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ON A NEW MEASURE ON INFINITE DIMENSIONAL UNITE CUBE

I. Sh. Jabbarov (Ganja)
jabbrovish@rambler.ru

Abstract

Measure Theory plays an important role in many questions of Mathematics. The notion of a measure being introduced as a generalization of a notion of the size of a segment made many of limiting processes be a formal procedure, and by this reason stood very productive in the questions of Harmonic analysis. Discovery of Haar measure was a valuable event for the harmonic analysis in topological groups. It stood clear that many of measures, particularly, the product of Lebesgue measure in finite dimensional cube $[0,1]^n$ could be considered as a Haar measure. The product measure has many important properties concerning projections (see [1,3]). The theorems of Fubini and Tonelly made it very useful in applications.

In this work we show that the coinsidence of considered measures, observed in finite dimensional case, impossible for infinite dimensional case, despite that such a representation was in use without proof. Considering infinite dimensional unite cube $\Omega = [0,1] \times [0,1] \times \cdots$, we define in this cube the Tichonoff metric by a special way despite that it induces the same topology. This makes possible to introduce a regular measure eliminating difficulties connected with concentration of a measure, with the progress of a dimension, around the bound. We use the metric to define a set function in the algebra of open balls defining their measure as a volume of open balls. By this way we introduce a new measure in infinite dimensional unite cube different from the Haar and product measures and discuss some differences between introduced measure and the product measure.

Main difference between the introduced measure and Haar measure consisted in non invariance of the first. The difference between the new measure and product measure connected with the property: let we are given with a infinite family of open balls every of which does not contain any other with total finite measure; then they have an empty intersection. Consequently, every point contained in by a finite number of considered balls only.

This property does not satisfied by cylindrical set. For example, let $D_1 = I_1 \times I \times I \times \cdots$, $D_2 = I_2 \times I_1 \times I \times I \times \cdots$,...

$$I = [0,1], I_k = \left[0, \frac{k}{k+1}\right], k = 1, 2, ...$$

It clear that every of these cylindrical sets does contain any other, but their intersection is not empty (contains zero). This makes two measures currently different.

Keywords: Measure theory, Lebesgue measure, Haar measure, Borel measure.

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1. Introduction

Let's enter in the unite cube

$$\Omega = \{(x_n) \,|0 \leqslant x_n \leqslant 1, n = 1, 2, ...\}$$

the Tichonoff metric as below:

$$d(x.y) = \sum_{n=1}^{\infty} e^{1-n} \,|x_n - y_n| \,; \tag{1}$$

here $x = (x_n), y = (y_n) \in \Omega$. Let's define the ball of a radius $r > 0$ and a centre $\theta \in \Omega$ by the equality

$$B(\theta, r) = \{E \in \Omega | d(x, \theta) < r\} \,.$$

It is best known that in the cube Ω a product Lebesgue measure may be introduced (see [1, p. 219]). There is also another construction of a measure called a Haar measure. The Haar measure is a measure defined in the locally compact topological groups. It was proven also uniqueness of this measure (see [2, p.241]). Many of measures used in various brunches of the mathematics could be considered as a Haar measure. Particularly, the product of Lebesgue measures in $[0, 1]^n$ is a Haar measure for any natural number n and, hence, is unique in this cube. Really, to prove this, consider the topological group R^n. The group $Z \oplus \cdots \oplus Z$ is a subgroup, so the factor group $T_n = R^n/(Z \oplus \cdots \oplus Z)$ as a compact group is locally compact. Therefore, the invariant measure in this group is unique. Let $A \subset [0, 1]^n$ is a measurable set in the product Lebesgue meaning in $[0, 1]^n$. Consider the union of intersections $(\bar{a} + A) \bigcap (\bar{m} + [0, 1)^n)$, $\bar{m} \in Z^n$ for any given vector $\bar{a} \in R^n$. Only no more than 2^n of these intersections are non-empty, and the sum of their measures is equal to the measure of the set A. Therefore, the product measure is invariant in regard to the transitions $\bar{x} \mapsto \bar{x} + \bar{a}(mod 1)$, $\bar{x} \in [0, 1)^n, \bar{a} \in R^n$.

