ЧЕБЫШЕВСКИЙ СБОРНИК
Том 24. Выпуск 1.

УДК 512 DOI 10.22405/2226-8383-2023-24-1-203-212

Когомологии де Рама алгебры полиномиальных функций на симплициальном комплексе[1]

И. С. Басков

Басков Игорь Сергеевич — Математический институт им. В. А. Стеклова РАН (г. Санкт-Петербург).
e-mail: baskovigor@pdmi.ras.ru

Аннотация

Мы рассматриваем алгебру $A^0(X)$ полиномиальных функций на симплициальном комплексе X, которая является компонентой степени 0 введенной Сулливаном dg-алгебры $A^\bullet(X)$ полиномиальных форм. Все рассматриваемые алгебры над произвольным полем k характеристики 0.

Нашей целью является вычисление когомологий де Рама алгебры $A^0(X)$, то есть когомологий универсальной dg-алгебры $\Omega^\bullet_{A^0(X)}$. Имеется канонический морфизм dg-алгебр $P : \Omega^\bullet_{A^0(X)} \to A^\bullet(X)$. Мы доказываем, что морфизм P является квазиизоморфизмом. Таким образом, когомологии де Рама алгебры $A^0(X)$ канонически изоморфны когомологиям симлициального комплекса X с коэффициентами в поле k. Более того, для $k = \mathbb{Q}$, dg-алгебра $\Omega^\bullet_{A^0(X)}$ служит моделью симплициального комплекса X в смысле рациональной теории гомотопий. Наш результат показывает, что для алгебры $A^0(X)$ верно утверждение теоремы сравнения Гротендика (доказанной им для гладких алгебр).

Для доказательства мы рассматриваем резольвенты Чеха, ассоциированные с покрытием симплициального комплекса звездами вершин.

Ранее Кан — Миллер доказали, что морфизм P сюръективен, а также описали его ядро. Другое описание ядра дали Сулливан и Феликс — Джессап — Паран.

Ключевые слова: когомологии де Рама алгебры, универсальная dg-алгебра, алгебра полиномиальных функций, dg-алгебра полиномиальных форм, рациональная теория гомотопий.

Библиография: 15 названий.

Для цитирования:
И. С. Басков. Когомологии де Рама алгебры полиномиальных функций на симплициальном комплексе // Чебышевский сборник, 2023, т. 24, вып. 1, с. 203–212.

[1]Данная работа была поддержана Санкт-Петербургским международным математическим Институтом имени Леонарда Эйлера, грантовое соглашение NN 075–15–2019–1620 от 08.11.2019 и 075-15-2022-289 от 06.04.2022.

CHEBYSHEVSKII SBORNIK
Vol. 24. No. 1.

UDC 512 DOI 10.22405/2226-8383-2023-24-1-203-212

The de Rham cohomology of the algebra of polynomial functions on a simplicial complex

I. S. Baskov

Baskov Igor Sergeevich — Steklov Mathematical Institute of Russian Academy of Sciences (St. Petersburg).
e-mail: baskovigor@pdmi.ras.ru

Abstract

We consider the algebra $A^0(X)$ of polynomial functions on a simplicial complex X. The algebra $A^0(X)$ is the 0th component of Sullivan's dg-algebra $A^\bullet(X)$ of polynomial forms on X. All algebras are over an arbitrary field k of characteristic 0.

Our main interest lies in computing the de Rham cohomology of the algebra $A^0(X)$, that is, the cohomology of the universal dg-algebra $\Omega^\bullet_{A^0(X)}$. There is a canonical morphism of dg-algebras $P : \Omega^\bullet_{A^0(X)} \to A^\bullet(X)$. We prove that P is a quasi-isomorphism. Therefore, the de Rham cohomology of the algebra $A^0(X)$ is canonically isomorphic to the cohomology of the simplicial complex X with coefficients in k. Moreover, for $k = \mathbb{Q}$ the dg-algebra $\Omega^\bullet_{A^0(X)}$ is a model of the simplicial complex X in the sense of rational homotopy theory. Our result shows that for the algebra $A^0(X)$ the statement of Grothendieck's comparison theorem holds (proved by him for smooth algebras).

