ЧЕБЫШЕВСКИЙ СБОРНИК
Том 23. Выпуск 5.

УДК 514.74 DOI 10.22405/2226-8383-2022-23-5-101-116

Приведение гладких функций к нормальным формам вблизи критических точек[1]

А. С. Оревкова

Оревкова Александра Степановна — Московский государственный университет имени М. В. Ломоносова; Московский Центр фундаментальной и прикладной математики (г. Москва).
e-mail: s15b3_orevkova@179.ru

Аннотация

Работа посвящена «равномерному» приведению гладких функций на двумерных многообразиях к каноническому виду вблизи критических точек этих функций. Функция $f(x, y)$ имеет особенность типа A_k, E_6 или E_8 в своей критической точке, если в некоторых локальных координатах с центром в этой точке ряд Тейлора функции имеет вид $x^2+y^{k+1}+R_{2,k+1}$, $x^3+y^4+R_{3,4}$, $x^3+y^5+R_{3,5}$ соответственно, где через $R_{m,n}$ обозначена сумма мономов более высокого порядка, т.е. $R_{m,n} = \sum a_{ij}x^iy^j$, где $\frac{i}{m} + \frac{j}{n} > 1$. Согласно результату В. И. Арнольда (1972), эти особенности просты и гладкой заменой переменных приводятся к каноническому виду, в котором член $R_{m,n}$ равен нулю.

Для особенностей типов A_k, E_6 и E_8 мы явно строим такую замену и оцениваем снизу (через C^r-норму функции, где $r = k + 3, 7$ и 8 соответственно) максимальный радиус окрестности, в которой определена замена. Наша замена является «равномерным» приведением к каноническому виду в том смысле, что построенные нами окрестность и замена координат в ней (а также все частные производные замены координат) непрерывно зависят от функции f и ее частных производных.

Ключевые слова: правая эквивалентность гладких функций, ADE-особенности, нормальные формы особенностей, равномерное приведение к нормальным формам.

Библиография: 14 названий.

Для цитирования:
А. С. Оревкова. Приведение гладких функций к нормальным формам вблизи критических точек // Чебышевский сборник, 2022, т. 23, вып. 5, с. 101–116.

[1]Автор является стипендиатом фонда развития теоретической физики и математики “БАЗИС”, договор № 21-8-9-9-1.

CHEBYSHEVSKII SBORNIK
Vol. 23. No. 5.

UDC 514.74 DOI 10.22405/2226-8383-2022-23-5-101-116

Reducing smooth functions to normal forms near critical points[2]

A. S. Orevkova

Orevkova Alexandra Stepanovna — Lomonosov Moscow State University; Moscow Center of Fundamental and Applied Mathematics (Moscow).
e-mail: s15b3_orevkova@179.ru

Abstract

The paper is devoted to "uniform" reduction of smooth functions on 2-manifolds to canonical form near critical points of the functions by some coordinate changes in some neighborhoods of these points. A function $f(x, y)$ has a singularity of the type A_k, E_6, or E_8 at its critical point if, in some local coordinate system centered at this point, the Taylor series of the function has the form $x^2 + y^{k+1} + R_{2,k+1}$, $x^3 + y^4 + R_{3,4}$, $x^3 + y^5 + R_{3,5}$ respectively, where $R_{m,n}$ stands for a sum of higher order terms, i.e., $R_{m,n} = \sum a_{ij}x^iy^j$ where $\frac{i}{m} + \frac{j}{n} > 1$. In according to a result by V. I. Arnold (1972), these singularities are simple and can be reduced to the canonical form with $R_{m,n} = 0$ by a smooth coordinate change.

For the singularity types A_k, E_6, and E_8, we explicitly construct such a coordinate change and estimate from below (in terms of C^r-norm of the function, where $r = k + 3$, 7, and 8 respectively) the maximal radius of a neighborhood in which the coordinate change is defined. Our coordinate change provides a "uniform" reduction to the canonical form in the sense that the radius of the neighborhood and the coordinate change we constructed in it (as well as all partial derivatives of the coordinate change) continuously depend on the function f and its partial derivatives.

Keywords: right equivalence of smooth functions, ADE-singularities, normal form of singularities, uniform reducing to normal form.

Bibliography: 14 titles.

For citation:
A. S. Orevkova, 2022, "Reducing smooth functions to normal forms near critical points", *Chebyshevskii sbornik*, vol. 23, no. 5, pp. 101–116.

1. Introduction

Definition 1. *A smooth function $f = f(u, v)$ has a singularity type E_k $(k = 6, 7, 8)$ at its critical point $P \in \mathbb{R}^2$ if*

(i) *the first and second differentials $df(P)$ and $d^2f(P)$ vanish, and the third differential $d^3f(P)$ is a perfect cube (non-zero);*

(ii) *one the following coefficients of the Taylor series of f at P does not vanish (the upper black points in Fig. 1): $f^{(4)}_{y^4}(P)$, $f^{(4)}_{xy^3}(P)$, and $f^{(5)}_{y^5}(P)$, where $(u, v) \to (x, y)$ is a linear coordinate change such that $d^3f(P) = 6(dx)^3$. More specifically: the singularity type is E_6 if $f^{(4)}_{y^4}(P) \neq 0$ (equivalently, there exists a tangent vector $v \in \operatorname{Ker} d^3f(P)$ at P such that $v^4f \neq 0$, where v^4f*

[2]The author is a Fellow of the Theoretical Physics and Mathematics Advancement Foundation "BASIS".

denotes the fourth derivative of f along the vector v); the singularity type is E_7 if $f^{(4)}_{y^4}(P) = 0$ and $f^{(4)}_{xy^3}(P) \neq 0$; the singularity type is E_8 if $f^{(4)}_{y^4}(P) = f^{(4)}_{xy^3}(P) = 0$ and $f^{(5)}_{y^5}(P) \neq 0$. The vanishing coefficients of the Taylor series correspond to the points beneath the oblique line (except the origin) in Fig. 1.

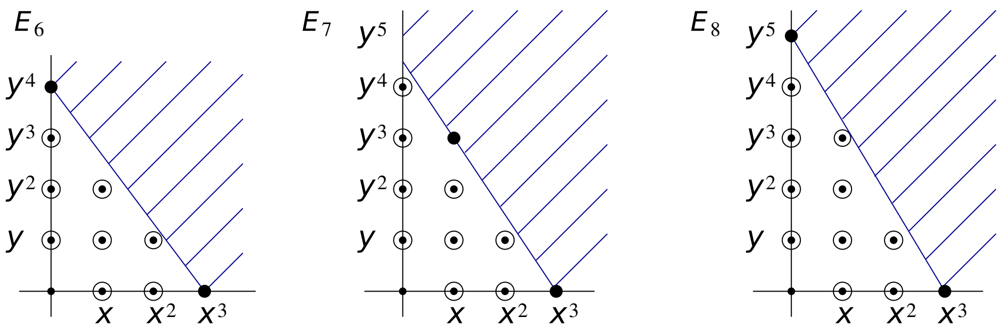

Fig 1: Taylor coefficients of a function with a singularity E_6, E_7, and E_8: the triangle of zeros

From the definition of E_k we have $f^{(a+b)}_{x^a y^b}(P) = 0$ for $0 < \frac{a}{3} + \frac{2b}{k+2} < 1$, $f'''_{x^3}(P) = 6$ $(k = 6, 7, 8)$ and $f^{(1+k/2)}_{y^{1+k/2}}(P) \neq 0$ $(k = 6, 8)$, see Fig. 1. We will assume that $P = \mathbf{0} = (0, 0)$ in the coordinates x, y.

Assumption 1. *For singularities E_k $(k = 6, 8)$, assume that $f^{(1+k/2)}_{y^{1+k/2}}(\mathbf{0}) = \pm(1 + \frac{k}{2})!$.*

Theorem 1 (Reducing E_k to normal form [1]). *Let a function $f(u, v)$ have a singularity E_k $(k = 6, 7, 8)$ at a critical point P. Then, in some neighborhood of P, there is a local coordinate system $\tilde{x}, \tilde{y}$ in which the point P is the origin, and the function has the normal form $f = f(P) + \tilde{x}^3 \pm \tilde{y}^{1+k/2}$ for $k = 6, 8$, $f = f(P) + \tilde{x}^3 + \tilde{x}\tilde{y}^3$ for $k = 7$.*

In [1], the existence of a coordinate change was proved using the Tougeron theorem [14]. In view of this, obtaining a formula for the corresponding coordinate change requires solving the Cauchy problem for a system of ODE's. We construct our coordinate change explicitly, without using the Tougeron theorem.

Lemma 1 ([1]). *In the case of a singularity E_k $(k = 6, 8)$, under the hypotheses of Assumption 1, there exist coordinate changes $(x, y) \to (x_1 = x + d_0 y^2, y) \to (x_1, y_1 = y + d_1 x_1) \to (x_2 = x_1 + d_2 y_1^{k/2-1}, y_1) \to (x_2, y_2 = y_1 + d_3 x_2^2)$ where $d_j \in \mathbb{R}$ and $d_0 = 0$ for $k = 6$, such that $f^{(a+b)}_{x_2^a y_2^b}(\mathbf{0}) = 0$ for all $a, b \in \mathbb{Z}_+$ with $a + b > 0$, $a < 3$, and $b < 1 + \frac{k}{2}$.*

A proof of Lemma 1 is schematically shown in Fig. 2.

