc}_2(\phi, I_{j,m}, I_{k,n}). \end{aligned} \tag{39}$$

Keeping notation (29) and (30) in mind, we may write

$$\mathrm{A}_9 = \sum_{j=1}^{m}\sum_{k=1}^{n}g(\theta_{j,m}, \theta_{k,n})(R^\alpha_{j,m} - R^\alpha_{j+1,m})(R^\beta_{k,n} - R^\beta_{k+1,n}),$$

whence a double summation by parts (see (23) of Lemma 2) with $a_{j,k} = g(\theta_{j,m}, \theta_{k,n})$ and $b_{j,k} = (R^\alpha_{j,m} - R^\alpha_{j+1,m})(R^\beta_{k,n} - R^\beta_{k+1,n})$ for $j = 0, 1, \dots, m$ and $k = 0, 1, \dots, n$, gives

$$\begin{aligned}\mathrm{A}_9 &= \sum_{j=1}^{m}\sum_{k=1}^{n}\Big\{g(\theta_{j,m}, \theta_{k,n}) - g(\theta_{j,m}, \theta_{k-1,n}) - g(\theta_{j-1,m}, \theta_{k,n}) + g(\theta_{j-1,m}, \theta_{k-1,n})\Big\} \\ &\qquad\qquad \times (R^\alpha_{j,m} - R^\alpha_{m+1,m})(R^\beta_{k,n} - R^\beta_{n+1,n}) \\ &= \sum_{j=1}^{m}\sum_{k=1}^{n}\{\phi(\theta_{j,m}, \theta_{k,n}) - \phi(\theta_{j,m}, \theta_{k-1,n}) - \phi(\theta_{j-1,m}, \theta_{k,n}) + \phi(\theta_{j-1,m}, \theta_{k-1,n})\}R^\alpha_{j,m}R^\beta_{k,n},\end{aligned}$$

because $a_{j,0} = a_{0,k} = 0$ by definition of $g(u,v)$, for all $j = 0, 1, \dots, m$ and $k = 0, 1, \dots, n$, by (31), (32), and by (36) with $u = \theta_{j-1,m}$ and $v = \theta_{k-1,n}$.

Thus, from (31) and (32), it follows that

$$|\mathrm{A}_9| \leqslant \frac{C_2^2 2^{2+\alpha+\beta}}{\pi^{2+\alpha+\beta}} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{1}{j^{1+\alpha} k^{1+\beta}} \mathrm{osc}_2(\phi, I_{j-1,m}, I_{k-1,n})$$

$$\leqslant \frac{C_2^2 2^{2+\alpha+\beta}}{\pi^{2+\alpha+\beta}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{1}{(j+1)^{1+\alpha}(k+1)^{1+\beta}} \mathrm{osc}_2(\phi, I_{j,m}, I_{k,n}). \tag{40}$$

Combining (25)–(28), (34)–(37), and (38)–(40) yields

$$|\mathrm{A}_{mn}| \leqslant C_\alpha C_\beta \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{1}{(j+1)^{1+\alpha}(k+1)^{1+\beta}} \mathrm{osc}_2(\phi, I_{j,m}, I_{k,n}), \tag{41}$$

where $C_\eta = \left(1 + \frac{2^{1+\eta}}{\pi} C_1 + \frac{2^{1+\eta}}{\pi^{2+\eta}} C_2\right)$ for $\eta = \alpha, \beta$.

In order to estimate B_m and C_n in (24), it is enough to apply the first inequality of (8) of Theorem 3 with the equality (3), which gives

$$|\mathrm{B}_m| = \left| \frac{1}{2\pi} \int_0^\pi \phi(u,0) K_m^\alpha(u) du \right| \leqslant C_\alpha \sum_{j=0}^{m} \frac{1}{(j+1)^{1+\alpha}} \mathrm{osc}_1(\phi(\cdot,0), I_{j,m}) \tag{42}$$

and

$$|\mathrm{C}_n| = \left| \frac{1}{2\pi} \int_0^\pi \phi(v,0) K_n^\beta(v) dv \right| \leqslant C_\beta \sum_{k=0}^{n} \frac{1}{(k+1)^{1+\beta}} \mathrm{osc}_1(\phi(0,\cdot), I_{k,n}). \tag{43}$$

Now, using (41)–(43) in (24), we get (18) to be proved. ■

Proof of Theorem 11. For fixed m and n in $\{0, 1, 2, \dots\}$, set

$$\mathrm{M}_{j,k} = \sum_{i=0}^{j} \sum_{l=0}^{k} \frac{1}{(i+1)^\gamma (l+1)^\delta} \mathrm{osc}_2\left(\phi_{xy}, I_{i,m}, I_{l,n}\right),$$

$$\mathrm{M}_j' = \sum_{i=0}^{j} \frac{1}{(i+1)^\gamma (n+1)^\delta} \mathrm{osc}_2\left(\phi_{xy}, I_{i,m}, I_{n,n}\right),$$

and

$$\mathrm{M}_k'' = \sum_{l=0}^{k} \frac{1}{(m+1)^\gamma (l+1)^\delta} \mathrm{osc}_2\left(\phi_{xy}, I_{m,m}, I_{l,n}\right),$$

where $j = 0, 1, ..., m$; $k = 0, 1, ..., n$. Then we have

$$\mathrm{M}_{j,k} \leqslant V_{\gamma\delta}\left(\phi_{xy}, \theta_{j+1,m}, \theta_{k+1,n}\right). \tag{44}$$