The situation is currently different in the infinite dimensional case. In this case the number of non-empty intersections $(\bar{a} + A) \bigcap (\bar{m} + \Omega')$, $A \subset \Omega'$ (here $\Omega' = \{(\omega_n) \,|0 \leqslant \omega_n < 1\}$) is non-countable. So, we can not state that the product measure in Ω' is invariant in regard to the transitions $\bar{x} \mapsto \bar{x} + \bar{a}(mod 1)$, $\bar{x} \in \Omega', \bar{a} \in \Omega$. Therefore, in Ω the product and Haar measures are different. Our goal here is to define a *new measure* different from the both mentioned above measures.

2. Construction of a new measure

DEFINITION 1. *Let $\sigma : N \to N$ be any one to one mapping of the set of natural numbers. If for any $n > m$ there is a natural number m such that $\sigma(n) = n$, then we call σ a finite permutation. A subset $A \subset \Omega$ is called to be finite-symmetrical if for any element $\theta = (\theta_n) \in A$ and any finite permutation σ one has $\sigma\theta = (\theta_{\sigma(n)}) \in A$.*

Let Σ to denote the set of all finite permutations. We shall define on this set a product of two finite permutations as a composition of mappings. Then Σ becomes a group which contains each group of n degree permutations as a subgroup (we consider each n degree permutation σ as a finite permutation in the sense of definition 1, i. e. we put $\sigma(m) = m$ when $m > n$). The set Σ is a countable set and we can arrange its elements in a sequence.

Let $\omega \in \Omega$,$\Sigma(\omega) = \{\sigma\omega | \sigma \in \Sigma\}$ and $\Sigma'(\omega)$ to mean the closed set of all limit points of the sequence $\Sigma(\omega)$. For real t we denote $\{t\Lambda\} = (\{t\lambda_n\})$ where$\Lambda = (\lambda_n)$. Let μ to denote the product of linear Lebesgue measures m given on the interval $[0, 1]$:$\mu = m \times m \times \cdots$.

To construct the measure let's consider the open ball

$$B(0, r) = \{E \in \Omega_0 | d(x, 0) < r\}$$

in the cube $\Omega_0 = \{x = (x_n) \,|\, |x_n| \leqslant 1\}$. Since $|x_n| \leqslant 1$ then for the natural number N we have

$$\sum_{n=N+1}^{\infty} e^{1-n}|x_n| \leqslant e^{-N} \sum_{n=0}^{\infty} e^{-n} < e^{1-N}.$$

Taking arbitrarily small real number $\varepsilon > 0$ we get

$$\sum_{n=1}^{N} e^{1-n}|x_n| \leqslant d(x, 0) \leqslant \sum_{n=1}^{N} e^{1-n}|x_n| + \varepsilon$$

when $N \geqslant \log e\varepsilon^{-1}$. Therefore,

$$B_N(0, r - \varepsilon) \times [0, 1] \times \cdots \subset B(0, r) \subset B_N(0, r) \times [0, 1] \times \cdots,$$

where$B_N(0, r)$ denotes the projection of the ball $B(0, r)$ to the subspace of first N coordinates. Then, for the volume $\mu_N(r)$ of the projection$B_N(0, r)$ we have (see [7] or [8, p.319]):

$$\mu_N(r) - \mu_N(r - \varepsilon) = \int\limits_{r-\varepsilon \leqslant \sum\limits_{n=1}^{N} e^{1-n}|x_n| \leqslant r} dx_1 \cdots dx_N =$$

$$= 2N \int\limits_{r-\varepsilon \leqslant u \leqslant r} du \int\limits_{\sum\limits_{n=1}^{N} e^{1-n}u_n = u} \frac{ds}{\|\nabla\|} \leqslant \varepsilon 2^N \int_M \frac{ds}{\|\nabla\|},$$

and the last integral is an surface integral over the surface M defined by the equation

$$\sum_{n=1}^{N} e^{1-n}u_n = u, \quad 0 \leqslant u_k \leqslant 1,\ 1 \leqslant k \leqslant N; \tag{2}$$

here ∇ is a norm of the gradient of the linear function on the left side of the latest equality, i.e.