In order to prove the statement we consider Čech resolution associated to the cover of the simplicial complex by the stars of the vertices.

Earlier, Kan–Miller proved that the morphism P is surjective and gave a description of its kernel. Another description of the kernel was given by Sullivan and Félix–Jessup–Parent.

Keywords: algebraic de Rham cohomology, universal dg-algebra, algebra of polynomial functions, dg-algebra of polynomial forms, rational homotopy theory.

Bibliography: 15 titles.

For citation:
I. S. Baskov, 2023“The de Rham cohomology of the algebra of polynomial functions on a simplicial complex” , *Chebyshevskii sbornik*, vol. 24, no. 1, pp. 203–212.

1. Introduction

All algebras and dg-algebras are commutative over a field k of characteristic 0. In [15, Section 7] Sullivan introduces the dg-algebra $A^\bullet(X)$ of *polynomial forms* on a simplicial complex X. The algebra $A^0(X)$ of the degree zero elements of $A^\bullet(X)$ is the algebra of *polynomial functions* on X. The cohomology of the dg-algebra $A^\bullet(X)$ is isomorphic to $H^\bullet(X, k)$. One can ask what natural dg-algebras are weakly equivalent to $A^\bullet(X)$. One such candidate is the universal dg-algebra $\Omega^\bullet_{A^0(X)}$ on the algebra $A^0(X)$ of polynomial functions on X. There is a canonical morphism of dg-algebras $P : \Omega^\bullet_{A^0(X)} \to A^\bullet(X)$.

The main result is Theorem 1, where we prove that P is a quasi-isomorphism.

In [12] the authors prove that the morphism P is surjective and give a description of its kernel. In [7] and [14, Appendix G(i)] another description of the kernel is given. In [8, Example 3.8], Gómez establishes that the morphism P is *not* a quasi-isomorphism, which contradicts our main result, Theorem 1. We were able to correct the erroneous computation of Gómez in Remark 4.

Grothendieck proved that for a smooth $\mathbb{C}$-algebra A the cohomology groups of the algebraic de Rham complex $\Omega^\bullet_A$ are isomorphic to the cohomology groups of the space $\operatorname{Spec} A$ with complex analytic topology, see [10, Theorem 1]. The algebra $A^0(X)$ is not smooth in general and the result of Grothendieck does not hold for general algebras, see [1, Example 4.4].

The result of this paper can be used in order to give another proof of the similar result for the algebra of piecewise polynomial functions on a polyhedron, which is known due to [2, Theorem 51].

Acknowledgement

I would like to thank Dr. Semёn Podkorytov for his immense help, fruitful discussions and the idea to consider Čech resolution in order to prove the result. I am grateful to the St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences for their financial assistance.

2. Simplicial complexes

Definition 1. *We call a set X of finite non-empty subsets of a finite set E a simplicial complex if for every $v \in E$ we have $\{v\} \in X$ and for every $s \in X$ and every non-empty subset $s' \subset s$ we have $s' \in X$. We denote by $V(X)$ the set E and call its elements vertices of X. The sets $s \in X$ of cardinality $m+1$ are called m-simplices. A simplicial complex Y is a subcomplex of X if for every $s \in Y$ we have $s \in X$.*

We denote by $T_p(X)$ the set of all sequences $u = (u_0, \ldots, u_p)$ of vertices of X for $p \geq -1$. We denote by $\partial_i u$ the sequence $(u_0, \ldots, \widehat{u_i}, \ldots, u_p)$. For a vertex v we denote by $v * u$ the sequence $(v, u_0, \ldots, u_p)$. The symmetric group Σ_{p+1} acts on $T_p(X)$.