Also, define functions $\mathrm{M}(u,v)$, $\mathrm{M}'(u)$, and $\mathrm{M}''(v)$ on the rectangle $\left[\frac{\pi}{m+1}, \pi\right) \times \left[\frac{\pi}{n+1}, \pi\right)$, and the intervals $\left[\frac{\pi}{m+1}, \pi\right)$ and $\left[\frac{\pi}{n+1}, \pi\right)$ respectively, by

$$\mathrm{M}(u,v) = \mathrm{M}_{\left[\frac{(m+1)u}{\pi}\right]-1, \left[\frac{(n+1)v}{\pi}\right]-1}, \tag{45}$$

$$\mathrm{M}'(u) = \mathrm{M}_{\left[\frac{(m+1)u}{\pi}\right]-1}, \tag{46}$$

and

$$\mathrm{M}''(v) = \mathrm{M}_{\left[\frac{(n+1)v}{\pi}\right]-1}. \tag{47}$$

Note that

$$\frac{(j+1)\pi}{m+1} \leqslant u < \frac{(j+2)\pi}{m+1} \implies j+1 \leqslant \frac{(m+1)u}{\pi} < j+2$$
$$\implies \left[\frac{(m+1)u}{\pi}\right] = j+1.$$

Similarly,

$$v \in \left[\frac{(k+1)\pi}{n+1}, \frac{(k+2)\pi}{n+1}\right) \implies \left[\frac{(n+1)v}{\pi}\right] = k+1.$$

Therefore, for each $j = 0, 1, \ldots, m-1$; $k = 0, 1, \ldots, n-1$, and for each (u, v) in $\left[\frac{(j+1)\pi}{m+1}, \frac{(j+2)\pi}{m+1}\right) \times$ $\times \left[\frac{(k+1)\pi}{n+1}, \frac{(k+2)\pi}{n+1}\right)$, by (45), we have

$$\mathrm{M}(u, v) = \mathrm{M}_{j,k}. \tag{48}$$

Now, using the double partial summation formula (see (22) of Lemma 2) with $a_{j,k} = \frac{1}{(j+1)^{1+\alpha-\gamma}(k+1)^{1+\beta-\delta}}$ and $b_{j,k} = \frac{1}{(j+1)^{\gamma}(k+1)^{\delta}}\mathrm{osc}_2\left(\phi_{xy}, I_{j,m}, I_{k,n}\right)$ for $j = 0, 1, \ldots, m$; $k = 0, 1, \ldots, n$, we get

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{n} \frac{1}{(j+1)^{1+\alpha}(k+1)^{1+\beta}} \mathrm{osc}_2\left(\phi_{xy}, I_{j,m}, I_{k,n}\right) \\
&\quad= \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{1}{(j+1)^{1+\alpha-\gamma}(k+1)^{1+\beta-\delta}} \frac{1}{(j+1)^{\gamma}(k+1)^{\delta}} \mathrm{osc}_2\left(\phi_{xy}, I_{j,m}, I_{k,n}\right) \\
&\quad= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \mathrm{M}_{j,k}\Bigg(\frac{1}{(j+1)^{1+\alpha-\gamma}(k+1)^{1+\beta-\delta}} - \frac{1}{(j+2)^{1+\alpha-\gamma}(k+1)^{1+\beta-\delta}} \\
&\qquad\qquad - \frac{1}{(j+1)^{1+\alpha-\gamma}(k+2)^{1+\beta-\delta}} + \frac{1}{(j+2)^{1+\alpha-\gamma}(k+2)^{1+\beta-\delta}}\Bigg) \\
&\qquad + \frac{1}{(n+1)^{1+\beta-\delta}} \sum_{j=0}^{m-1} \mathrm{M}_{j,n}\left(\frac{1}{(j+1)^{1+\alpha-\gamma}} - \frac{1}{(j+2)^{1+\alpha-\gamma}}\right) \\
&\qquad + \frac{1}{(m+1)^{1+\alpha-\gamma}} \sum_{k=0}^{n-1} \mathrm{M}_{m,k}\left(\frac{1}{(k+1)^{1+\beta-\delta}} - \frac{1}{(k+2)^{1+\beta-\delta}}\right) \\
&\qquad + \frac{\mathrm{M}_{m,n}}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}} \\
&\quad= \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D}, \text{ say.}
\end{aligned} \tag{49}$$

We will use properties of the Riemann-Stieltjes integral to estimate A, B, and C. First, we estimate A. Since $\alpha > \gamma - 1$ and $\beta > \delta - 1$, the functions $-u^{-1-\alpha+\gamma}$ and $-v^{-1-\beta+\delta}$ are continuous and nondecreasing for $u, v > 0$. Therefore, we have

$$
\begin{aligned}
\mathrm{A} &= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \mathrm{M}_{j,k}\left(\frac{1}{(j+1)^{1+\alpha-\gamma}} - \frac{1}{(j+2)^{1+\alpha-\gamma}}\right)\left(\frac{1}{(k+1)^{1+\beta-\delta}} - \frac{1}{(k+2)^{1+\beta-\delta}}\right) \\
&= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \mathrm{M}_{j,k}\left(\int_{j+1}^{j+2} d\left(-u^{-1-\alpha+\gamma}\right)\right)\left(\int_{k+1}^{k+2} d\left(-v^{-1-\beta+\delta}\right)\right) \\
&= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \mathrm{M}_{j,k}\int_{j+1}^{j+2}\int_{k+1}^{k+2}(1+\alpha-\gamma)(1+\beta-\delta)u^{-2-\alpha+\gamma}v^{-2-\beta+\delta}dudv \\
&= (1+\alpha-\gamma)(1+\beta-\delta)\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \mathrm{M}_{j,k}\left(\int_{j+1}^{j+2} u^{-2-\alpha+\gamma}du\right)\left(\int_{k+1}^{k+2} v^{-2-\beta+\delta}dv\right). \qquad (50)
\end{aligned}
$$