Theorem 2 (Estimating the radius of a neighborhood for the coordinate change). *In the case of singularities E_k $(k = 6, 8)$, under the hypotheses of Assumption 1, let $(x, y) \to (x_2, y_2)$ be the coordinate change from Lemma 1. Suppose that, in a neighborhood $U_0 = \{(x_2, y_2) \mid \max(|x_2|, |y_2|) < R_0\}$ of $\mathbf{0}$, the following estimates hold: $C_{\alpha\beta} := \sup_{U_0} \left| f^{(\alpha+\beta)}_{x_2^\alpha y_2^\beta}(x_2, y_2) \right| \leqslant M$ for*

$$(\alpha, \beta) \in \{(0,5), (1,4), (3,1), (3,2), (3,3), (4,0), (4,1), (4,2), (4,3)\} \quad \textit{if } k = 6,$$

$$(\alpha, \beta) \in \{(0,6), (1,5), (3,1), (3,2), (3,3), (3,4), (4,0), (4,1), (4,2), (4,3), (4,4)\} \quad \textit{if } k = 8,$$

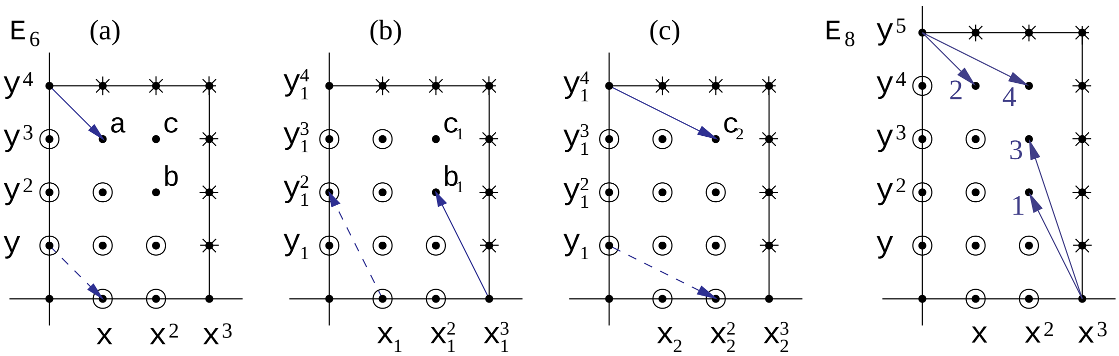

Fig 2: Coordinate changes killing the coefficients inside the rectangle: E_6: (a) $y = y_1 - d_1 x$, $d_1 = \pm\frac{a}{4}$; (b) $x = x_2 - d_2 y_1^2$, $d_2 = \frac{b}{3} \mp \frac{a^2}{8}$; (c) $y_1 = y_2 - d_3 x_2^2$, $d_3 = \pm\frac{c_2}{4}$; E_8

where $R_0 > 0$, $M \geqslant 0$. Then, in the neighborhood $U = \{(x_2, y_2) \mid \max(|x_2|, |y_2|) < R\}$, with $R = \min\{R_0, \frac{2}{M+2}\}$, there is a coordinate change of the form

$$\phi : (x_2, y_2) \to (\tilde{x} = x_2 \sqrt[3]{h(x_2, y_2)}, \tilde{y} = y_2 g(x_2, y_2)^{\frac{2}{2+k}})$$

that reduces f to the normal form $f = f(P) + \tilde{x}^3 \pm \tilde{y}^{1+k/2}$ of E_k. In more detail:

(a) *the functions $h(\boldsymbol{x})$ and $g(\boldsymbol{x})$ are positive in U, thus the change $\phi|_U$ is well-defined and is C^∞-smooth;*

(b) $\|\phi'(\boldsymbol{x}) - I\| < C < 1$ *for all $\boldsymbol{x} \in U$, where $C = \frac{2}{5}$, thus $\phi|_U$ is C^1-close to the identity;*

(c) *the coordinate change $\phi|_U$ is injective and regular, i.e., it is an embedding and $\det \phi'(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in U$, moreover $\phi(U)$ contains the open disk of radius $(1 - C)R$ centered at $\mathbf{0}$.*

Further in Section 4, Theorem 3, we prove a similar result for the singularity type A_k. Our coordinate change $\phi|_U$ from Theorems 2 and 3 provides a “uniform” reduction of the function f at a singular point of type E_k, $k = 6, 8$, and A_k, $k \geqslant 1$, to the canonical form $f = f(P) + \tilde{x}^3 \pm \tilde{y}^{1+k/2}$ and $f = f(P) \pm \tilde{x}^2 \pm \tilde{y}^{k+1}$ in the sense that the neighborhood radius and the coordinate change we constructed in this neighborhood (as well as all partial derivatives of the coordinate change) continuously depend on the function f and its partial derivatives. A uniform reduction of smooth functions near critical points to a canonical form was known earlier for several singularity types [2, 11, 12], [13, Sec. 8].

The uniform Morse lemma [4] was applied in [5]–[8] for studying topology of the spaces of Morse functions on surfaces and decomposition of these spaces into classes of topological equivalence. Our results have similar applications for studying topology of the spaces of smooth functions [9] and gradient-like flows [10] with prescribed local singularities of ADE-types.

The author is grateful to Elena Kudryavtseva for suggesting the problem and useful discussions, advice and help in writing of this paper.

2. Key lemmas

Lemma 2. *Let $f^{(a+b)}_{x^a y^b}(\mathbf{0}) = 0$ for all $a, b \in \mathbb{Z}_+$ with $a < m$ and $b < n$, where $m, n \in \mathbb{N}$. Then the function $f|_U$ has the form $f(x, y) = x^m h(x, y) + y^n g(x, y)$ where U is a neighborhood of $\mathbf{0}$ in $\mathbb{R}^2$ and*

$$g(x,y) = \frac{1}{(n-1)!}\int_0^1 f^{(n)}_{y^n}(x,sy)(1-s)^{n-1}\,ds, \tag{1}$$

$$h(x,y) = \frac{1}{x^m}\sum_{i=0}^{n-1}\frac{y^i}{i!}f^{(i)}_{y^i}(x,0) = \frac{1}{(m-1)!}\sum_{i=0}^{n-1}\frac{y^i}{i!}\int_0^1 f^{(m+i)}_{x^m y^i}(sx,0)(1-s)^{m-1}\,ds. \tag{2}$$

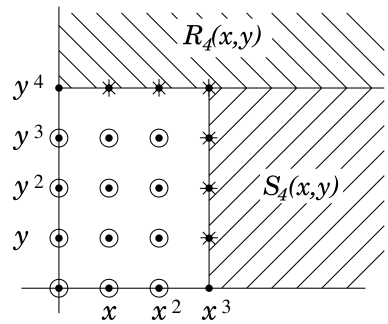

Fig 3: Constructing functions $S_n(x,y) = x^m h(x,y)$ and $R_n(x,y) = y^n g(x,y)$ (for $m = 3$ and $n = 4$)

Proof. We represent f as a function of y with a parameter x. We write down the Taylor formula at the point $y = 0$ with the remainder term in the integral form:

$$f(x,y) = S_n(x,y) + R_n(x,y) = x^m h(x,y) + y^n g(x,y),$$

$$S_n(x,y) = f(x,0) + f'_y(x,0)y + f''_{y^2}(x,0)\frac{y^2}{2!} + \cdots + f^{(n-1)}_{y^{n-1}}(x,0)\frac{y^{n-1}}{(n-1)!} = x^m h(x,y),$$

$$R_n(x,y) = \frac{1}{(n-1)!}\int_0^y f^{(n)}_{y^n}(x,t)(y-t)^{n-1}\,dt = \frac{y^n}{(n-1)!}\int_0^1 f^{(n)}_{y^n}(x,sy)(1-s)^{n-1}\,ds = y^n g(x,y).$$

The functions $S_4(x,y) = x^3 h(x,y)$ and $R_4(x,y) = y^4 g(x,y)$ appearing in the case of a singularity type E_6 are schematically shown in Fig. 3. The lemma is proved. □

Lemma 3. *Let $\phi : U \to \mathbb{R}^n$ be a smooth mapping, where U is a convex open subset of $\mathbb{R}^n$. Let the differential of ϕ have the form $\phi'(\boldsymbol{x}) = I + A(\boldsymbol{x})$, where I is the unit matrix of dimension n, $\|A(\boldsymbol{x})\| < c$, $0 < c < 1$. Then ϕ is injective and $\det\phi'(\boldsymbol{x}) \neq 0$ at every point $\boldsymbol{x} \in U$, i.e., ϕ is a diffeomorphism to its image $\phi(U)$. Moreover, $\langle\phi(\boldsymbol{x}) - \phi(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y}\rangle \geqslant (1-c)\|\boldsymbol{x} - \boldsymbol{y}\|^2$ for any pair of points $\boldsymbol{x}, \boldsymbol{y} \in U$.*