$$\|\nabla\| = \sqrt{1 + e^{-2} + \cdots + e^{2-2N}}.$$

Defining u_1 from (2) we get

$$\int_M \frac{ds}{\|\nabla\|} \leqslant \int_0^1 \cdots \int_0^1 du_2 \cdots du_N = 1.$$

So, we have

$$\mu_N(r) - \mu_N(r-\varepsilon) \leqslant \varepsilon 2^N.$$

By taking the greatest N, satisfying the condition $N \geqslant \log e\varepsilon^{-1}$, i.e. $N = [\log e\varepsilon^{-1}] + 1$, we may write $\varepsilon \leqslant e^{2-N}$. Then from the last inequality it is follows that

$$\mu_N(r) - \mu_N(r-\varepsilon) \leqslant 2^N e^{2-N} \to 0,$$

as $N \to \infty$, or as $\varepsilon \to 0$. Since $B_{N+1}(0,r) \subset B_N(0,r) \times [0,1]$ then the sequence $(\mu_N(r))$ is monotonically decreasing. So, it is bounded below by $\mu_{N_0}(r/2)$, with $N_0 = [\log 2er^{-1}] + 1$. Therefore, there exists a limit

$$\lim_{\varepsilon \to 0} B_N(0, r-\varepsilon) = \lim_{N \to \infty} B_N(0,r) = \mu_0(r)$$

which we are accepting as a measure of the ball $B(0,r)$.

The measure of the ball in Ω we define as a limit of the measures of intersections $\Omega \bigcap B_N(\theta, r) \times [0,1] \times \cdots$ as $N \to \infty$, where N defined for given ε as above. The measure of the complement of the ball $B(\theta, r)$ is defined simply as $1 - \mu(B(\theta, r))$ or as a limit of measures of complements $[0,1]^N \backslash B_N(\theta, r-\varepsilon)$, as $\varepsilon \to 0$.

Consider now a union $A_1 \bigcup \cdots \bigcup A_k$ every component A_i of which is some ball in Ω (naturally, with the different centers and radiuses) or it's complement. The measure of this union we define as above. Fixing $\varepsilon > 0$ we replace the components A_i by cylindrical sets with a tower $B_N(\theta, r)$, if it is a ball, or with a tower

$$[0,1]^N \backslash B_N(\theta, r-\varepsilon),$$

if it is a complement of the ball. We get then a union

$$\bigcup_i C_N(\theta_i, r_i),$$

with components $C_N(\theta_i, r_i)$ every of which being either $B_N(\theta, r)$ or $[0,1]^N \backslash B_N(\theta, r-\varepsilon)$. The set we have got has an N-dimensional Lebesgue measure. The ε is undertaken less than $\min_i r_i$. As well as above, the error is estimated as a value $\leqslant k\varepsilon 2^N$. The demanded mesure we get after of passing to the limit.

Let's denote by Π the algebra generated by the class of open balls in Ω. So we have constructed some finite additive set function μ_0 in Π. It is easy to note that μ_0 is a restriction of a product measure into the Π. The algebra Π can uniquely be extended to the σ-algebra Ξ of subsets in Ω (see [1-6]). Corresponding continuation of a set function defined above to the σ-algebra Ξ gives in Ω some Borel measure and we get a measure space $(\Omega,\Xi,\ \mu_0)$. So, the introduced Borel measure is a measure induced by the product measure, but main difference consisted in the definition of a metric by (1) and in that fact that σ-algebra Ξ is mainly narrow than the σ- algebra of cylindrical subsets. The algebra Π is a sub algebra of the σ- algebra of cylindrical subsets.

The inner and outer measures μ_{0*} and μ_0^* are defined by a known way (see [1-5]). Since each open ball can be enclosed in some cylindrical set with a measure enough close to the measure of the ball, we have for any subset $A\in \Omega$

$$\mu_{0*}(A) \leqslant \mu_*(A) \leqslant \mu^*(A) \leqslant \mu_0^*(A).$$

We call now a subset A to be *μ_0-measurable* in Ω if and only if the equality $\mu_{0*}(A) = \mu_0^*(A)$ is satisfied. The defined Lebesgue extension of a measure μ_0 is a regular measure and every μ_0-measurable subset is measurable in the sense of product measure.