Put $s = \frac{u\pi}{m+1}$. Then $\frac{du}{ds} = \frac{m+1}{\pi}$; $u \to j+1 \Leftrightarrow s \to \frac{(j+1)\pi}{m+1}$, $u \to j+2 \Leftrightarrow s \to \frac{(j+2)\pi}{m+1}$. Therefore

$$
\begin{aligned}
\int_{j+1}^{j+2} u^{-2-\alpha+\gamma}du &= \int_{\frac{(j+1)\pi}{m+1}}^{\frac{(j+2)\pi}{m+1}} \left(\frac{(m+1)s}{\pi}\right)^{-2-\alpha+\gamma}\left(\frac{m+1}{\pi}\right)ds \\
&= \left(\frac{m+1}{\pi}\right)^{-1-\alpha+\gamma}\int_{\frac{(j+1)\pi}{m+1}}^{\frac{(j+2)\pi}{m+1}} s^{-2-\alpha+\gamma}ds \\
&= \frac{\pi^{1+\alpha-\gamma}}{(m+1)^{1+\alpha-\gamma}}\int_{\frac{(j+1)\pi}{m+1}}^{\frac{(j+2)\pi}{m+1}} u^{-2-\alpha+\gamma}du. \qquad (51)
\end{aligned}
$$

Similarly,

$$
\int_{k+1}^{k+2} v^{-2-\beta+\delta}dv = \frac{\pi^{1+\beta-\delta}}{(n+1)^{1+\beta-\delta}}\int_{\frac{(k+1)\pi}{n+1}}^{\frac{(k+2)\pi}{n+1}} v^{-2-\beta+\delta}dv. \qquad (52)
$$

Using (51) and (52) in (50), we get

$$
\mathrm{A} = \frac{(1+\alpha-\gamma)(1+\beta-\delta)\pi^{2+\alpha+\beta-\gamma-\delta}}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\int_{\frac{(j+1)\pi}{m+1}}^{\frac{(j+2)\pi}{m+1}}\int_{\frac{(k+1)\pi}{n+1}}^{\frac{(k+2)\pi}{n+1}} \mathrm{M}_{j,k}\, u^{-2-\alpha+\gamma}v^{-2-\beta+\delta}dudv.
$$

Since $\mathrm{M}(u,v) = \mathrm{M}_{j,k}$ for all $(u,v) \in \left[\frac{(j+1)\pi}{m+1}, \frac{(j+2)\pi}{m+1}\right) \times \left[\frac{(k+1)\pi}{n+1}, \frac{(k+2)\pi}{n+1}\right)$, we get

$$
\begin{aligned}
\mathrm{A} &= \frac{(1+\alpha-\gamma)(1+\beta-\delta)\pi^{2+\alpha+\beta-\gamma-\delta}}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\int_{\frac{(j+1)\pi}{m+1}}^{\frac{(j+2)\pi}{m+1}}\int_{\frac{(k+1)\pi}{n+1}}^{\frac{(k+2)\pi}{n+1}} \mathrm{M}(u,v)u^{-2-\alpha+\gamma}v^{-2-\beta+\delta}dudv \\
&= \frac{(1+\alpha-\gamma)(1+\beta-\delta)\pi^{2+\alpha+\beta-\gamma-\delta}}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\int_{\frac{\pi}{m+1}}^{\pi}\int_{\frac{\pi}{n+1}}^{\pi} \mathrm{M}(u,v)u^{-2-\alpha+\gamma}v^{-2-\beta+\delta}dudv.
\end{aligned}
$$

Put $u = \frac{\pi}{s}$ and $v = \frac{\pi}{t}$. Then $\frac{du}{ds} = -\pi s^{-2}$, $\frac{dv}{dt} = -\pi t^{-2}$, $u \to \frac{\pi}{m+1} \Leftrightarrow s \to m+1$, $u \to \pi \Leftrightarrow s \to 1$, $v \to \frac{\pi}{n+1} \Leftrightarrow t \to n+1$, and $v \to \pi \Leftrightarrow t \to 1$. Therefore

$$
\begin{aligned}
\mathrm{A} = & \frac{(1+\alpha-\gamma)(1+\beta-\delta)\pi^{2+\alpha+\beta-\gamma-\delta}}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}} \times \\
& \times \int_{m+1}^{1}\int_{n+1}^{1} \mathrm{M}\left(\frac{\pi}{s},\frac{\pi}{t}\right)\left(\frac{\pi}{s}\right)^{-2-\alpha+\gamma}\left(\frac{\pi}{t}\right)^{-2-\beta+\delta}\pi^2 s^{-2}t^{-2}dsdt = \\
& = \frac{(1+\alpha-\gamma)(1+\beta-\delta)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\int_1^{m+1}\int_1^{n+1}\mathrm{M}\left(\frac{\pi}{s},\frac{\pi}{t}\right)s^{\alpha-\gamma}t^{\beta-\delta}dsdt = \\
& = \frac{(1+\alpha-\gamma)(1+\beta-\delta)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\sum_{j=1}^{m}\sum_{k=1}^{n}\int_j^{j+1}\int_k^{k+1}\mathrm{M}\left(\frac{\pi}{s},\frac{\pi}{t}\right)s^{\alpha-\gamma}t^{\beta-\delta}dsdt = \\
& \leqslant \frac{(1+\alpha-\gamma)(1+\beta-\delta)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\sum_{j=1}^{m}\sum_{k=1}^{n}\mathrm{M}\left(\frac{\pi}{j},\frac{\pi}{k}\right)j^{\alpha-\gamma}k^{\beta-\delta}\int_j^{j+1}\int_k^{k+1}dsdt = \\
& = \frac{(1+\alpha-\gamma)(1+\beta-\delta)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\sum_{j=1}^{m}\sum_{k=1}^{n}\mathrm{M}\left(\frac{\pi}{j},\frac{\pi}{k}\right)j^{\alpha-\gamma}k^{\beta-\delta}. \qquad (53)
\end{aligned}
$$