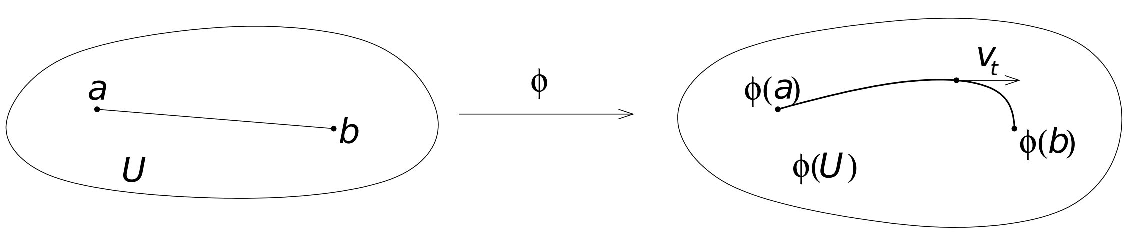

Fig 4: The mapping ϕ on the segment $[\boldsymbol{a}, \boldsymbol{b}]$

Proof. Take any two points $\boldsymbol{a}, \boldsymbol{b} \in U$, $\boldsymbol{a} \neq \boldsymbol{b}$, and consider the mapping ϕ on the segment $[\boldsymbol{a}, \boldsymbol{b}]$. Consider the velocity vector

$$v_t = \tfrac{d}{dt}\big(\phi(\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}))\big)$$

along the ϕ-image of this segment, see Fig. 4. Let us look at the projection of the velocity vector v_t onto $\boldsymbol{b}-\boldsymbol{a}$:

$$\begin{aligned}\langle v_t, \boldsymbol{b}-\boldsymbol{a}\rangle &= \bigl\langle \tfrac{d}{dt}\bigl(\phi(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}))\bigr), \boldsymbol{b}-\boldsymbol{a}\bigr\rangle = \\ &= \langle \phi'(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}))(\boldsymbol{b}-\boldsymbol{a}), \boldsymbol{b}-\boldsymbol{a}\rangle = \langle (I + A(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a})))(\boldsymbol{b}-\boldsymbol{a}), \boldsymbol{b}-\boldsymbol{a}\rangle = \\ &= \langle \boldsymbol{b}-\boldsymbol{a}, \boldsymbol{b}-\boldsymbol{a}\rangle + \langle A(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}))(\boldsymbol{b}-\boldsymbol{a}), \boldsymbol{b}-\boldsymbol{a}\rangle.\end{aligned}$$

Let us find an upper bound for the absolute value of the second term. Note that by the Cauchy-Schwarz inequality and by the definition of the matrix norm:

$$\bigl|\langle A(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}))(\boldsymbol{b}-\boldsymbol{a}), \boldsymbol{b}-\boldsymbol{a}\rangle\bigr| \leqslant \|A(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}))(\boldsymbol{b}-\boldsymbol{a})\| \cdot \|\boldsymbol{b}-\boldsymbol{a}\| \leqslant \|A(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}))\|\,\|\boldsymbol{b}-\boldsymbol{a}\|^2.$$

Let us go back to the estimation of $\langle v_t, \boldsymbol{b}-\boldsymbol{a}\rangle$:

$$\begin{aligned}\langle v_t, \boldsymbol{b}-\boldsymbol{a}\rangle &= \langle \boldsymbol{b}-\boldsymbol{a}, \boldsymbol{b}-\boldsymbol{a}\rangle + \langle A(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}))(\boldsymbol{b}-\boldsymbol{a}), \boldsymbol{b}-\boldsymbol{a}\rangle \geqslant \\ &\geqslant \|\boldsymbol{b}-\boldsymbol{a}\|^2 - |\langle A(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}))(\boldsymbol{b}-\boldsymbol{a}), \boldsymbol{b}-\boldsymbol{a}\rangle| \geqslant \\ &\geqslant \|\boldsymbol{b}-\boldsymbol{a}\|^2 - \|A(\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}))\|\,\|\boldsymbol{b}-\boldsymbol{a}\|^2 > \|\boldsymbol{b}-\boldsymbol{a}\|^2(1-c) > 0,\end{aligned}$$

from which $v_t \neq 0$, thus $\operatorname{Ker}\phi'(\boldsymbol{a}) = 0$, therefore $\det\phi'(\boldsymbol{a}) \neq 0$.

By hypothesis $\|A(\boldsymbol{x})\| < c < 1$. Let us look at the dot product $\langle \phi(\boldsymbol{b}) - \phi(\boldsymbol{a}), \boldsymbol{b}-\boldsymbol{a}\rangle$. As $\langle \boldsymbol{x}(t), \boldsymbol{a}\rangle'_t = \bigl[\sum x_i a_i\bigr]'_t = \sum x'_t a_i = \langle \boldsymbol{x}', \boldsymbol{a}\rangle$, we have

$$\langle \phi(\boldsymbol{b}) - \phi(\boldsymbol{a}), \boldsymbol{b}-\boldsymbol{a}\rangle = \int_0^1 \langle v_t, \boldsymbol{b}-\boldsymbol{a}\rangle\,dt > (1-c)\int_0^1 \|\boldsymbol{b}-\boldsymbol{a}\|^2\,dt = (1-c)\,\|\boldsymbol{b}-\boldsymbol{a}\|^2 > 0.$$

In other words, the injectivity condition is satisfied: $\phi(\boldsymbol{a}) \neq \phi(\boldsymbol{b})$ for each point $\boldsymbol{b} \in U \setminus \{\boldsymbol{a}\}$. Lemma 3 is proved. □

Lemma 4. *Under the notations of Theorem 2, the following inequality is true for all points* $(x_2, y_2) \in U$*:* $|f^{(k+l)}_{x_2^k y_2^l}(x_2, y_2) - f^{(k+l)}_{x_2^k y_2^l}(0,0)| \leqslant R(C_{k+1,l} + C_{k,l+1}) \leqslant 2MR$.

Proof. By abusing notations, denote (x_2, y_2) by (x, y). We have

$$\begin{aligned}|f^{(k+l)}_{x^k y^l}(x,y) - f^{(k+l)}_{x^k y^l}(0,0)| &= |f^{(k+l)}_{x^k y^l}(x,y) - f^{(k+l)}_{x^k y^l}(0,y) - f^{(k+l)}_{x^k y^l}(0,0) + f^{(k+l)}_{x^k y^l}(0,y)| \leqslant \\ &\leqslant |f^{(k+l)}_{x^k y^l}(x,y) - f^{(k+l)}_{x^k y^l}(0,y)| + |f^{(k+l)}_{x^k y^l}(0,y) - f^{(k+l)}_{x^k y^l}(0,0)| = \\ &= \Bigl|\int_0^x f^{(k+l+1)}_{x^{k+1} y^l}(t,y)\,dt\Bigr| + \Bigl|\int_0^y f^{(k+l+1)}_{x^k y^{l+1}}(0,t)\,dt\Bigr| \leqslant R(C_{k+1,l} + C_{k,l+1}).\end{aligned}$$

The lemma is proved. □

Lemma 5. *For any square* $n \times n$ *matrix* $A = \{a_{ij}\}_{i,j=1}^n$*, the Euclidean norm of the linear operator given by this matrix can be estimated as follows:* $\|A\| \leqslant \sqrt{\sum_{i,j=1}^n a_{ij}^2} = \sqrt{\operatorname{tr}(AA^t)}$. ■

3. Proof of Theorem 1 for E_6, E_8 and Theorem 2

By Lemma 1, after the change $(x, y) \to (x_2, y_2)$, we have $f^{(a+b)}_{x_2^a y_2^b}(\mathbf{0}) = 0$ for all $a, b \in \mathbb{Z}_+$ with $a + b > 0$, $a < 3$, and $b < 1 + \frac{k}{2}$. By Lemma 2, $f = f(P) + x_2^3 h(x_2, y_2) + \eta y_2^{1+k/2} g(x_2, y_2)$ for some functions $h, g \in C^\infty(U_0)$, where $\eta = \pm 1$ is defined as $\eta = \operatorname{sgn} f^{(1+k/2)}_{y^{1+k/2}}(\mathbf{0})$. Observe that $h(\mathbf{0}) = g(\mathbf{0}) = 1$ due to the definition of E_k singularity ($k = 6, 8$) and Assumption 1. This immediately reduces the function to the required normal form $f = f(P) + \tilde{x}^3 + \eta\tilde{y}^{1+k/2}$ by the coordinate change $\phi : \mathbb{R}^2_{x_2,y_2} \to \mathbb{R}^2_{\tilde{x},\tilde{y}}$ with $\tilde{x} = x_2\sqrt[3]{h(x_2, y_2)}$, $\tilde{y} = y_2 g(x_2, y_2)^{\frac{2}{2+k}}$. Thus we proved Theorem 1 for $k = 6, 8$.