3. Supplementary basic result

LEMMA 1. *Let $A \subset \Omega$ be a finite-symmetric subset of zero measure and $\Lambda = (\lambda_n)$ is an unbounded, monotonically increasing sequence of positive real numbers any finite subfamily of elements of which is linearly independent over the field of rational numbers. Let $B \supset A$ be any open, in the Tychonoff metric, subset with$\mu(B) < \varepsilon$,*

$$E_0 = \{0 \leqslant t \leqslant 1 | \{t\Lambda\} \in A \wedge \Sigma'\{t\Lambda\} \subset B\}.$$

Then, we have$m(E_0) \leqslant c_0\varepsilon$ where $c_0 > 0$ is an absolute constant, m designates the Lebesgue measure.

PROOF. Let ε be any small positive number. As the numbers λ_n are linearly independent, for any finite permutation σ, one has $(\{t_1\lambda_n\}) \neq (\{t_2\lambda_{\sigma(n)}\})$ when

$t_1 \neq t_2$; otherwise we should receive the equality $\{t_1\lambda_s\} = \{t_2\lambda_s\}$ for some natural s which is invariant for σ. Then one has $(t_1 - t_2)\lambda_s = k$, $k \in Z$. Further, writing out the similar equality for natural $r > s$, we get the relation

$$k_1/\lambda_r - k/\lambda_s = \frac{k_1\lambda_s - k\lambda_r}{\lambda_r\lambda_s} = 0$$

for some integral k_1 which contradicts the linear independence of the numbers λ_n. Hence, for any pair of various numbers t_1 and t_2 one has $(\{t_1\lambda_n\}) \notin \{(\{t_2\lambda_{\sigma(n)}\})|\sigma \in \Sigma\}$ (the outer brackets denote a set). By the conditions of the lemma 1, there exists a family of open balls $B_1, B_2, ...$ such that each ball does not contain any other one from this family (the ball contained in by other one can be deleted) and

$$A \subset B \subset \bigcup_{j=1}^{\infty} B_j, \sum \mu(B_j) < 1.5\varepsilon.$$

Now we take some permutation $\sigma \in \Sigma$ satisfying the equalities $\sigma(1) = n_1, \ldots$ $\ldots, \sigma(k) = n_k$ where natural numbers n_j are taken as below. Let B'_N to denote the projection of the ball B_1, with $\mu(B_1) = \varepsilon_1$, into the subspace of first N co-ordinate axes where the number N is taken so that

$$\mu(B'_N) < 2\varepsilon_1.$$

Let B'_N be enclosed by a union of cubes with edge δ and a total measure not exceeding $3\varepsilon_1$ having intersections over their boundary only. We put down $k = N$ and define numbers $n_1, ..., n_k$ by using the following constraints

$$1 < \lambda_{n_1},\ \lambda_{n_2}^{-1} < (1/4)\delta\lambda_{n_1}^{-1},\ \lambda_{n_3}^{-1} < (1/4)\delta\lambda_{n_2}^{-1}, ..., \lambda_{n_k}^{-1} < (1/4)\delta\lambda_{n_{k-1}}^{-1},\ \delta < 0.1. \quad (3)$$

Now we take any cube with the edge δ and with the centre in some point $(\alpha_m)_{1\leqslant m\leqslant k}$. Then the point $(\{t\lambda_{n_m}\})$ belongs to this cube, if

$$|\{t\lambda_{n_m}\} - \alpha_m| \leqslant \frac{\delta}{2}. \quad (4)$$

Since the interval $(\alpha_m - \delta/2, \alpha_m + \delta/2)$ for sufficiently small δ has a length < 0.1 then the real numbers $t\lambda_{n_m}$fractional parts of which lie in this interval have one and the same integral parts during continuous variation of t. So at $m = 1$, for some whole r, one has:

$$\frac{r + \alpha_1 - \delta/2}{\lambda_{n_1}} \leqslant t \leqslant \frac{r + \alpha_1 + \delta/2}{\lambda_{n_1}}. \quad (5)$$