Consider a sequence of vertices $u \in T_p(X)$. We denote by $\mathrm{St}u$ the *star* of u, that is the smallest subcomplex of X containing all the simplices containing the vertices u_i. If $p = -1$, we have $\mathrm{St}u = X$. If the sequence u spans a simplex in X, then $\mathrm{St}u$ is the star of this simplex. If $p \geq 0$ and the sequence u does not span a simplex, we have $\mathrm{St}u = \varnothing$. For a subcomplex Y of X we denote by $\mathrm{St}_Y u$ the smallest subcomplex of Y containing all the simplices containing the vertices u_i. If $u \notin T_p(Y)$ then $\mathrm{St}_Y u = \varnothing$.

3. Sullivan's dg-algebra of polynomial forms

For a simplicial complex X we define the dg-algebra $A^\bullet(X)$ following Sullivan, see [15, Section 7], [5], [11], [9]. For an m-simplex a consider the dg-algebra

$$A^\bullet(a) := \frac{\Lambda\left(t_v, dt_v \mid \deg(t_v) = 0,\, \deg(dt_v) = 1,\, v \in a\right)}{\left(\sum_{v\in a} t_v - 1, \sum_{v \in a} dt_v\right)}$$

with the differential $t_v \mapsto dt_v$ for $v \in a$.

For a simplex b such that $b \subset a$ one has a natural morphism of dg-algebras

$$|_b : A^\bullet(a) \to A^\bullet(b), \qquad t_v \mapsto \begin{cases} 0,\ v \notin b, \\ t_v,\ v \in b. \end{cases}$$

Then an element $\omega = (\omega_a)_{a \in X}$ of $A^\bullet(X)$ is a collection of elements $\omega_a \in A^\bullet(a)$ such that for two simplices $b \subset a$ one has $\omega_a|_b = \omega_b$.

We call the algebra $A^0(X)$ the *algebra of polynomial functions* on X. This algebra has another description as a quotient of a Stanley-Reisner algebra, see [3].

An inclusion of simplicial complexes $Y \subset X$ gives rise to the restriction morphism of dg-algebras

$$|_Y : A^\bullet(X) \to A^\bullet(Y).$$

Lemma 1. *The above restriction morphism is surjective.*

Proof. This fact is quite nontrivial, see [15, Section 7].
□

We introduce the double graded vector space $\mathcal{D}^{p,q}$, $p, q \in \mathbb{Z}$ as follows. For $p \le -2$ set $\mathcal{D}^{p,q} = 0$ and for $p \ge -1$, we define $\mathcal{D}^{p,q}$ as the subspace of

$$\prod_{u \in T_p(X)} A^q(\mathrm{St}u)$$

consisting of families of forms $\omega_u \in A^q(\mathrm{St}u)$, such that for any $\sigma \in \Sigma_{p+1}$ we have

$$\omega_{\sigma u} = (\operatorname{sgn}\sigma)\omega_u.$$

We define the linear map

$$\delta : \mathcal{D}^{p,q} \to \mathcal{D}^{p+1,q}.$$

For $p \ge -1$ and for

$$\omega = (\omega_u)_{u \in T_p(X)} \in \mathcal{D}^{p,q}$$

we set the value of $\delta\omega$ on $s \in T_{p+1}(X)$ as

$$(\delta\omega)_s = \sum_{i=0}^{p+1} (-1)^i \omega_{\partial_i s}|_{\mathrm{St}s}.$$

The differential d on $A^\bullet(\mathrm{St}u)$ gives rise to a differential d on $\mathcal{D}^{p,\bullet}$ for each p.

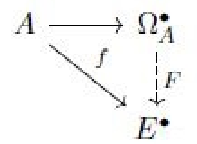
Proposition 1. *The map δ is a differential on $\mathcal{D}^{\bullet,q}$ for each $q \in \mathbb{Z}$. Moreover, the double graded vector space $\mathcal{D}^{\bullet,\bullet}$ together with δ and d forms a double complex in the sense that $d\,\delta = \delta\, d$.*

Proposition 2. *The complex*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}^{-1,q} & \xrightarrow{\delta} & \mathcal{D}^{0,q} & \xrightarrow{\delta} & \mathcal{D}^{1,q} & \xrightarrow{\delta} & \dots \\ & & \| & & & & & & \\ & & A^q(X) & & & & & & \end{array}$$

is exact.