Now, we estimate B. Proceeding as above, we have

$$
\begin{aligned}
\mathrm{B} & = \frac{1}{(n+1)^{1+\beta-\delta}}\sum_{j=0}^{m-1}\mathrm{M}_{j,n}\left(\frac{1}{(j+1)^{1+\alpha-\gamma}}-\frac{1}{(j+2)^{1+\alpha-\gamma}}\right) \\
& = \frac{1}{(n+1)^{1+\beta-\delta}}\sum_{j=0}^{m-1}\mathrm{M}_{j,n}\left(\int_{j+1}^{j+2}d\left(-u^{-1-\alpha+\gamma}\right)\right) \\
& = \frac{1}{(n+1)^{1+\beta-\delta}}\sum_{j=0}^{m-1}\left(\mathrm{M}_{j,n-1}+\mathrm{M}'_j\right)\left(\int_{j+1}^{j+2}(1+\alpha-\gamma)u^{-2-\alpha+\gamma}du\right) \\
& = \frac{(1+\alpha-\gamma)\pi^{1+\alpha-\gamma}}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\sum_{j=0}^{m-1}\left(\mathrm{M}_{j,n-1}+\mathrm{M}'_j\right)\int_{\frac{(j+1)\pi}{m+1}}^{\frac{(j+2)\pi}{m+1}}u^{-2-\alpha+\gamma}du. \qquad (54)
\end{aligned}
$$

Note that if $u \in \left[\frac{(j+1)\pi}{m+1}, \frac{(j+2)\pi}{m+1}\right)$ and $v = \frac{n\pi}{n+1}$, then as $\left[\frac{(n+1)v}{\pi}\right] = \left[\frac{n+1}{\pi}\cdot\frac{n\pi}{n+1}\right] = n$, by (48), we have $\mathrm{M}\left(u, \frac{n\pi}{n+1}\right) = \mathrm{M}_{j,n-1}$. Also, for $u \in \left[\frac{(j+1)\pi}{m+1}, \frac{(j+2)\pi}{m+1}\right)$, $\left[\frac{(m+1)u}{\pi}\right] - 1 = j$, so that $\mathrm{M}'(u) = \mathrm{M}'_j$. So from (54), we get

$$
\begin{aligned}
\mathrm{B} & = \frac{(1+\alpha-\gamma)\pi^{1+\alpha-\gamma}}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\sum_{j=0}^{m-1}\int_{\frac{(j+1)\pi}{m+1}}^{\frac{(j+2)\pi}{m+1}}\left(\mathrm{M}\left(u,\frac{n\pi}{n+1}\right)+\mathrm{M}'(u)\right)u^{-2-\alpha+\gamma}du \\
& = \frac{(1+\alpha-\gamma)\pi^{1+\alpha-\gamma}}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\int_{\frac{\pi}{m+1}}^{\pi}\left(\mathrm{M}\left(u,\frac{n\pi}{n+1}\right)+\mathrm{M}'(u)\right)u^{-2-\alpha+\gamma}du \\
& = \frac{(1+\alpha-\gamma)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\int_1^{m+1}\left(\mathrm{M}\left(\frac{\pi}{s},\frac{n\pi}{n+1}\right)+\mathrm{M}'\left(\frac{\pi}{s}\right)\right)s^{\alpha-\gamma}ds \\
& = \frac{(1+\alpha-\gamma)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\sum_{j=1}^{m}\int_j^{j+1}\left(\mathrm{M}\left(\frac{\pi}{s},\frac{n\pi}{n+1}\right)+\mathrm{M}'\left(\frac{\pi}{s}\right)\right)s^{\alpha-\gamma}ds \\
& \leqslant \frac{(1+\alpha-\gamma)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}}\sum_{j=1}^{m}\left(\mathrm{M}\left(\frac{\pi}{j},\frac{n\pi}{n+1}\right)+\mathrm{M}'\left(\frac{\pi}{j}\right)\right)j^{\alpha-\gamma}, \qquad (55)
\end{aligned}
$$

and similarly, we can prove

$$C \leqslant \frac{(1+\beta-\delta)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}} \sum_{k=1}^{n} \left(\mathrm{M}\left(\frac{m\pi}{m+1}, \frac{\pi}{k}\right) + \mathrm{M}''\left(\frac{\pi}{k}\right)\right) k^{\beta-\delta}. \tag{56}$$