The measure of a set of such values of t does not exceed the size $\delta\lambda_{n_1}^{-1}$. The number of such intervals corresponding to different values of $r = [t\lambda_{n_1}] \leqslant \lambda_{n_1}$ does not exceed

$$[\lambda_{n_1}] + 2 \leqslant \lambda_{n_1} + 2.$$

So, the total measure of intervals satisfying (4) at $m = 1$is less or equal to

$$(\lambda_{n_1}+2)\delta\lambda_{n_1}^{-1} \leqslant (1+2\lambda_{n_1}^{-1})\delta.$$

Consider now the case $m=2$. Taking one of intervals of a view (4), we will have

$$\frac{s+\alpha_2-\delta/2}{\lambda_{n_2}} \leqslant t \leqslant \frac{s+\alpha_2+\delta/2}{\lambda_{n_2}}, \tag{6}$$

with some $s=[t\lambda_{n_2}] \leqslant \lambda_{n_2}$. As we consider the conditions (4) for the values $m=1$ and $m=2$ simultaneously, we should estimate a total measure of intervals (6) which have nonempty intersections with intervals of a kind (5), using conditions (3). Every interval of a kind (6) is placed only in the one interval with the length $\lambda_{n_2}^{-1}$ where the expression $t\lambda_{n_2}$ has one and the same integral part s. The number of intervals with the length $\lambda_{n_2}^{-1}$ having a nonempty intersection with one fixed interval of a kind (5), does not exceed the size

$$[\delta\lambda_{n_1}^{-1}\lambda_{n_2}]+2 \leqslant \delta\lambda_{n_1}^{-1}\lambda_{n_2}+2.$$

So, the measure of a set of values of t for which intervals (6) have nonempty intersections only with one of intervals of a kind (5) is bounded by the value $(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\delta\lambda_{n_2}^{-1}$. Since, the number of intervals (5) is no more than $\lambda_{n_1}+2$ then the measure of a set of values t for which the condition (4) are satisfied simultaneously for $m=1$ and $m=2$ will be less or equal than

$$(\lambda_{n_1}+2)(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\delta\lambda_{n_2}^{-1}.$$

It is possible to continue these reasoning considering all of conditions of a kind

$$\frac{l+\alpha-\delta/2}{\lambda_{n_m}} \leqslant t \leqslant \frac{l+\alpha+\delta/2}{\lambda_{n_m}}, m=1,...,k.$$

Then we find the following estimation for the measure $m(\delta)$ of a set of those t for which the points $(\{t\lambda_{n_m}\})$ located in the given cube with the edge δ:

$$m(\delta) \leqslant (2+\lambda_{n_1})(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\cdots(2+\delta\lambda_{n_{k-1}}^{-1}\lambda_{n_k})\delta\lambda_{n_k}^{-1} \leqslant \delta^k\prod_{m=1}^{\infty}(1+2m^{-2}).$$

Summarising over all such cubes, we receive the final estimation of a kind $\leqslant 3c\varepsilon_1$ for the measure of a set of those t for which $(\{t\lambda_{n_m}\}) \in B_1$ with the absolute constant

$$c=\prod_{m=1}^{\infty}(1+2m^{-2}).$$

We notice that the sequence $\Lambda=(\lambda_n)$, satisfying the conditions (3) defined above, depends on δ. For each ball B_k we fix some sequenceΛ_k using conditions (3). Considering all such balls, we designate $\Delta_0=\{\Lambda_k|k=1,2,...\}$.

Let's prove that for any point $t \in E_0$ the set $\Sigma(\{t\Lambda\})$ is contained in the finite union $\bigcup_{k\leqslant n} B_k$ for some n. Really, let at some $t \in E_0$ all members of the sequence $\Sigma(\{t\Lambda\})$ does not contained in the union $\bigcup_{k\leqslant n} B_k$, for any natural n. Two cases are possible: 1) there will be a point $\bar{\theta} \in \Sigma(\{t\Lambda\})$ belonging to infinite number of spheresB_k; 2) there will be a subsequence of elements $\bar{\theta}_j, \bar{\theta}_j \in \Sigma(\{t\Lambda\})$ which does not contained in any finite union of ballsB_k. We shall consider both possibilities separately and shall prove that they lead to the contradiction.