Proof. The proof is similar to that of Proposition 4 below and relies on Lemma 1. In this case one can use the partition of unity t_v, $v \in V(X)$, instead of $\rho_v, v \in V(X)$. □



4. The dg-algebra of de Rham forms

To a k-algebra A one associates the commutative dg-algebra $\Omega^\bullet_A$ ([13, Theorem 3.2]) with $\Omega^0_A = A$. It has the following universal property: for any dg-algebra $E^\bullet$ and any algebra homomorphism $f : A \to E^0$ there exists a unique morphism of dg-algebras $F : \Omega^\bullet_A \to E^\bullet$ such that $F|_A = f$:

The elements of Ω^q_A are called algebraic q-forms. The dg-algebra $\Omega^\bullet_A$ is covariant in the algebra A. We will simply write $\Omega^\bullet(X)$ for the dg-algebra $\Omega^\bullet_{A^0(X)}$.

Inclusion of simplicial complexes $Y \subset X$ gives rise to the restriction morphism of dg-algebras

$$|_Y : \Omega^\bullet(X) \to \Omega^\bullet(Y).$$

LEMMA 2. *Suppose A and B are k-algebras and $\varphi : A \to B$ is a surjective homomorphism of algebras. Then the induced morphism $\Omega_\varphi : \Omega^\bullet_A \to \Omega^\bullet_B$ is surjective and its kernel is the ideal of $\Omega^\bullet_A$ generated by* $\operatorname{Ker}\varphi$ *and* $d(\operatorname{Ker}\varphi)$.

From this and Lemma 1 it follows that for an inclusion of simplicial complexes $Y \subset X$ the restriction morphism $|_Y : \Omega^\bullet(X) \to \Omega^\bullet(Y)$ is surjective.

PROOF. See [2, Lemma 6]. □

Let us introduce the following elements t_v of $A^0(X)$. For a vertex $v \in V(X)$ and an m-simplex $a \in X$ set $(t_v)_a = 0$ if $v \notin a$ and $(t_v)_a = t_v \in A^0(a)$ if $v \in a$.

LEMMA 3. *Take a simplicial complex X and a subcomplex $Y \subset X$. Suppose $\omega \in \Omega^q(Y)$ is such that $\omega|_{\mathrm{St}_Y(v)} = 0$ for a vertex $v \in V(X)$. Then $t_v^2|_Y\,\omega = 0$.*

PROOF. First, if $v \notin V(Y)$, then $t_v|_Y = 0$ and the claim follows.

Assume $v \in V(Y)$. By Lemma 1 and Lemma 2, the form ω lies in the dg-ideal I of $\Omega^\bullet(X)$ generated by the elements $m \in A^0(X)$ with the restriction to $\mathrm{St}_Y(v)$ being zero, therefore $t_v|_Y\, m = 0$. It is enough to consider the cases $\omega = m$ and $\omega = dm$. We have $t_v^2|_Y\, m = 0$ and $t_v^2|_Y\, dm = t_v|_Y\, d(t_v|_Y\, m) - t_v|_Y\, m dt_v|_Y = 0$. □