By abusing notations, we will denote (x_2, y_2) by (x, y). By Lemma 2, we have explicit formulas (1), (2) for g, h (with $m = 3$ and $n = 1 + k/2$), namely:

$$\eta g(x, y) = \frac{1}{(k/2)!}\int_0^1 f^{(1+k/2)}_{y^{1+k/2}}(x, sy)(1-s)^{k/2}\,ds, \quad h(x, y) = \frac{1}{2}\sum_{i=0}^{k/2}\frac{y^i}{i!}\int_0^1 f^{(3+i)}_{x^3y^i}(sx, 0)(1-s)^2\,ds.$$

It remains to apply Lemma 3 to the coordinate transformation ϕ and to the neighborhood U from the formulation of Theorem 2. In other words, it remains to check the fulfillment of the assumptions of Lemma 3. By using the above formulas for the coordinate change ϕ and the functions g, h, the bound $C_{\alpha\beta} \leqslant M$ and Lemmas 4 and 5, as well as the Taylor expansion formula with a remainder in the Lagrange form or an integral remainder, we will prove the required bound $\|\phi'(\boldsymbol{x}) - I\| = \|A(\boldsymbol{x})\| < C < 1$ for each point $\boldsymbol{x} \in U$.

Let us proceed with detailed estimations, separately for the cases of E_6 and E_8.

3.1. The case of E_6

For simplifying notations, we will give the proof for the case $\eta = 1$, i.e. $f^{(1+k/2)}_{y^{1+k/2}}(\mathbf{0}) > 0$. In the case $\eta = -1$, the proof is similar.

We compute the elements of the Jacobi matrix of ϕ:

$$\frac{\partial\tilde{x}}{\partial x} = h^{\frac{1}{3}} + x\cdot\frac{1}{3}h^{-\frac{2}{3}}\cdot h'_x, \tag{3}$$

$$\frac{\partial\tilde{x}}{\partial y} = x\cdot\frac{1}{3}h^{-\frac{2}{3}}\cdot h'_y, \tag{4}$$

$$\frac{\partial\tilde{y}}{\partial x} = y\cdot\frac{1}{4}g^{-\frac{3}{4}}\cdot\left(\frac{1}{6}\int\limits_0^1 f^{(5)}_{xy^4}(x, sy)(1-s)^3\,ds\right), \tag{5}$$

$$\frac{\partial\tilde{y}}{\partial y} = g^{\frac{1}{4}} + y\cdot\left(\frac{1}{4}g^{-\frac{3}{4}}\cdot\frac{1}{6}\int\limits_0^1 s f^{(5)}_{y^5}(x, sy)(1-s)^3\,ds\right). \tag{6}$$

If $M = 0$, then $\phi = \mathrm{id}$ and everything is proved. Let further $M > 0$, and therefore $R < 1$. By Assumption 1, we have $h(0, 0) = 1$, $g(0, 0) = 1$. Thus, the Jacobi matrix at $\mathbf{0}$ is the unit matrix I.

By using the above formulas (3)–(6) for Jacobi matrix' elements, let us estimate the elements of the Jacobi matrix $\frac{\partial(\tilde{x}-x,\tilde{y}-y)}{\partial(x,y)}$ and prove item (a) for the case of E_6.

Step 1. Here we find an upper bound for $|1 - \frac{\partial\tilde{x}}{\partial x}|$. Remind that:

$$\tilde{x} = x\sqrt[3]{h(x, y)} = \sqrt[3]{f(x, 0) + yf'_y(x, 0) + \frac{y^2}{2}f''_{y^2}(x, 0) + \frac{y^3}{6}f'''_{y^3}(x, 0)}.$$

Denote $\tilde{h} = \tilde{h}(x, y) = h(x, y) - 1$.

Let us estimate $\tilde{h}(x,y)$. In the expression for $h(x,y)$, we apply the Taylor expansion formula in x with a remainder in the Lagrange form to the coefficients of powers of y:

$$|h(x,y)-1| = \left|\frac{f'''_{x^3}(0,0)}{6} + x\frac{f^{(4)}_{x^4}(c_0,0)}{24} + \sum_{k=1}^{3}\frac{f^{(k+3)}_{x^3y^k}(c_k,0)}{6}\cdot\frac{y^k}{k!} - 1\right| =$$
$$= \left|1 + x\frac{f^{(4)}_{x^4}(c_0,0)}{24} + y\frac{f^{(4)}_{x^3y}(c_1,0)}{6} + y^2\frac{f^{(5)}_{x^3y^2}(c_2,0)}{12} + y^3\frac{f^{(6)}_{x^3y^3}(c_3,0)}{36} - 1\right| \leqslant$$
$$\leqslant \frac{MR}{12}\left(\frac{5}{2} + R + \frac{R^2}{3}\right) < \frac{5}{24}\cdot\frac{MR}{1-R} \leqslant \frac{5}{24}\cdot\frac{2M}{M+2}\cdot\frac{1}{1-2/(M+2)} = \frac{5}{24}\cdot 2 < \frac{1}{2}.$$

Hence $h(x,y)\in(0.5,1.5)$. Therefore $\sqrt[3]{h(x,y)}\in(0.79,1.21)$.

Item (a) of Theorem 2 is proved for the function $h(x,y)$. Set $c=0.5$. Then $|\tilde{h}(x,y)|<c$.

By the formula (3) we have

$$\frac{\partial\tilde{x}}{\partial x} = h^{\frac{1}{3}} + x\cdot\frac{1}{3}h^{-\frac{2}{3}}\cdot h'_x = (1+\tilde{h})^{\frac{1}{3}} + x\cdot\frac{1}{3}(1+\tilde{h})^{-\frac{2}{3}}\cdot h'_x.$$

Let us estimate $|h'_x|$. We use Taylor's formula with an integral remainder:

$$|h'_x| = \left|\left(\sum_{k=0}^{3}\frac{y^k}{k!}\frac{f^{(k)}_{y^k}(x,0)}{x^3}\right)'_x\right| = \left|\left(\sum_{k=0}^{3}\frac{y^k}{k!}\frac{1}{2}\int_0^1 f^{(k+3)}_{y^kx^3}(sx,0)(1-s)^2\,ds\right)'_x\right| =$$
$$= \left|\sum_{k=0}^{3}\frac{y^k}{k!\cdot 2}\int_0^1 s f^{(k+4)}_{y^kx^4}(sx,0)(1-s)^2\,ds\right| \leqslant \sum_{k=0}^{3}\frac{R^kM}{k!\cdot 2}\int_0^1(s-2s^2+s^3)\,ds <$$
$$< \frac{M}{24}\left(1+R+\frac{R^2}{2}+\frac{R^3}{6}\right) < \frac{1}{24}\cdot\frac{M}{1-R} \leqslant \frac{1}{12R}.$$

Set $c_x = \frac{1}{12R}$. We obtain a bound for $|\frac{\partial\tilde{x}}{\partial x}-1|$ using the estimate of $\sqrt[3]{h}$:

$$\left|\frac{\partial\tilde{x}}{\partial x}-1\right| = \left|\sqrt[3]{h}-1+x\cdot\tfrac{1}{3}(1+\tilde{h})^{-\frac{2}{3}}h'_x\right| \leqslant |\sqrt[3]{h}-1| + \left|x\cdot\tfrac{1}{3}(1+\tilde{h})^{-\frac{2}{3}}h'_x\right| <$$
$$< 0.21 + \frac{c_xR}{3}(1-c)^{-\frac{2}{3}} = 0.21 + \frac{2^{\frac{2}{3}}}{3\cdot 12} < 0.21+0.05 = 0.26.$$

Step 2. Let us estimate $\frac{\partial\tilde{x}}{\partial y}$ from the formula (4).

a) First we estimate $|h'_y|$ when $x\neq 0$. We use Taylor's formula with an integral remainder:

$$|h'_y| = \left|(f(x,0)+yf'_y(x,0)+\frac{y^2}{2}f''_{y^2}(x,0)+\frac{y^3}{6}f'''_{y^3}(x,0))'_y/x^3\right| =$$
$$= \left|(f'_y(x,0)+yf''_{y^2}(x,0)+\frac{y^2}{2}f'''_{y^3}(x,0))/x^3\right| =$$
$$= \left|\frac{1}{2}\int_0^1\left(f^{(4)}_{x^3y}(sx,0)+yf^{(5)}_{x^3y^2}(sx,0)+\frac{y^2}{2}f^{(6)}_{x^3y^3}(sx,0)\right)(1-s)^2\,ds\right| \leqslant$$
$$\leqslant \frac{M}{6}(1+R+R^2) < \frac{1}{6}\cdot\frac{M}{1-R} \leqslant \frac{1}{3R}.$$

Set $c_y = \frac{1}{3R}$.