1) Let B_{k_1},B_{k_2}, B_{k_3}, ... be sequence of all balls into which the element $\bar{\theta}$ belongs. We shall denote d the distance from $\bar{\theta}$ to the bound ofB_{k_1}. As $B_{k_1}$is open set, then $d > 0$. Let B_k be any ball of a radius $< d/2$ from the list above, containing the point$\bar{\theta}$. From the told it follows that the ball B_k should contained in the ballB_{k_1}. But it contradicts the agreement accepted above.

2) Let $\bar{\theta}$ be some limit point of the sequence $(\bar{\theta}_j)$. According to the condition of the lemma 3 $\bar{\theta} \in B_s$ for some$s$. Let d denotes the distance from $\bar{\theta}$ to the bound ofB_s. As $\bar{\theta}$ is a limit point, then a ball with the centre in the point $\bar{\theta}$ and radius $d/4$ contains an infinite set of members of the sequence $(\bar{\theta}_j)$, say members $\bar{\theta}_{j_1}, \bar{\theta}_{j_2}, ...$ According to 1), each point of this sequence can belong only to finite number of balls. So, the specified sequence will be contained in a union of infinite subfamily of balls B_k. Among them will be found infinitely many number of balls having radius $< d/4$. All of them, then, should contained in the ball B_s. The received contradiction excludes the case 2) also.

So, for any$t \in E_0$ it will be found such n for which $\Sigma(\{t\Lambda\}) \subset \bigcup_{k\leqslant n} B_k$. From here it follows that the set E_0 can be represented as a union of subsets $E_k, k = 1, 2, ...,$ where

$$E_k = \{t \in E_0 | \Sigma(t\Lambda) \subset \bigcup_{s\leqslant k} B_s\}.$$

So,

$$E_0 = \bigcup_{k=1}^{\infty} E_k; \;\; E_k \subset E_{k+1} (k \geqslant 1).$$

Further, $m(E_0) = \lim\limits_{k\to\infty} m(E_k)$, in agree with [4, p. 368]. As the set E_k is a finite symmetrical, then the measure of a set of values t, interesting us, is possible to estimate by using of any sequence Λ_k, since, as it has been shown above, the sets $\Sigma(\{t\Lambda\})$ for different values of t have empty intersection. So,

$$m(E_k) \leqslant \limsup_{\Lambda' \in \Delta_0} m(E_k(\Lambda')),$$

where $E_k(\Lambda') = \{t \in E_k | (\{t\Lambda'\}) \in \bigcup_{s\leqslant k} B_s\}$. Hence,

$$m(E_k(\Lambda')) \leqslant \sum_{s\leqslant k} m(E^{(s)}(\Lambda')),$$

where$E^{(s)}(\Lambda') = \{t \in E_0|(\{t\Lambda'\}) \in B_s\}$. Applying the inequality found above, we receive:

$$m(E_k(\Lambda')) \leqslant 6c(\varepsilon_1 + \cdots + \varepsilon_k).$$

This result invariable for all $\Lambda' = \Lambda_r$ beginnig from some natural $r = r(k)$. Taking limsup, as $k \to \infty$, we receive the demanded result. The proof of the lemma 1 is finished. □

4. Main theorem

The projection of the curve $(\{t\lambda_n\})_{n\geqslant 1}$in two dimensional plane, i. e. the curve $(\{t\lambda_1\}, \{t\lambda_2\})$ has zero measure. By the theorem of Fubini, then, the product Lebesgue measure of the curve $(\{t\lambda_n\})_{n\geqslant 1}$ is also equal to zero. The following theorem shows that the introduced measure is different from the product measure.