In view of (44)–(47), we have

$$\mathrm{M}\left(\frac{\pi}{j}, \frac{\pi}{k}\right) = \mathrm{M}_{\left[\frac{m+1}{j}\right]-1, \left[\frac{n+1}{k}\right]-1} \leqslant V_{\gamma\delta}\left(\phi_{xy}, \theta_{\left[\frac{m+1}{j}\right],m}, \theta_{\left[\frac{n+1}{k}\right],n}\right) \leqslant V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right) \tag{57}$$

and

$$\begin{aligned}
\mathrm{M}\left(\frac{\pi}{j}, \frac{n\pi}{n+1}\right) + \mathrm{M}'\left(\frac{\pi}{j}\right) &= \mathrm{M}_{\left[\frac{m+1}{j}\right]-1, n-1} + \mathrm{M}'_{\left[\frac{m+1}{j}\right]-1} \\
&= \sum_{i=0}^{\left[\frac{m+1}{j}\right]-1} \sum_{l=0}^{n-1} \frac{1}{(i+1)^{\gamma}(l+1)^{\delta}} \operatorname{osc}_2\left(\phi_{xy}, I_{i,m}, I_{l,n}\right) \\
&\quad + \sum_{i=0}^{\left[\frac{m+1}{j}\right]-1} \frac{1}{(i+1)^{\gamma}(n+1)^{\delta}} \operatorname{osc}_2\left(\phi_{xy}, I_{i,m}, I_{n,n}\right) \\
&= \sum_{i=0}^{\left[\frac{m+1}{j}\right]-1} \sum_{l=0}^{n} \frac{1}{(i+1)^{\gamma}(l+1)^{\delta}} \operatorname{osc}_2\left(\phi_{xy}, I_{i,m} I_{l,n}\right) \\
&= \mathrm{M}_{\left[\frac{m+1}{j}\right]-1, n} \\
&\leqslant V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \pi\right).
\end{aligned} \tag{58}$$

In a similar way, we can prove the inequalities

$$\mathrm{M}\left(\frac{m\pi}{m+1}, \frac{\pi}{k}\right) + \mathrm{M}''\left(\frac{\pi}{k}\right) \leqslant V_{\gamma\delta}\left(\phi_{xy}, \pi, \frac{\pi}{k}\right) \tag{59}$$

and

$$\mathrm{M}_{m,n} \leqslant V_{\gamma\delta}\left(\phi_{xy}, \pi, \pi\right). \tag{60}$$

Using (57) into (53), (58) into (55), (59) into (56), and then the results and (60) into (49), we get

$$\begin{aligned}
&\sum_{j=0}^{m} \sum_{k=0}^{n} \frac{1}{(j+1)^{1+\alpha}(k+1)^{1+\beta}} \operatorname{osc}_2\left(\phi_{xy}, I_{j,m}, I_{k,n}\right) \\
&\leqslant \frac{(1+\alpha-\gamma)(1+\beta-\delta)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}} \sum_{j=1}^{m} \sum_{k=1}^{n} j^{\alpha-\gamma} k^{\beta-\delta} V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right) \\
&\quad + \frac{(1+\alpha-\gamma)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}} \sum_{j=1}^{m} j^{\alpha-\gamma} V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \pi\right) \\
&\quad + \frac{(1+\beta-\delta)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}} \sum_{k=1}^{n} k^{\beta-\delta} V_{\gamma\delta}\left(\phi_{xy}, \pi, \frac{\pi}{k}\right) \\
&\quad + \frac{1}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}} V_{\gamma\delta}\left(\phi_{xy}, \pi, \pi\right).
\end{aligned} \tag{61}$$

Note that

$$\sum_{j=1}^{m} j^{\alpha-\gamma} V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \pi\right) \leqslant \sum_{j=1}^{m}\sum_{k=1}^{n} j^{\alpha-\gamma} k^{\beta-\delta} V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right),$$

$$\sum_{k=1}^{n} k^{\beta-\delta} V_{\gamma\delta}\left(\phi_{xy}, \pi, \frac{\pi}{k}\right) \leqslant \sum_{j=1}^{m}\sum_{k=1}^{n} j^{\alpha-\gamma} k^{\beta-\delta} V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right),$$

and

$$V_{\gamma\delta}\left(\phi_{xy}, \pi, \pi\right) \leqslant \sum_{j=1}^{m}\sum_{k=1}^{n} j^{\alpha-\gamma} k^{\beta-\delta} V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right).$$

Therefore, from (61) we get

$$\sum_{j=0}^{m}\sum_{k=0}^{n} \frac{1}{(j+1)^{1+\alpha}(k+1)^{1+\beta}} \operatorname{osc}_2\left(\phi_{xy}, I_{j,m}, I_{k,n}\right)$$
$$\leqslant \frac{(2+\alpha-\gamma)(2+\beta-\delta)}{(m+1)^{1+\alpha-\gamma}(n+1)^{1+\beta-\delta}} \sum_{j=1}^{m}\sum_{k=1}^{n} j^{\alpha-\gamma} k^{\beta-\delta} V_{\gamma\delta}\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right). \tag{62}$$

Second, in view of the second inequality of (8) of Theorem 3, we get the inequalities

$$\sum_{j=0}^{m} \frac{1}{(j+1)^{1+\alpha}} \operatorname{osc}_1\left(\phi_{xy}(\cdot,0), I_{j,m}\right) \leqslant \frac{2+\alpha-\gamma}{(m+1)^{1+\alpha-\gamma}} \sum_{j=1}^{m} \frac{1}{j^{\gamma-\alpha}} V_{\gamma}\left(\phi_{xy}(\cdot,0), \frac{\pi}{j}\right) \tag{63}$$

and

$$\sum_{k=0}^{n} \frac{1}{(k+1)^{1+\beta}} \operatorname{osc}_1\left(\phi_{xy}(0,\cdot), I_{k,n}\right) \leqslant \frac{2+\beta-\delta}{(n+1)^{1+\beta-\delta}} \sum_{k=1}^{n} \frac{1}{k^{\delta-\beta}} V_{\delta}\left(\phi_{xy}(0,\cdot), \frac{\pi}{k}\right). \tag{64}$$