b) In the formula (4) for $\frac{\partial\tilde{x}}{\partial y}$ we expand $h^{-\frac{2}{3}}$ in a Taylor series in $\tilde{h}$ and take $c_{\tilde{h}} = c_{\tilde{h}}(x, y) \in$ $\in [0, \tilde{h}(x, y)]$ from the Taylor-Lagrange formula:

$$\Big|\frac{\partial\tilde{x}}{\partial y}\Big| = \big|x\cdot\tfrac{1}{3}(1+\tilde{h})^{-\frac{2}{3}}\cdot h'_y\big| = \Big|\frac{x}{3}\big(1-\tfrac{2}{3}\tilde{h}+\tfrac{5}{9}(1+c_{\tilde{h}})^{-\frac{8}{3}}\tilde{h}^2\big)h'_y\Big| \leqslant$$
$$\leqslant \frac{R}{3}\big(1+\tfrac{2}{3}c+\tfrac{5}{9}(1+c_{\tilde{h}})^{-\frac{8}{3}}c^2\big)c_y < \frac{c_yR}{3}\big(1+\tfrac{2}{3}c+\tfrac{5}{9}(1-c)^{-\frac{8}{3}}c^2\big) =$$
$$= \frac{1}{9}\big(1+\tfrac{1}{3}+\tfrac{5}{9}2^{\frac{8}{3}}/4\big) < 0.25.$$

Step 3. Let us estimate $\frac{\partial\tilde{y}}{\partial x}$ from the formula (5). Let us first estimate separately the factors of this expression.

a) Auxiliary Assessment. By Lemma 4, we have

$$\Big|\int\limits_0^1\Big(f^{(4)}_{y^4}(x,sy)-f^{(4)}_{y^4}(0,0)\Big)(1-s)^3\,ds\Big| \leqslant \int\limits_0^1 2MR(1-s)^3\,ds = \frac{MR}{2} < 1.$$

b) By Assumption 1, we have $f^{(4)}_{y^4}(0,0) = 24$. We get the following lower bound:

$$6|g| = \Big|\int\limits_0^1 f^{(4)}_{y^4}(x,sy)(1-s)^3\,ds\Big| = \Big|\int\limits_0^1\Big(f^{(4)}_{y^4}(x,sy)-f^{(4)}_{y^4}(0,0)+f^{(4)}_{y^4}(0,0)\Big)(1-s)^3\,ds\Big|$$
$$\geqslant 6-\frac{MR}{2} > 5 > 0.$$

Item (a) of Theorem 2 has been completely proved for E_6. Set $\tilde{c} = 1-\frac{MR}{12}$. Then

$$\Big|\frac{\partial\tilde{y}}{\partial x}\Big| = \Big|y\frac{1}{4}\Big(\frac{1}{6}\int\limits_0^1 f^{(4)}_{y^4}(x,sy)(1-s)^3\,ds\Big)^{-\frac{3}{4}}\Big(\frac{1}{6}\int\limits_0^1 f^{(5)}_{xy^4}(x,sy)(1-s)^3\,ds\Big)\Big| \leqslant$$
$$\leqslant R\frac{1}{4}\tilde{c}^{-\frac{3}{4}}\frac{M}{24} < \frac{MR}{96}\Big(1-\frac{MR}{12}\Big)^{-\frac{3}{4}} < \frac{1}{48}\Big(1-\frac{1}{6}\Big)^{-\frac{3}{4}} < 0.03.$$

Step 4. Let us estimate $|1-\frac{\partial\tilde{y}}{\partial y}|$ from formula (6).

$$\Big|\frac{\partial\tilde{y}}{\partial y}-1\Big| = \Big|\sqrt[4]{\frac{1}{6}\int_0^1 f^{(4)}_{y^4}(x,sy)(1-s)^3\,ds}-1+$$
$$+\frac{y}{4}\Big(\frac{1}{6}\int\limits_0^1 f^{(4)}_{y^4}(x,sy)(1-s)^3\,ds\Big)^{-\frac{3}{4}}\frac{1}{6}\int\limits_0^1 sf^{(5)}_{y^5}(x,sy)(1-s)^3\,ds\Big| \leqslant$$
$$< \Big(1+\frac{MR}{12}\Big)^{\frac{1}{4}}-1+\frac{MR}{80}\Big(1-\frac{MR}{12}\Big)^{-\frac{3}{4}} < \Big(1+\tfrac{1}{6}\Big)^{\frac{1}{4}}-1+\tfrac{1}{40}\Big(1-\tfrac{1}{6}\Big)^{-\frac{3}{4}} < 0.07.$$

Step 5. By Steps 1–4, for each point $\boldsymbol{x} \in U$, we have:

$$\|\phi'(\boldsymbol{x})-I\| = \|A(\boldsymbol{x})\| \leqslant \sqrt{0.26^2+0.25^2+0.03^2+0.07^2} = \sqrt{0.1359} < 0.4 = C < 1,$$

that proves item (b) of Theorem 2 for $C = \frac{2}{5}$. By Lemma 3 the coordinate change ϕ is injective in U, that proves item (c), except for the properties of $\phi(U)$.

From the last assertion of Lemma 3 and [5, Cor. 8.3, Step 1], we conclude that $\phi(U)$ contains the open disk of radius $(1-C)R$ centered at the origin.

This completes our proof of Theorem 2 for the case of a singularity E_6.

3.2. The case of E_8

For a singularity E_8, the coordinate change ϕ is given by the formulas $\tilde{x} = x\sqrt[3]{h(x,y)} = \sqrt[3]{\sum_{k=0}^{4} \frac{y^k}{k!} f^{(k)}_{y^k}(x,0)}$, $\tilde{y} = y\sqrt[5]{g(x,y)} = y\left[\eta\frac{1}{24}\int_0^1 f^{(5)}_{y^5}(x,sy)(1-s)^4\,ds\right]^{\frac{1}{5}}$, where $\eta = \operatorname{sgn} f^{(5)}_{y^5}(\mathbf{0})$.

Like in the case of E_6, we will further assume that $\eta = 1$ (for $\eta = -1$, the proof is similar).

Step 1. As in the case of E_6, we find a bound for the term $\tilde{h}(x,y)$, in order to estimate $|1 - \frac{\partial\tilde{x}}{\partial x}|$. The difference from the case of E_6 is an additional term in the sum of absolute values that can be bound by $\frac{MR^4}{144}$, then the sum can be bound by the same geometric progression. In more detail:

$$|h(x,y) - 1| = \left|\frac{f'''_{x^3}(0,0)}{6} + x\frac{f^{(4)}_{x^4}(c_0,0)}{24} + \sum_{k=1}^{4}\frac{f^{(k+3)}_{x^3y^k}(c_k,0)}{6}\cdot\frac{y^k}{k!} - 1\right| \leqslant \frac{RC_{40}}{24} + \\ + \frac{RC_{31}}{6} + \frac{R^2C_{32}}{12} + \frac{R^3C_{33}}{36} + \frac{R^4C_{34}}{144} \leqslant \frac{MR}{12}\left(\frac{5}{2} + R + \frac{R^2}{3} + \frac{R^3}{12}\right) < \frac{5}{24}\cdot 2 < \frac{1}{2}.$$

The bound from before holds: $h(x,y) \in (0.5, 1.5)$ or $|\tilde{h}| < 0.5$. Then $\sqrt[3]{h(x,y)} \in (0.79, 1.21)$.

Similar arguments work for the estimation of the term $|h'_x|$. Thus we can bound it as we did it in the case of E_6:

$$|h'_x| < \frac{1}{12R}.$$

Set $c_x = \frac{1}{12R}$.

The element $\frac{\partial\tilde{x}}{\partial x}$ has the same representation in terms of the function h as in the case E_6 and the same estimation for every term. Hence the following bound holds:

$$\left|\frac{\partial\tilde{x}}{\partial x} - 1\right| < 0.26.$$

Step 2. a) For estimating the term $|h'_y|$ when $x \neq 0$, we can use the bound from the case of E_6:

$$|h'_y| = \left|\left(\sum_{k=0}^{4}\frac{y^k}{k!}\frac{f^{(k)}_{y^k}(x,0)}{x^3}\right)'_y\right| \leqslant \frac{1}{3R}.$$

b) The element $\frac{\partial\tilde{x}}{\partial y}$ has the same representation in terms of the function h as in the case of E_6. Hence the following bound holds:

$$\left|\frac{\partial\tilde{x}}{\partial y}\right| < 0.25.$$

Step 3. In this step, we have orders of partial derivatives and a constant $\tilde{c}$ that differ from the case of E_6.

a) Auxiliary Assessment. By Lemma 4, we have

$$\left|\int_0^1\left(f^{(5)}_{y^5}(x,sy) - f^{(5)}_{y^5}(0,0)\right)(1-s)^4\,ds\right| \leqslant \int_0^1 R(C_{15}+C_{06})(1-s)^4\,ds = \frac{(C_{15}+C_{06})R}{5} \leqslant \frac{2MR}{5}.$$

b) By Assumption 1, we have $f^{(5)}_{y^5}(0,0) = 120$. We get the following lower bound for $24|g|$:

$$\left|\int_0^1 f^{(5)}_{y^5}(x,sy)(1-s)^4\,ds\right| = \left|\int_0^1\left(f^{(5)}_{y^5}(x,sy) - f^{(5)}_{y^5}(0,0) + f^{(5)}_{y^5}(0,0)\right)(1-s)^4\,ds\right| \geqslant 24 - \frac{2MR}{5}.$$