THEOREM 1. *Let the sequence (λ_n) be an unbounded sequence of positive real numbers every finite subfamily of elements of which is linearly independent over the field of rational numbers. Then the curve $(\{t\lambda_n\}), t \in [0,1]$ is not μ_0-measurable set in Ω.*

PROOF. Let's suppose the converse statement. Let the curve $(\{t\lambda_n\}), t \in [0,1]$ be measurable. Then it has a zero measure. Therefore, the union $U = \bigcup_{0\leqslant t\leqslant 1} \Sigma(\{t\Lambda\})$ as a set constructed from the curve $(\{t\lambda_n\}), t \in [0,1]$ by an action of the group Σ of all finite permutations has zero measure also, since it is a countable union of sets of zero measure. The set U is a finite-symmetric. Let n be any natural number. If we take a projection of the set U to Ω by omitting the first n coordinates (restricting the sequence $(\{t\lambda_n\})$), we get again the set U_n of zero measure. Really, the set U can be overlapped by the union of balls with the total measure of not exceeding $\varepsilon > 0$. Restricting the ball $B(\theta_0, \lambda)$ by omitting the first n coordinates, and denoting the projection by

S_N, we get

$$S_N = \left\{ (\theta_n) \,|\, \sum_{n=N+1}^{\infty} \left|\theta_n - \theta_n^0\right| e^{1-n} < \lambda \right\}.$$

Since

$$\sum_{n=N+1}^{\infty} \left|\theta_n - \theta_n^0\right| e^{1-N} = e^{-N} \sum_{n=1}^{\infty} \left|\theta_n - \theta_n^0\right| e^{1-n},$$

then denoting the projection of the point θ_0 by θ_0', we have $S_N = B(\theta_0', e^N\lambda)$, and $e^N\lambda \to 0$ as $\lambda \to 0$ for any fixed N. From this one deduces the demanded statement.

Consider the sequence of sets $V_n = [0,1]^n \times U_n$ for all natural n. It is obviously that $V_n \subset V_{n+1}$. Let $V = \bigcup_{n=1}^{\infty} V_n$. We have $\mu(V_n) = 0$ for all values of n. Therefore,

also $\mu(V) = 0$, and the set V is finite symmetrical. Then, there will be found some countable family of balls B_r with a total measure not exceeding ε the union of which contains the set V. For every fixed natural n we define the set $\Sigma'_n(t\Lambda)$ as a closed set of all limit points of the sequence $\Sigma_n(\bar{\omega}) = \{\sigma\bar{\omega}|\sigma \in \Sigma \wedge\ \sigma(1) = 1 \wedge \cdots \wedge \sigma(n) = n\}$. Let

$$B^{(n)} = \{t|\{t\Lambda\} \in V \wedge \sum_n{}'(\{t\Lambda\}) \subset \bigcup_{r=1}^{\infty} B_r\}, n = 1, 2, ...$$

For every t the sequence $\sum_{n+1}(\{t\Lambda\})$ is a subsequence of the sequence $\sum_n(\{t\Lambda\})$. Therefore, $\sum'_{n+1}(\{t\Lambda\}) \subset \sum'_n(\{t\Lambda\})$ and we have $B^{(n)} \subset B^{(n+1)}$. Then, one gets the inequality $m(B) \leqslant \sup\limits_n m(B^{(n)})$ denoting $B = \bigcup_n B^{(n)}$.

Let's estimate $m(B^{(n)})$. The set $\sum'_n(\{t\Lambda\})$ is a closed set. Clearly, if we will "truncate"the sequences $\{t\Lambda\}$, remaining only components $\{t\lambda_k\}$ with indexes greater than n, and will denote the truncated sequence as $\{t\Lambda\}' \in \Omega$, then the set $\sum'(\{t\Lambda\}')$ also will be closed. Now we consider the products $[0,1]^n \times \{\{t\Lambda\}'\}$ (external brackets designate the set of one element) for every t. We have