Using (62)–(64) in the Inequality (18) of Theorem 10 we get (19). This completes the proof of Theorem 11. ■

Proof of Corollary 2. For any δ_1 and δ_2 greater than zero, existence of positive integers m and n satisfying $\frac{1}{m+2} \leqslant \frac{\delta_1}{2\pi} < \frac{1}{m+1}$ and $\frac{1}{n+2} \leqslant \frac{\delta_2}{2\pi} < \frac{1}{n+1}$ respectively, implies that

$$\operatorname{osc}_2(\phi_{xy}, I_{j,m}, I_{k,n}) \leqslant 4\omega(f, \delta_1, \delta_2),$$

$$\operatorname{osc}_1(\phi_{xy}(\cdot,0), I_{j,m}) \leqslant 2\omega_x(f, \delta_1), \text{ and } \operatorname{osc}_1(\phi_{xy}(0,\cdot), I_{k,n}) \leqslant 2\omega_y(f, \delta_2).$$

Also, for $\alpha \in (-1, 0)$, we have

$$\sum_{j=0}^{m} \frac{1}{(j+1)^{1+\alpha}} = \sum_{j=0}^{m} \frac{1}{(j+1)^{1+\alpha}} \int_j^{j+1} dt \leqslant 1 + \sum_{j=1}^{m} \int_j^{j+1} \frac{1}{t^{1+\alpha}} dt = 1 + \int_1^{m+1} \frac{1}{t^{1+\alpha}} dt$$
$$= 1 - \frac{1}{\alpha(m+1)^{\alpha}} + \frac{1}{\alpha} = O\left(\frac{1}{(m+1)^{\alpha}}\right) = O(\delta_1^{\alpha}).$$

Similarly for $\beta \in (-1, 0)$, we have

$$\sum_{k=0}^{n} \frac{1}{(k+1)^{1+\beta}} = O(\delta_2^{\beta}).$$

Now, using the inequality of Theorem 10, we have

$$\left|\sigma_{m,n}^{\alpha,\beta}(f,x,y)-s(f,x,y)\right| \leqslant 4C_\alpha C_\beta \sum_{j=0}^{m}\sum_{k=0}^{n}\frac{1}{(j+1)^{1+\alpha}(k+1)^{1+\beta}}\omega(f,\delta_1,\delta_2)$$
$$+2C_\alpha\sum_{j=0}^{m}\frac{1}{(j+1)^{1+\alpha}}\omega_x(f,\delta_1)+2C_\beta\sum_{k=0}^{n}\frac{1}{(k+1)^{1+\beta}}\omega_y(f,\delta_2) \quad (65)$$
$$=O(\delta_1^{\alpha}\delta_2^{\beta})\omega(f,\delta_1,\delta_2)+O(\delta_1^{\alpha})\omega_x(f,\delta_1)+O(\delta_2^{\beta})\omega_y(f,\delta_2). \quad (66)$$

Now, if f satisfies conditions in (i), then from (66)

$$\left|\sigma_{m,n}^{\alpha,\beta}(f,x,y)-s(f,x,y)\right| = O(\delta_1^{\alpha}\delta_2^{\beta})o(\delta_1^{-\alpha}\delta_2^{-\beta})+O(\delta_1^{\alpha})o(\delta_1^{-\alpha})+O(\delta_2^{\beta})o(\delta_2^{-\beta})$$
$$=o(1), \quad \text{as } \delta_1,\delta_2\to 0.$$

For the proof of (ii) and (iii), first we have

$$\omega(f,\delta_1,\delta_2)=\sup_{|u-u'|\leqslant\delta_1,\ |v-v'|\leqslant\delta_2}|f(u,v)-f(u',v)-f(u,v')+f(u',v')|$$
$$\leqslant \sup_{|u-u'|\leqslant\delta_1,\ v\in\mathbb{T}}|f(u,v)-f(u',v)|+\sup_{|u-u'|\leqslant\delta_1,\ v'\in\mathbb{T}}|f(u,v')-f(u',v')|$$
$$=2\omega_x(f,\delta_1),$$

similarly, we have $\omega(f,\delta_1,\delta_2)\leqslant 2\omega_y(f,\delta_2)$, and

$$\omega(f,\delta_1,\delta_2)=\sqrt{\omega(f,\delta_1,\delta_2)}\sqrt{\omega(f,\delta_1,\delta_2)}\leqslant 2\sqrt{\omega_x(f,\delta_1)}\sqrt{\omega_y(f,\delta_2)}.$$

Now, if f satisfies conditions in (ii), then by (65), we have

$$\left|\sigma_{m,n}^{\alpha,\beta}(f,x,y)-s(f,x,y)\right| \leqslant 8C_\alpha C_\beta \sum_{j=0}^{m}\sum_{k=0}^{n}\frac{1}{(j+1)^{1+\alpha}(k+1)^{1+\beta}}\sqrt{\omega_x(f,\delta_1)}\sqrt{\omega_y(f,\delta_2)}$$
$$+2C_\alpha\sum_{j=0}^{m}\frac{1}{(j+1)^{1+\alpha}}\omega_x(f,\delta_1)+2C_\beta\sum_{k=0}^{n}\frac{1}{(k+1)^{1+\beta}}\omega_y(f,\delta_2) \quad (67)$$
$$=O(\delta_1^{\alpha}\delta_2^{\beta})O(\delta_1^{-\alpha+\epsilon/2}\delta_2^{-\beta+\epsilon/2})+O(\delta_1^{\alpha})O(\delta_1^{-2\alpha+\epsilon})+O(\delta_2^{\beta})\delta_2^{-2\beta+\epsilon}$$
$$=O(\delta_1^{\epsilon/2}\delta_2^{\epsilon/2})+O(\delta_1^{-\alpha+\epsilon})+O(\delta_2^{-\beta+\epsilon})=o(1), \quad \text{as } \delta_1,\delta_2\to 0.$$