Thus $|g| > 0$, and item (a) of Theorem 2 is proved for E_8. Set $\tilde{c} = 1 - \frac{MR}{60}$. Then

$$\left|\frac{\partial \tilde{y}}{\partial x}\right| = \left| y \cdot \frac{1}{5}\left(\frac{1}{24}\int_0^1 f^{(5)}_{y^5}(x,sy)(1-s)^4\,ds\right)^{-\frac{4}{5}} \cdot \left(\frac{1}{24}\int_0^1 f^{(6)}_{xy^5}(x,sy)(1-s)^4\,ds\right)\right| \leqslant$$
$$\leqslant R\frac{1}{5}\tilde{c}^{-\frac{4}{5}}\frac{C_{15}}{120} < \frac{MR}{600}\left(1-\frac{MR}{60}\right)^{-\frac{4}{5}} < \frac{1}{300}\left(1-\frac{1}{30}\right)^{-\frac{4}{5}} < 0.004 < 0.03.$$

Step 4. Let us estimate $|1 - \frac{\partial \tilde{y}}{\partial y}|$:

$$\left|\frac{\partial \tilde{y}}{\partial y} - 1\right| = \left| \sqrt[5]{\frac{1}{24}\int_0^1 f^{(5)}_{y^5}(x,sy)(1-s)^4\,ds} - 1 + \right.$$
$$\left. + \frac{y}{5}\left(\frac{1}{24}\int_0^1 f^{(5)}_{y^5}(x,sy)(1-s)^4\,ds\right)^{-\frac{4}{5}}\frac{1}{24}\int_0^1 s f^{(6)}_{y^6}(x,sy)(1-s)^4\,ds\right| \leqslant$$
$$\leqslant \left|\left(\frac{1}{24}\int_0^1\Big(f^{(5)}_{y^5}(x,sy) - f^{(5)}_{y^5}(0,0) + f^{(5)}_{y^5}(0,0)\Big)(1-s)^4\,ds\right)^{\frac{1}{5}} - 1\right| + |\ldots| \leqslant$$
$$\leqslant \left|\left(1 - \frac{(C_{15}+C_{06})R}{120}\right)^{\frac{1}{5}} - 1\right| + R\tilde{c}^{-\frac{4}{5}}\frac{C_{06}}{5\cdot 24\cdot 30} <$$
$$< 1 - \left(1 - \frac{MR}{60}\right)^{\frac{1}{5}} + \frac{MR}{3600}\left(1 - \frac{MR}{60}\right)^{-\frac{4}{5}} <$$
$$< 1 - \left(1 - \tfrac{1}{30}\right)^{\frac{1}{5}} + \tfrac{1}{1800}\left(1 - \tfrac{1}{30}\right)^{-\frac{4}{5}} < 0.008 < 0.07.$$

Step 5. From Steps 1–4, we get the required bound for a singularity of the type E_8:

$$\|\phi'(\boldsymbol{x}) - I\| < \sqrt{0.26^2 + 0.25^2 + 0.004^2 + 0.008^2} < 0.4 = C < 1.$$

The rest of the proof is the same as in the case of E_6.

Theorem 2 is completely proved. □

4. The case of A_k

Definition 2. *A smooth function $f(u,v)$ has a singularity type A_k ($k \geqslant 2$) at its critical point $P \in \mathbb{R}^2$ if*

(i) *the first differential $df(P)$ vanishes and the second differential $d^2 f(P)$ is a perfect square (non-zero);*

(ii) *some condition on coefficients of the Taylor series of f at P holds (see below).*

Consider a linear change $(u,v) \to (x,y)$ such that $d^2 f(P) = \pm 2(dx)^2$. By using a non-linear change $(x,y) \to (x_1 = x + y^2 Q(y), y_1 = y)$ for some polynomial $Q(y)$ of degree $\leqslant \frac{k-3}{2}$, one can achieve that $f^{(i+2)}_{x_1 y_1^{i+1}}(P) = 0$ for all i with $1 \leqslant i \leqslant [\frac{k-1}{2}]$. Notice that $f''_{x_1^2}(P) = \pm 2$. The condition on the Taylor series coefficients is as follows: $f'''_{y_1^3}(P) = \dots = f^{(k)}_{y_1^k}(P) = 0$ and $f^{(k+1)}_{y_1^{k+1}}(P) \neq 0$ (if $k = 2$, then this condition means that $f'''_{y^3}(P) \neq 0$).

We will assume that $P = (0,0) = \mathbf{0}$ in the coordinates x, y.

Due to the splitting lemma [1, Lemma 4.1], there exists a C^∞-smooth coordinate change $(x,y) \to (X,Y)$ (which in fact can be constructed by an explicit formula by using a parametric version of the Morse lemma) such that $P=(0,0)$ in the coordinates x,y, and $f=f(P)\pm X^2+F(Y)$ where $F(y)$ is a smooth function such that $F'(0)=F''(0)=0$. The condition (ii) in the above definition is in fact equivalent to the following: $F'''(0)=\dots=F^{(k)}(0)=0$ and $F^{(k+1)}(0)\neq 0$ (see also [3, Assertion 6.1] for another condition equivalent to this condition). It is an easy exercise that there exists a C^∞-smooth coordinate change $Y\to\widetilde Y$ (which in fact can be constructed by an explicit formula) centered at the origin that reduces f to the normal form $f=f(P)\pm X^2\pm\widetilde Y^{k+1}$.

We say that a smooth function $f(u,v)$ has a singularity type A_1 at its critical point $P\in\mathbb{R}^2$ if this point is non-degenerate (i.e., Morse). For $k=1$, we can find a linear change $(u,v)\to(x_1,y_1)$ such that $f''_{x_1^2}(P)=\pm 2$ and $f''_{x_1y_1}(P)=0$, thus $f''_{y_1^2}(P)\neq 0$.

Assumption 2. *For singularities A_k $(k\geqslant 1)$, assume that $f^{(k+1)}_{y_1^{k+1}}(0,0)=\pm(k+1)!$.*

Similarly to Lemma 1, one can construct a coordinate change $(x_1,y_1)\to(x_2=x_1+y_1^2Q_1(y_1),$ $y_2=y_1)$ for some polynomial $Q_1(y_1)$ of degree $\leqslant k$, such that $f^{(a+b)}_{x_2^ay_2^b}(P)=0$ for all $a,b\in\mathbb{Z}_+$ with $a+b>0$, $a<2$, and $b<k+1$.

The following theorem extends the uniform Morse lemma [4, 5] to the case of the infinite series of singularities A_k for all $k\geqslant 1$.

Theorem 3 (Estimating the radius of a neighborhood for the coordinate change). *In the case of a singularity A_k $(k\geqslant 1)$, under the hypotheses of Assumption 2, let $(x_1,y_1)\to(x_2,y_2)$ be the coordinate change from above. Suppose that, in a neighborhood $U_0=\{(x_2,y_2)\mid \max(|x_2|,|y_2|)<R_0\}$ of $\mathbf{0}$, the following estimates hold: $C_{\alpha\beta}:=\sup_{U_0}\left|f^{(\alpha+\beta)}_{x_2^\alpha y_2^\beta}(x_2,y_2)\right|\leqslant M$ for all*

$$(\alpha,\beta)\in\{(0,k+2),(1,k+1),(2,i),(3,0),(3,i)\mid 1\leqslant i\leqslant k\},$$

where $R_0>0$, $M\geqslant 0$. Then, in the neighborhood $U=\{(x_2,y_2)\mid\max(|x_2|,|y_2|)<R\}$, with $R=\min\{R_0,\frac{3}{4M+3}\}$, there is a C^∞-smooth coordinate change of the form

$$\phi:(x_2,y_2)\to(\tilde x=x_2\sqrt{h(x_2,y_2)},\tilde y=y_2g(x_2,y_2)^{\frac{1}{k+1}})$$

that reduces f to the normal form $f=f(P)\pm\tilde x^2\pm\tilde y^{k+1}$ of A_k. In more detail: the coordinate change $\phi|_U$ satisfies the conditions (a), (b), (c) *from Theorem* 2 with $C=0.93$.

Proof. By abusing notations, we will denote (x_2,y_2) by (x,y). Let $\eta_h=\operatorname{sgn}f''_{x^2}(P)$, $\eta_g=$ $=\operatorname{sgn}f^{(k+1)}_{y^{k+1}}(P)$. Due to Lemma 2, we have $f=f(P)+\eta_hx^2h(x,y)+\eta_gy^{k+1}g(x,y)$, where the functions η_hh,η_gg are given by the explicit formulas (1), (2) (with $m=2$ and $n=k+1$). Thus $h(\mathbf{0})=g(\mathbf{0})=1$ due to the definition of A_k singularity and Assumption 2.

Like in the case of E_k $(k=6,8)$, we will further assume that $\eta_h=\eta_g=1$ (for $\eta_h=-1$ or $\eta_g=-1$, the proof is similar).

Consider the coordinate change

$$\tilde x=x\sqrt{h(x,y)}=\sqrt{\sum_{i=0}^k\frac{y^i}{i!}f^{(i)}_{y^i}(x,0)},\quad \tilde y=y\sqrt[k+1]{g(x,y)}=y\left[\frac{1}{k!}\int_0^1 f^{(k+1)}_{y^{k+1}}(x,sy)(1-s)^k\,ds\right]^{\frac{1}{k+1}}.$$

We extend our proof of Theorem 2 (about singularities E_6,E_8) to the case of singularities A_k.