$$\{t\Lambda\} \in [0,1]^n \times \{\{t\Lambda\}'\} \subset V.$$

Let $(\theta_1, ..., \theta_n) \in [0,1]^n$ is any point. There exist a neighborhood $V' \subset [0,1]^n$ of this point such that $(\theta_1, ..., \theta_n, \{t\Lambda\}') \in V' \times W \subset \bigcup_r B_r$, for some neighborhood W of the point $\{t\Lambda\}'$. We, therefore, supplied every point $(\theta_1, ..., \theta_n) \in [0,1]^n$ with some pare of open sets (V', W). Since the set $[0,1]^n$ is closed, then they can be found a finite number of open sets V' the union of which contains $[0,1]^n$. The intersection of corresponding open sets W, being an open set, contains the point $\{t\Lambda\}'$. Therefore, for some finite set of indexes R we have

$$[0,1]^n \times \{\{t\Lambda\}'\} \subset \bigcup V \times \bigcap W = [0,1]^n \times \bigcap W \subset \bigcup_{r\in R} B_r, \tag{7}$$

for each considered point$t$. It is clear that the set R depends on the point t and $\bigcap W \subset \bigcap_{r\in R} B'_r$ when B'_r denotes the open set of truncated elements of B_r. The similar to (7) relationship is fair in the case when the point $\{t\Lambda\}$ would be replaced by any limit point $\bar{\omega}$ of the sequence $\Sigma(\{t\Lambda\})$ also, because $\bar{\omega} \in B_r$. If one denotes by B' the union of all open sets of a kind $\bigcap_{r\in R} B'_r$ corresponding to every possible values of t and of a limit point $\bar{\omega}$, we shall receive the relation

$$\{t\Lambda\} \in [0,1]^n \times \{\{t\Lambda\}'\} \subset A \subset [0,1]^n \times B' \subset \bigcup_{r=1}^{\infty} B_r,$$

for each considered values of t and

$$\{\bar{\omega}\} \in [0,1]^n \times \{\bar{\omega}\}' \subset A \subset [0,1]^n \times B' \subset \bigcup_{r=1}^{\infty} B_r,$$

for each limit point$\bar{\omega}$. From this it follows the inequality

$$\mu_0^* \left([0,1]^n \times B'\right) = \mu_0^*(B') \leqslant \varepsilon,$$

where μ^* means an outer measure. The set B' is open and $\Sigma'(\{t\Lambda\}') \in B'$. Now we can apply the lemma 1 and receive an estimation$m(B^{(n)}) \leqslant 6c\varepsilon$. Thus, we have$m(B) \leqslant 6c\varepsilon$. Since ε could be chosen arbitrarily small then there exist t such that $t \notin B$. So, $t \notin B^{(k)}$ for every $k = 1, 2, ...$ Consequently, for every k, there is a limit point $\bar{\omega}_k \in \Omega \backslash \bigcup_r B_r$ of the sequence $\sum_n(\{t\Lambda\})$. As the set $\Omega \backslash \bigcup_r B_r$ is closed, the limit point $\overline{\omega} = (\{t\Lambda\})$ of the sequence $(\bar{\omega}_k)$ will belong to the set $\Omega \backslash \bigcup_r B_r$. Therefore, $\{t\Lambda\} \notin \bigcup_{r \geqslant 1} B_r$ which contradicts the conditions of the theorem. Then the curve $(\{t\lambda_n\}), t \in [0,1]$ could not be μ_0-measurable. The proof of the theorem is finished. □

5. Conclusion

The lemma 1 deliveries the first fundamental difference between introduced and known measures. The main tool in the proof of this lemma is that fact that if we have some covering of the set $\Sigma\bar{\omega} \bigcup \Sigma'\bar{\omega}$, $\bar{\omega} \in \Omega$ by a union of a family of balls with a finite total measure and none of which containing other then there is *a finite number of balls only* containing the set $\Sigma\bar{\omega}$. This is somewhat different property than the compactness, and the same property is not satisfied by cylindrical sets.

Another difference stands clear after the theorem proved above. But in applications it is very important that every measurable set in a new meaning is measurable in the meaning of product measure. In the measure space with the product measure Suslin sets are wider than σ- algebra. According to sed above, in our case these two sets are coinsident. At last we can state that there exists a cylindrical set being nonmeasurable in the new meaning. It means that a given cylindrical set could represented as a union of noncounbable familiy of open balls only.

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Ganja State University
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