Similarly, if f satisfies conditions in (iii), then by (67), we have

$$\left|\sigma_{m,n}^{\alpha,\beta}(f,x,y)-s(f,x,y)\right| = O(\delta_1^{\alpha}\delta_2^{\beta})o(\delta_1^{-\alpha}\delta_2^{-\beta})+O(\delta_1^{\alpha})o(\delta_1^{-2\alpha})+O(\delta_2^{\beta})o(\delta_2^{-2\beta})$$
$$=o(1)+o(\delta_1^{-\alpha})+o\left(\delta_2^{-\beta}\right)=o(1), \quad \text{as } \delta_1,\delta_2\to 0.$$

Finally, we will prove (iv) by contradiction. Suppose for any continuous function f on $\mathbb{T}^2$ satisfying (21), its Fourier series is (C,α,β) summable in restricted sense. Then it is $(C,0,0)$-summable in the Pringsheim sense. Now, we will show that (21) implies (15) (for the case $m=2$). First, we have

$$\lim_{\delta\to 0}\frac{\delta^{-\alpha}}{(\log\left(\frac{1}{\delta}\right))^{-1}}=0 \quad (\alpha>-1).$$

Therefore for any fixed $\epsilon > 0$, there is $\delta_0 > 0$ such that for any $0 < \delta_1, \delta_2 \leqslant \delta_0$, we have

$$\delta_1^{-\alpha} \leqslant \epsilon \left(\log\left(\frac{1}{\delta_1}\right)\right)^{-1} \text{ and } \delta_2^{-\beta} \leqslant \epsilon \left(\log\left(\frac{1}{\delta_2}\right)\right)^{-1}.$$

Therefore, in view of (21), we have

$$\omega_x(f, \delta_1) = O\left(\delta_1^{-\alpha}\right) = O\left(\left(\log\left(\frac{1}{\delta_1}\right)\right)^{-1}\right) \quad (\delta_0 \geqslant \delta_1 \to 0) \tag{68}$$

and

$$\omega_y(f, \delta_2) = O\left(\delta_2^{-\beta}\right) = O\left(\left(\log\left(\frac{1}{\delta_2}\right)\right)^{-1}\right) \quad (\delta_0 \geqslant \delta_2 \to 0). \tag{69}$$

That means, for any function $f \in C(\mathbb{T}^2)$ satisfying the conditions (68) and (69), its Fourier series converges in the Pringsheim sense. Which contradicts the theorem of D'yachenko [9, Theorem 1.2.4]. This completes the proof. ■

Funding

The first author would like to thank Council of Scientific and Industrial Research (CSIR) for financial support through JRF (File No. 09/114(0233)/2019-EMR-I).

Data availability

Not applicable.

Declarations

Ethical conduct

This article is original, it has not been previously published, and it has not been simultaneously submitted for evaluation to another journal.

Informed consent

The research does not involve human participants and/or animals.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Acknowledgements

The authors are thankful to the referee for his/her valuable suggestions for the present form of the paper.

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Бахвалов А. Н. Расхождение всюду ряда Фурье непрерывных функций нескольких переменных // *Математический сборник*, **188** (8), 1997, С. 1153–1170.

2. Бахвалов А. Н. λ-расходимость ряда Фурье непрерывных функций нескольких переменных // *Математика. Примечания*, **72** (4), 2002, С. 454–465.

3. Бахвалов А. Н. Суммирование рядов Фурье функций из многомерных классов Уотермана методами Чезаро // *Докл. Акад. Науки*, **437** (6), 2011, С. 731–733. Английский перевод. в *Докл. Математика*, **83** (2), С. 247–249.

4. Боянич Р. Оценка скорости сходимости рядов Фурье функций ограниченной вариации // *Опубл. Инст. Математика. (београд)*,**26** (40), 1979, С. 57–60.

5. Боянич Р., Уотерман, Д. О скорости сходимости рядов Фурье функций обобщенной ограниченной вариации // *Акад. Наука Умжет. Босне Герцегов. Рад. Оджель. Природа. Мат. Наука*, **22**, 1983, С. 5–11.

6. Боянич Р., Мажар С. М. Оценка скорости сходимости средних Чезаро рядов Фурье функций ограниченной вариации // *Математический анализ и его приложения (Кувейт, 1985)*, 1988, С. 17–22, *KFAS Proc. Ser., 3, Пергам, Оксфорд.*

7. Бера Р. К., Годадра Б. Л. О скорости сходимости двойных рядов Фурье функций обобщенной ограниченной вариации // *Acta Sci. Math. (Сегед)*, **88**, 2022, С. 723–737.

8. Бахдух М., Никишин Е. М. Сходимость двойного ряда Фурье непрерывных функций // *Сибирск. Мат. ж.*, **14**, 1973, С. 1189–1199; Английский перевод. в Сибирской математике. J., **14** (6), 832–839.

9. Дьяченко М. И. Некоторые проблемы теории множественных тригонометрических рядов // *Успехи мат. Науки.*, **47** (5), 1992, С. 97–162. Английский перевод. по русской математике. Опросы **47**.