We introduce the double graded vector space $\mathcal{C}^{p,q}$, $p, q \in \mathbb{Z}$ as follows. For $p \leq -2$ set $\mathcal{C}^{p,q} = 0$ and for $p \geq -1$, we define $\mathcal{C}^{p,q}$ as the subspace of

$$\prod_{u \in T_p(X)} \Omega^q(\mathrm{St}u)$$

consisting of families of forms $\omega_u \in \Omega^q(\mathrm{St}u)$, such that for any $\sigma \in \Sigma_{p+1}$ we have

$$\omega_{\sigma u} = (\operatorname{sgn}\sigma)\omega_u.$$

We define the linear map

$$\delta : \mathcal{C}^{p,q} \to \mathcal{C}^{p+1,q}.$$

For $p \geq -1$ and for

$$\omega = (\omega_u)_{u \in T_p(X)} \in \mathcal{C}^{p,q}$$

we set the value of $\delta\omega$ on $s \in T_{p+1}(X)$ as

$$(\delta\omega)_s = \sum_{i=0}^{p+1} (-1)^i \omega_{\partial_i s}|_{\mathrm{St}s}.$$

The differential d on $\Omega^\bullet(\mathrm{St}u)$ gives rise to a differential d on $\mathcal{C}^{p,\bullet}$ for each p.

Proposition 3. *The map δ is a differential on $\mathcal{C}^{\bullet,q}$ for each $q \in \mathbb{Z}$. Moreover, the double graded vector space $\mathcal{C}^{\bullet,\bullet}$ together with δ and d forms a double complex in the sense that $d\,\delta = \delta\, d$.*

Lemma 4. *There exist elements $p_v \in A^0(X)$, $v \in V(X)$, such that*

$$\sum_{v\in V(X)} p_v t_v^2 = 1.$$

Proof. In $A^0(X)$ we have the equality

$$\sum_{v\in V(X)} t_v = 1.$$

Raise both the sides to a big enough power and obtain the needed equality. □

We put $\rho_v = p_v t_v^2$.

For an inclusion of simplicial complexes $Y \subset X$ we choose a linear map, the distinguished "extension",

$$[-] : \Omega^q(Y) \to \Omega^q(X),$$

such that $[\omega]|_Y = \omega$. Such an extension exists by Lemma 1 applied to $A^0(X)$ and Lemma 2.

Lemma 5. *For an inclusion of simplicial complexes $Y \subset X$ and a form $\omega \in \Omega^q(Y)$ we have*

$$\sum_{v\in V(X)} \rho_v [\omega|_{\mathrm{St}_Y(v)}]|_Y = \omega.$$

Proof. As $\sum_v \rho_v = 1$ in $A^0(X)$ we have

$$\sum_{v\in V(X)} \rho_v [\omega|_{\mathrm{St}_Y(v)}]|_Y - \omega = \sum_{v\in V(X)} \rho_v|_Y \left([\omega|_{\mathrm{St}_Y(v)}]|_Y - \omega\right).$$

We have

$$\left([\omega|_{\mathrm{St}_Y(v)}]|_Y - \omega\right)|_{\mathrm{St}_Y(v)} = \omega|_{\mathrm{St}_Y(v)} - \omega|_{\mathrm{St}_Y(v)} = 0.$$

Hence, by Lemma 3 we have $\rho_v|_Y \left([\omega|_{\mathrm{St}_Y(v)}]|_Y - \omega\right) = 0$. □

Proposition 4. *The complex*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}^{-1,q} & \xrightarrow{\delta} & \mathcal{C}^{0,q} & \xrightarrow{\delta} & \mathcal{C}^{1,q} & \xrightarrow{\delta} & \ldots \\ & & \Big| & & & & & & \\ & & \Omega^q(X) & & & & & & \end{array}$$

is exact.

The proof follows the proof of [4, Proposition 8.5].

Proof. First, notice that

$$\mathrm{St}v * u = \mathrm{St}_{\mathrm{St}u}(v).$$

For $u \in T_p(X)$ and $\omega \in \Omega^q(\mathrm{St}u)$ by Lemma 5 we have

$$\sum_{v\in V(X)} \rho_v [\omega|_{\mathrm{St}v*u}]|_{\mathrm{St}u} = \omega. \tag{1}$$

For $\omega \in \Omega^q(Z)$, where Z is a subcomplex of X such that $\mathrm{St}v * u \subset Z$, by Lemma 3, we have

$$\rho_v\left([\omega] - [\omega|_{\mathrm{St}v*u}]\right)|_{\mathrm{St}u} = 0. \tag{2}$$