Step 1. a) Let us estimate $\tilde h(x,y)$:

$$|h(x,y)-1|=\left|\frac{f''_{x^2}(0,0)}{2}+x\frac{f'''_{x^3}(c_0,0)}{6}+\sum_{i=1}^k\frac{f^{(i+2)}_{x^2y^i}(c_i,0)}{2}\frac{y^i}{i!}-1\right|\leqslant$$
$$\leqslant\frac{RC_{30}}{6}+\sum_{i=1}^k\frac{C_{2i}R^i}{2\cdot i!}\leqslant\frac{2MR}{3(1-R)}\leqslant\frac{1}{2}.$$

Thus $h(x,y) \in (0.5, 1.5)$, or $|\tilde{h}| < 0.5 =: c$. Then $\sqrt{h(x,y)} \in (0.7, 1.3)$.

We have
$$\frac{\partial \tilde{x}}{\partial x} = h^{\frac{1}{2}} + x \cdot \frac{1}{2} h^{-\frac{1}{2}} \cdot h'_x = (1+\tilde{h})^{\frac{1}{2}} + x \cdot \frac{1}{2}(1+\tilde{h})^{-\frac{1}{2}} \cdot h'_x.$$

Let us estimate $|h'_x|$:
$$|h'_x| = \left| \left(\sum_{i=0}^{k} \frac{y^i}{i!} \frac{f^{(i)}_{y^i}(x,0)}{x^2} \right)'_x \right| = \left| \left(\sum_{i=0}^{k} \frac{y^i}{i!} \int_0^1 f^{(i+2)}_{y^i x^2}(sx,0)(1-s)\, ds \right)'_x \right| =$$
$$= \left| \sum_{i=0}^{k} \frac{y^i}{i!} \int_0^1 s f^{(i+3)}_{y^i x^3}(sx,0)(1-s)\, ds \right| \leqslant \sum_{i=0}^{k} \frac{R^i C_{3i}}{i!} \int_0^1 (s-s^2)\, ds \leqslant \frac{M}{6(1-R)} \leqslant \frac{1}{8R}.$$

Set $c_x = \frac{1}{8R}$. Let us estimate $|\frac{\partial \tilde{x}}{\partial x} - 1|$:
$$\left| \frac{\partial \tilde{x}}{\partial x} - 1 \right| = \left| \sqrt{h} - 1 + x \cdot \tfrac{1}{2}(1+\tilde{h})^{-\frac{1}{2}} h'_x \right| \leqslant |\sqrt{h} - 1| + \left| x \cdot \tfrac{1}{2}(1+\tilde{h})^{-\frac{1}{2}} h'_x \right| <$$
$$< 0.3 + \frac{c_x R}{2}(1-c)^{-\frac{1}{2}} < 0.3 + \frac{2^{\frac{1}{2}}}{2 \cdot 8} < 0.3 + 0.1 = 0.4.$$

Step 2. a) Let us estimate $|h'_y|$ when $x \neq 0$:
$$|h'_y| = \left| \left(\sum_{i=0}^{k} \frac{y^i}{i!} \frac{f^{(i)}_{y^i}(x,0)}{x^2} \right)'_y \right| = \left| \left(\sum_{i=0}^{k-1} \frac{y^i}{i!} \frac{f^{(i+1)}_{y^{i+1}}(x,0)}{x^2} \right) \right| =$$
$$= \left| \int_0^1 \left(\sum_{i=0}^{k-1} \frac{y^i}{i!} f^{(i+3)}_{x^2 y^{i+1}}(sx,0) \right)(1-s)\, ds \right| \leqslant \frac{1}{2} \sum_{i=0}^{k-1} \frac{R^i C_{2,i+1}}{i!} \leqslant \frac{M}{2(1-R)} \leqslant \frac{3}{8R}.$$

Set $c_y = \frac{3}{8R}$.

b) Let us estimate $|\frac{\partial \tilde{x}}{\partial y}|$:
$$\left| \frac{\partial \tilde{x}}{\partial y} \right| = \left| x \cdot \tfrac{1}{2}(1+\tilde{h})^{-\frac{1}{2}} \cdot h'_y \right| = \left| \frac{x}{2}\left(1 - \tfrac{1}{2}\tilde{h} + \tfrac{3}{8}(1+c_{\tilde{h}})^{-\frac{5}{2}} \tilde{h}^2\right) h'_y \right| \leqslant$$
$$\leqslant \frac{R}{2}\left(1 + \tfrac{1}{2}c + \tfrac{3}{8}(1+c_{\tilde{h}})^{-\frac{5}{2}} c^2\right) c_y < \frac{c_y R}{2}\left(1 + \tfrac{1}{2}c + \tfrac{3}{8}(1-c)^{-\frac{5}{2}} c^2\right) =$$
$$= \frac{3}{16}\left(1 + \tfrac{1}{4} + \tfrac{3}{8} 2^{\frac{5}{2}}/4\right) < 0.4.$$

Step 3. a) Auxiliary Assessment. By Lemma 4:
$$\left| \int_0^1 \left(f^{(k+1)}_{y^{k+1}}(x,sy) - f^{(k+1)}_{y^{k+1}}(0,0) \right)(1-s)^k\, ds \right| \leqslant \int_0^1 R(C_{1,k+1} + C_{0,k+2})(1-s)^k\, ds \leqslant \frac{2MR}{k+1}.$$

b) By Assumption 2 we have $f^{(k+1)}_{y^{k+1}}(0,0) = (k+1)!$. We get the following lower bound for $k!|g(x,y)|$:
$$\left| \int_0^1 f^{(k+1)}_{y^{k+1}}(x,sy)(1-s)^k\, ds \right| = \left| \int_0^1 \left(f^{(k+1)}_{y^{k+1}}(x,sy) - f^{(k+1)}_{y^{k+1}}(0,0) + f^{(k+1)}_{y^{k+1}}(0,0) \right)(1-s)^k\, ds \right| \geqslant$$
$$\geqslant k! - \frac{2MR}{k+1} \geqslant k!\left(1 - \frac{3}{2(k+1)!}\right) > 0.$$

Thus $|g| > 0$, and we proved item (a) of Theorem 3.

Set $\tilde{c} = 1 - \frac{2MR}{(k+1)!}$. Then the following sequence of inequalities holds:

$$\Big|\frac{\partial\tilde{y}}{\partial x}\Big| = \left|y\cdot\frac{1}{k+1}\left(\frac{1}{k!}\int_0^1 f^{(k+1)}_{y^{k+1}}(x,sy)(1-s)^k\,ds\right)^{-\frac{k}{k+1}}\cdot\left(\frac{1}{k!}\int_0^1 f^{(k+2)}_{xy^{k+1}}(x,sy)(1-s)^k\,ds\right)\right| \leqslant$$
$$\leqslant R\frac{1}{k+1}\tilde{c}^{-\frac{k}{k+1}}\frac{C_{1,k+1}}{(k+1)!} \leqslant$$
$$\leqslant \frac{MR}{(k+1)(k+1)!}\left(1-\frac{2MR}{(k+1)!}\right)^{-\frac{k}{k+1}} < \frac{3}{4\cdot 4}\left(1-\frac{3}{2(k+1)!}\right)^{-\frac{k}{k+1}} \leqslant \frac{3}{8}.$$

Let us prove the latter inequality in this sequence. For $k = 1$ this inequality is in fact an equality. For $k > 1$ we have $\tilde{c}^{-\frac{k}{k+1}} < \tilde{c}^{-1}$, moreover the function $\tilde{c}^{-1}|_{MR=\frac{3}{4}}$ of k is monotone decreasing. Since $\tilde{c}^{-1}|_{MR=\frac{3}{4}} < 2$ for $k = 2$, we have the same for all $k > 2$. This proves the desired inequality.