10. Харди, Г. Х. О двойных рядах Фурье // *Quart. J. Math.*, **35**, 1906, С. 53–79.

11. Джордан, С. Серия Фурье // *Парижс. Акад. Науки*, **92**, 1881, С. 228–230.

12. Лимайе, Б. В., Горпаде, С. Р. Курс многомерного исчисления и анализа // Спрингер, Нью-Йорк, Дордрехт, Гейдельберг, Лондон, 2010.

13. Мажар, С. М. О скорости сходимости средних Чезаро рядов Фурье функций обобщенной ограниченной вариации // *Китайский Дж. Математика.*, **13** (3), 1985, С. 195–202.

14. Мóрич, Ф. Количественная версия теста Дирихле-Жордана для двойных рядов Фурье // *Журнал Теор. Приближ.*, **71**, 1992, С. 344–358.

15. Мóрич, Ф. Аппроксимация прямоугольными частичными суммами двойных сопряженных рядов Фурье // *урнал Теор. Приближ.*, **103**, 2000, С. 130–150.

16. Перлман, С. Функции обобщенной вариации // *Польская академия наук. Инст. Математика.*, **105** (3), 1979, С. 199–211.

17. Зигмунд, А. Тригонометрический ряд I // Кембриджский университет, 1959.

18. Жижиашвили Л. В. Сходимость и суммируемость рядов Фурье // *Сообщение. Акад. Наука Грузин. SSR*, **29**, 1962, С. 257–261.

19. Жижиашвили Л. В. Сопряженные функции и тригонометрические ряды // *Изд. Тбилиси. Университет, Тбилиси.*, 1969.

20. Жижиашвили Л. В. О некоторых задачах теории простых и кратных тригонометрических рядов // *Русская математика. Опросы*, **28** (2), 1973, С. 65–119.

REFERENCES

1. Bakhvalov, A. N. 1997, "Divergence everywhere of the Fourier series of continuous functions of several variables", *Sb. Math.*, **188** (8), 1153–1170.

2. Bakhvalov, A. N. 2002, "λ-divergence of the Fourier series of continuous functions of several variables", *Math. Notes*, **72** (4), 454–465.

3. Bakhvalov, A. N. 2011, "Summation of Fourier series of functions from multidimensional Waterman classes by Cesàro methods", *Dokl. Akad. Nauk*, **437** (6), 731–733. English transl. in *Dokl. Math.*, **83** (2), 247–249.

4. Bojanić, R. 1979, "An estimate of the rate of convergence of Fourier series of functions of bounded variation", *Publ. Inst. Math. (beograd)*, **26** (40), 57–60.

5. Bojanić, R. & Waterman, D. 1983, "On the rate of convergence of Fourier series of functions of generalized bounded variation", *Akad. Nauka Umjet. Bosne Hercegov. Rad. Odjelj. Prirod. Mat. Nauka*, **22**, 5–11.

6. Bojanić, R. & Mazhar, S. M. 1988, "An estimate of the rate of convergence of Cesàro means of Fourier series of functions of bounded variation", *Mathematical analysis and its applications (Kuwait, 1985)*, 17–22, *KFAS Proc. Ser., 3, Pergamon, Oxford.*

7. Bera, R. K. & Ghodadra, B. L. 2022, "On the rate of convergence of double Fourier series of functions of generalized bounded variation", *Acta Sci. Math. (Szeged)*, **88**, 723–737.

8. Bahduh, M. & Nikishin E. M. 1973, "The convergence of the double Fourier series of continuous functions', *Sibirsk. Mat. Zh.*, **14**, 1189–1199; English transl. in Siberian Math. J., **14** (6), 832-839.

9. D'yachenko, M. I. 1992, "Some problems in the theory of multiple trigonometric series", *Uspekhi Mat. Nauk.*, **47** (5), 97–162. English transl. in Russian Math. Surveys, **47**.

10. Hardy, G. H. 1906, "On double Fourier series", *Quart. J. Math.*, **35**, 53–79.

11. Jordan, C. 1881, "Sur la series de Fourier", *C. R. Acad. Sci. Paris.*, **92**, 228–230.

12. Limaye, B. V. & Ghorpade, S. R. 2010, "A course in multivariable calculus and analysis", Springer, New York, Dordrecht Heidelberg, London.

13. Mazhar, S. M. 1985, "On the rate of convergence of Cesàro means of Fourier series of functions of generalized bounded variation", *Chinese J. Math.*, **13** (3), 195–202.

14. Móricz, F. 1992, "A quatitiative version of the Dirichlet-Jordan test for double Fourier series", *Journal of Approx. Theory*, **71**, 344–358.

15. Móricz, F. 2000, “Approximation by rectangular partial sums of double conjugate Fourier series”, *Journal of Approx. Theory*, **103**, 130–150.

16. Perlman, S. 1979, “Functions of generalized variation”, *Polish Acad. Sci. Inst. Math.*, **105** (3), 199–211.

17. Zygmund, A. 1959, “Trigonometric series I”, Cambridgee Univ. Press.

18. Zhizhiashvili, L. V. 1962, “Convergence and summability of Fourier series”, *Soobšč. Akad. Nauk Gruzin. SSR*, **29**, 257–261.

19. Zhizhiashvili, L. V. 1969, “Conjugate functions and trigonometric series”, Izdat. Tbilis. Univ., Tbilisi. (In Russian)

20. Zhizhiashvili, L. V. 1973, “On some problems of the theory of simple and multiple trigonometric series”, *Russian Math. Surveys*, **28** (2), 65–119.

Получено: 28.01.2023
Принято в печать: 14.06.2023