We construct a cochain homotopy

$$K : \mathcal{C}^{p,q} \to \mathcal{C}^{p-1,q}.$$

For $p \geq 0$ and $\omega \in \mathcal{C}^{p,q}$ and $w \in T_{p-1}(X)$ put

$$(K\omega)_w := \sum_{v \in V(X)} \rho_v[\omega_{v*w}]|_{\mathrm{St}w}.$$

By Lemma 3 this map does not depend on the choice of the distinguished extension.

Let us check that $\delta K + K\delta = 1$. For $p \geq -1$ and

$$\omega = (\omega_u)_{u \in T_p(X)} \in \mathcal{C}^{p,q} \subset \prod_{u \in T_p(X)} \Omega^q(\mathrm{St}u),$$

where $\omega_u \in \Omega^q(\mathrm{St}u)$, we have

$$(\delta K\omega)_u = \sum_{i=0}^{p} (-1)^i (K\omega)_{\partial_i u}|_{\mathrm{St}u} = \sum_{i=0}^{p} (-1)^i \sum_{v \in V(X)} \rho_v[\omega_{v*\partial_i u}]|_{\mathrm{St}u},$$

and

$$(K\delta\omega)_u = \sum_{v \in V(X)} \rho_v[(\delta\omega)_{v*u}]|_{\mathrm{St}u} =$$

$$= \sum_{v \in V(X)} \rho_v[\sum_{i=0}^{p+1} (-1)^i \omega_{\partial_i(v*u)}|_{\mathrm{St}v*u}]|_{\mathrm{St}u} =$$

$$= \sum_{v \in V(X)} \rho_v[\omega_u|_{\mathrm{St}v*u}]|_{\mathrm{St}u} + \sum_{v \in V(X)} \rho_v[\sum_{i=1}^{p+1} (-1)^i \omega_{\partial_i(v*u)}|_{\mathrm{St}v*u}]|_{\mathrm{St}u} \overset{\text{by (1)}}{=}$$

$$= \omega_u - \sum_{v \in V(X)} \rho_v[\sum_{i=0}^{p} (-1)^i \omega_{v*\partial_i u}|_{\mathrm{St}v*u}]|_{\mathrm{St}u}$$

$$= \omega_u - \sum_{i=0}^{p} (-1)^i \sum_{v \in V(X)} \rho_v[\omega_{v*\partial_i u}|_{\mathrm{St}v*u}]|_{\mathrm{St}u}.$$

Hence,

$$((\delta K + K\delta)\omega)_u = \omega_u + \sum_{i=0}^{p} (-1)^i \sum_{v \in V(X)} \rho_v\left([\omega_{v*\partial_i u}] - [\omega_{v*\partial_i u}|_{\mathrm{St}v*u}]\right)|_{\mathrm{St}u} \overset{\text{by (2)}}{=} \omega_u.$$

□

5. The morphism $P : \Omega^\bullet(X) \to A^\bullet(X)$

For a simplicial complex X, by the universal property of $\Omega^\bullet(X)$, there is a canonical morphism dg-algebras
$$P : \Omega^\bullet(X) \to A^\bullet(X),$$
which is the identity in degree 0.

We denote by $k[0]$ the complex with the 0th term k and the others zero. An element of k gives rise to a constant function in $A^0(X)$, hence, there are morphisms of complexes $\epsilon : k[0] \to \Omega^\bullet(X)$ and $\bar{\epsilon} : k[0] \to A^\bullet(X)$ such that $\bar{\epsilon} = P \circ \epsilon$.

Proposition 5. *For a sequence $u \in T_p(X)$, $p \geq 0$, the commutative diagram*

$$\begin{array}{ccc} k[0] & \xrightarrow{\epsilon} & \Omega^\bullet(\mathrm{St}\,u) \\ & \searrow^{\bar{\epsilon}} & \downarrow P \\ & & A^\bullet(\mathrm{St}\,u) \end{array}$$

consists of quasi-isomorphisms.