Step 4. Let us estimate $|1 - \frac{\partial\tilde{y}}{\partial y}|$:

$$\Big|\frac{\partial\tilde{y}}{\partial y} - 1\Big| = \Bigg|\sqrt[k+1]{\frac{1}{k!}\int_0^1 f^{(k+1)}_{y^{k+1}}(x,sy)(1-s)^k\,ds} - 1 +$$
$$+\frac{y}{k+1}\left(\frac{1}{k!}\int_0^1 f^{(k+1)}_{y^{k+1}}(x,sy)(1-s)^k\,ds\right)^{-\frac{k}{k+1}}\frac{1}{k!}\int_0^1 sf^{(k+2)}_{y^{k+2}}(x,sy)(1-s)^k\,ds\Bigg| \leqslant$$
$$\leqslant \left|\left(\frac{1}{k!}\int_0^1\Big(f^{(k+1)}_{y^{k+1}}(x,sy) - f^{(k+1)}_{y^{k+1}}(0,0) + f^{(k+1)}_{y^{k+1}}(0,0)\Big)(1-s)^k\,ds\right)^{\frac{1}{k+1}} - 1\right| + |\ldots| \leqslant$$
$$\leqslant \left|\left(1 - \frac{(C_{1,k+1}+C_{0,k+2})R}{(k+1)!}\right)^{\frac{1}{k+1}} - 1\right| + R\tilde{c}^{-\frac{k}{k+1}}\frac{C_{0,k+2}}{(k+2)(k+1)(k+1)!} <$$
$$< 1 - \left(1-\frac{2MR}{(k+1)!}\right)^{\frac{1}{k+1}} + \cdots = 1 - \tilde{c}^{\frac{1}{k+1}} + \frac{MR\tilde{c}^{-\frac{k}{k+1}}}{(k+2)(k+1)(k+1)!} <$$
$$< \frac{3}{2\cdot(k+1)!} + \frac{3\cdot 2}{4\cdot 3\cdot 2\cdot 2} \leqslant \frac{1}{2} \leqslant \frac{5}{8}$$

if $k > 1$. And when $k = 1$, we get $1 - \tilde{c}^{\frac{1}{2}} \leqslant \frac{1}{2}$, therefore $|\frac{\partial\tilde{y}}{\partial y} - 1| < \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$.

Step 5. From Steps 1–4, for A_k we have $\|\phi'(\boldsymbol{x}) - I\| < \sqrt{0.4^2 + 0.4^2 + \frac{3}{8}^2 + \frac{5}{8}^2} < 0.93 < 1$ for all $k \geqslant 1$. The rest of the proof is the same as for Theorem 2. Theorem 3 is proved. □

5. Conclusion

For singularities A_k, E_6, and E_8 there are explicitly constructed (in Theorems 1–3) a coordinate change reducing a function to normal form and estimated (in terms of C^r-norm of the function, where $r = k+3$, 7, and 8 respectively) from below a radius of a neighborhood where this coordinate change is well-defined.

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Арнольд В. И. Нормальные формы функций вблизи вырожденных критических точек, группы Вейля A_k, D_k, E_k и лагранжевы особенности // Функц. анализ и его прил. 1972. Т. 6, № 4. С. 3–25. Engl. version: Arnol'd V. I. Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities // Funct. Anal. Appl. 1972. Vol. 6, № 4. P. 254–272.

2. Brodersen H. M-t topologically stable mappings are uniformly stable // Math. Scand. 1983. Vol. 52. P. 61–68.

3. Kudryavtseva E. A., Lakshtanov E. L. Classification of singularities and bifurcations of critical points of even functions // In: *Topological Methods in the Theory of Integrable Systems.* Eds. Bolsinov, A.V., Fomenko, A.T., and Oshemkov, A.A.; Cambridge Scientific Publishers, Springer. 2006. P. 173–214. http://arxiv.org/abs/1212.4302.

4. Кудрявцева Е. А. Равномерная лемма Морса и критерий изотопности функций Морса на поверхностях // Вестн. Моск. ун-та. Сер. I Матем. Мех. 2009. № 4, С. 13-22 Engl. version: Kudryavtseva E. A. Uniform Morse lemma and isotope Morse functions on surfaces // Moscow Univ. Math. Bull. 2009. Vol. 64, no. 4. P. 150–158.

5. Кудрявцева Е. А., Пермяков Д. А. Оснащенные функции Морса на поверхностях // Матем. Сб. 2010. Т. 201, № 4. С. 501-567. Engl. version: Kudryavtseva E. A., Permyakov D. A. Framed Morse functions on surfaces // Sbornik Math. 2010. Vol. 201, no. 4. P. 501–567.

6. Кудрявцева Е. А. Топология пространств функций Морса на поверхностях // Матем. Заметки. 2012. Т. 92, № 2. С. 241-261. Engl. version: Kudryavtseva E. A. The Topology of Spaces of Morse Functions on Surfaces // Math. Notes. 2012. Vol. 92, no. 2. P. 219–236. http://arxiv.org/abs/1104.4792.

7. Кудрявцева Е. А. Специальные оснащенные функции Морса на поверхностях // Вестн. Моск. ун-та. Матем. Механ. 2012. № 4, С. 14–20. Engl. version: Kudryavtseva E. A. Special framed Morse functions on surfaces // Moscow Univ. Math. Bull. 2012. Vol. 67, no. 4. P. 151–157. http://arxiv.org/abs/1106.3116.

8. Кудрявцева Е. А. О гомотопическом типе пространств функций Морса на поверхностях // Матем. сб. 2013. Т. 204, № 1. С. 79-118 Engl. version: Kudryavtseva E. A. On the homotopy type of spaces of Morse functions on surfaces // Sb. Math. 2013. Vol. 204, no. 1. P. 75–113. http://arxiv.org/abs/1104.4796.

9. Кудрявцева Е. А. Топология пространств функций с заданными особенностями на поверхностях // Докл. Акад. Наук 2016. Т. 468, № 1. С. 139-142. Engl. version: Kudryavtseva E. A. Topology of the spaces of functions with prescribed singularities on surfaces // Doklady Math. 2016. Vol. 93, no. 3. P. 264–266.

10. Kudryavtseva E. A. Topology of the spaces of functions and gradient-like flows with prescribed singularities on surfaces // Arxiv. 2021. http://arxiv.org/abs/2106.03017.

11. Mather J. N. Infinitesimal stability implies stability // Ann. of Math. 1969. Vol. 89. P. 254–291.

12. Самойленко А. М. Об эквивалентности гладкой функции полиному Тейлора в окрестности критической точки конечного типа // Функц. анализ, 1968. Т. 2, № 4. С. 63-69. Engl. version: Samoilenko A. M. The equivalence of a smooth function to a Taylor polynomial in the neighborhood of a finite-type critical point // Funct. Anal. Appl. 1968. Vol. 2, no. 4. P. 318–323.

13. Sergeraert F. Un théorème de fonctions implicites sur certains espaces de Fréchet et quelques applications // Annales scientifiques de l'É.N.S. 4e série. 1972. Vol. 5, no. 4. P. 599–660.

14. Tougeron J. C. Ideaux de fonctions differentiables. I // Ann. Inst. Fourier. 1968. Vol. 18, no. 1. P. 177–240.

REFERENCES

1. Arnol'd, V. I. 1972, "Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities", *Funct. Anal. Appl.*, vol. 6, no. 4, pp. 254–272.

2. Brodersen, H. 1983, "M-t topologically stable mappings are uniformly stable", *Math. Scand.*, vol. 52, pp. 61–68.

3. Kudryavtseva, E. A. & Lakshtanov, E. L. 2006, "Classification of singularities and bifurcations of critical points of even functions", In: *Topological Methods in the Theory of Integrable Systems.* Eds. Bolsinov, A.V., Fomenko, A.T., and Oshemkov, A.A.; Cambridge Scientific Publishers, Springer, pp. 173–214. http://arxiv.org/abs/1212.4302.

4. Kudryavtseva, E. A. 2009, "Uniform Morse lemma and isotope Morse functions on surfaces", *Moscow Univ. Math. Bull.*, vol. 64, no. 4, pp. 150–158.

5. Kudryavtseva, E. A. & Permyakov, D. A. 2010, "Framed Morse functions on surfaces", *Sbornik Math.*, vol. 201, no. 4, pp. 501–567.

6. Kudryavtseva, E. A. 2012, "The Topology of Spaces of Morse Functions on Surfaces", *Math. Notes.*, vol. 92, no. 2, pp. 219–236. http://arxiv.org/abs/1104.4792.

7. Kudryavtseva, E. A. 2012, "Special framed Morse functions on surfaces", *Moscow Univ. Math. Bull.*, vol. 67, no. 4, pp. 151–157. http://arxiv.org/abs/1106.3116.

8. Kudryavtseva, E. A. 2013, "On the homotopy type of spaces of Morse functions on surfaces", *Sb. Math.*, vol. 204, no. 1, pp. 75–113. http://arxiv.org/abs/1104.4796.

9. Kudryavtseva, E. A. 2016, "Topology of the spaces of functions with prescribed singularities on surfaces", *Doklady Math.*, vol. 93, no. 3, pp. 264–266.

10. Kudryavtseva, E. A. 2021,"Topology of the spaces of functions and gradient-like flows with prescribed singularities on surfaces", Available at http://arxiv.org/abs/2106.03017.

11. Mather, J. N. 1969, "Infinitesimal stability implies stability", *Ann. of Math.*, vol. 89, pp. 254–291.

12. Samoilenko, A. M. 1968, "The equivalence of a smooth function to a Taylor polynomial in the neighborhood of a finite-type critical point", *Funct. Anal. Appl.*, vol. 2, no. 4, pp. 318–323.

13. Sergeraert, F. 1972, "Un théorème de fonctions implicites sur certains espaces de Fréchet et quelques applications", *Annales scientifiques de l'É.N.S. 4e série.*, vol. 5, no. 4, pp. 599–660.

14. Tougeron, J. C. 1968, "Ideaux de fonctions differentiables. I", *Ann. Inst. Fourier.*, vol. 18, no. 1, pp. 177–240.

Получено: 08.09.2022
Принято в печать: 22.12.2022