Proof. The map ϵ is a quasi-isomorphism by [2, Corollary 47]. The map $\bar{\epsilon}$ is a quasi-isomorphism by [6, Theorem 10.9]. Hence, the morphism P is a quasi-isomorphism. □

Theorem 1. *The natural map*
$$P : \Omega^\bullet(X) \to A^\bullet(X)$$
is a quasi-isomorphism.

Proof. The morphism P on stars gives rise to the maps $\pi_p : \mathcal{C}^{p,\bullet} \to \mathcal{D}^{p,\bullet}$ for each $p \geq 0$. We have the following commutative diagram of non-negative complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet(X) & \xrightarrow{\delta} & \mathcal{C}^{0,\bullet} & \xrightarrow{\delta} & \mathcal{C}^{1,\bullet} & \xrightarrow{\delta} & \dots \\ & & \downarrow P & & \downarrow \pi_0 & & \downarrow \pi_1 & & \\ 0 & \longrightarrow & A^\bullet(X) & \xrightarrow{\delta} & \mathcal{D}^{0,\bullet} & \xrightarrow{\delta} & \mathcal{D}^{1,\bullet} & \xrightarrow{\delta} & \dots \end{array}$$

The vertical arrows π_p, $p \geq 0$ are quasi-isomorphisms by Proposition 5. The first row is exact by Proposition 4. The second row is exact by Proposition 2.

Hence, the map P is also a quasi-isomorphism. □

Замечание 4. *As was said in the introduction, the paper [8] suggests that Theorem* 1 is false. Namely, in [8, Example 3.8], one considers the simplicial complex X corresponding to the boundary of a triangle on the vertices 1, 2, 3. The dg-algebra $\Omega^\bullet(X)$ is generated by the elements t_1, t_2, t_3 modulo the dg-ideal generated by $t_1 + t_2 + t_3 - 1$ and $t_1 t_2 t_3$. Next, the author considers the form $t_1^2 t_2^2 dt_3$ and claims that this form is not zero. However, this form is zero, which can be seen as follows: applying the differential d to the equality $t_1 t_2 t_3 = 0$ we get
$$t_2 t_3 dt_1 + t_1 t_3 dt_2 + t_1 t_2 dt_3 = 0.$$
Next, we multiply this equality by $t_1 t_2$ and get
$$0 = t_1 t_2^2 t_3 dt_1 + t_1^2 t_2 t_3 dt_2 + t_1^2 t_2^2 dt_3 = t_1^2 t_2^2 dt_3.$$

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Arapura, Donu and Kang, Su-Jeong, Kähler–de Rham cohomology and Chern classes // Communications in Algebra 2011. Vol. 39, № 4. P. 1153-1167.
2. Baskov, I., The de Rham cohomology of soft function algebras // preprint 2022, https://arxiv.org/abs/2208.11431.
3. Billera, Louis J., The algebra of continuous piecewise polynomials // Advances in Mathematics 1989. Vol. 76, № 2. P. 170-183.
4. Bott, Raoul and Tu, Loring W., Differential forms in algebraic topology // Springer 1982.
5. Bousfield, Aldridge Knight and Gugenheim, Victor K.A.M., On PL De Rham theory and rational homotopy type // Memoirs American Mathematical Soc. 1976. Vol. 179.
6. Félix, Yves and Halperin, Stephen and Thomas, Jean-Claude, Rational homotopy theory // Springer 2012.
7. Félix, Yves and Jessup, Barry and Parent, Paul-Eugène, The combinatorial model for the Sullivan functor on simplicial sets // Journal of Pure and Applied Algebra 2009. Vol. 213, № 2. P. 231-240.
8. Gómez, Francisco, Simplicial types and polynomial algebras // Archivum Mathematicum 2002. Vol. 38, № 1. P. 27-36.
9. Griffiths, Phillip and Morgan, John, Rational Homotopy Theory and Differential Forms // Birkhäuser 1981.
10. Grothendieck, Alexander, On the de Rham cohomology of algebraic varieties // Publications Mathématiques de l'IHÉS, 1966. Vol. 29, № 1. P. 95-103.
11. Hess, Kathryn, Rational Homotopy Theory //Interactions between Homotopy Theory and Algebra: Summer School on Interactions Between Homotopy Theory and Algebra, University of Chicago, July 26-August 6, 2004, Chicago, Illinois. – 2007. – P. 175-202.
12. Kan, Daniel M. and Miller, Edward Y., Sullivan's de Rham complex is definable in terms of its 0-forms // Proceedings of the American Mathematical Society 1976. Vol. 57, № 2. P. 337–339.
13. Kunz, Ernst, Kähler differentials // 1986, Friedr. Vieweg & Sohn.
14. Sullivan, Dennis, Differential forms and the topology of manifolds // Manifolds Tokyo 1973. P. 37-49.
15. Sullivan, Dennis, Infinitesimal computations in topology // Publications Mathématiques de l'IHÉS 1977. Vol. 47, P. 269–331.

REFERENCES

1. Arapura, Donu and Kang, Su-Jeong, 2011, "Kähler–de Rham cohomology and Chern classes" // *Communications in Algebra*. Vol. 39, № 4. pp. 1153–1167.
2. Baskov, I., 2022, "The de Rham cohomology of soft function algebras" // preprint, https://arxiv.org/abs/2208.11431.

3. Billera, Louis J., 1989, “The algebra of continuous piecewise polynomials” // *Advances in Mathematics*. Vol. 76, № 2. pp. 170–183.

4. Bott, Raoul and Tu, Loring W., 1982, “Differential forms in algebraic topology” // *Springer*.

5. Bousfield, Aldridge Knight and Gugenheim, Victor K.A.M., 1976, “On PL De Rham theory and rational homotopy type” // *Memoirs American Mathematical Soc.*. Vol. 179.

6. Félix, Yves and Halperin, Stephen and Thomas, Jean-Claude, 2012, “Rational homotopy theory” // *Springer*.

7. Félix, Yves and Jessup, Barry and Parent, Paul-Eugène, 2009, “The combinatorial model for the Sullivan functor on simplicial sets” // *Journal of Pure and Applied Algebra*. Vol. 213, № 2. pp. 231–240.

8. Gómez, Francisco, 2002, “Simplicial types and polynomial algebras” // *Archivum Mathematicum*. Vol. 38, № 1. pp. 27–36.

9. Griffiths, Phillip and Morgan, John, 1981, “Rational Homotopy Theory and Differential Forms” // *Birkhäuser*.

10. Grothendieck, Alexander, 1966, “On the de Rham cohomology of algebraic varieties” // *Publications Mathématiques de l’IHÉS*. Vol. 29, № 1. pp. 95–103.

11. Hess, Kathryn, 2007, “Rational Homotopy Theory” // *Interactions between Homotopy Theory and Algebra: Summer School on Interactions Between Homotopy Theory and Algebra, University of Chicago*, July 26-August 6, 2004, Chicago, Illinois. – pp. 175–202.

12. Kan, Daniel M. and Miller, Edward Y., 1976, “Sullivan’s de Rham complex is definable in terms of its 0-forms” // *Proceedings of the American Mathematical Society*. Vol. 57, № 2. pp. 337–339.

13. Kunz, Ernst, 1986, “Kähler differentials” // Friedr. Vieweg & Sohn.

14. Sullivan, Dennis, 1973, “Differential forms and the topology of manifolds” // *Manifolds Tokyo*. pp. 37–49.

15. Sullivan, Dennis, 1977, “Infinitesimal computations in topology” // *Publications Mathématiques de l’IHÉS*. Vol. 47, pp. 269–331.

Получено: 20.01.2023
Принято в печать: 24.04.2023