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Формула Карлемана в матричных областях Зигеля

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Аннотация

Верхняя полуплоскость не является ограниченной областью, но формулы Карлемана для нее играют важную роль в дальнейшем изложении. В данной работе найдена формула Карлемана для матричных областях Зигеля.

Ключевые слова: Классические области, Формула Карлемана, граница Шилова, ядро Коши, матричный единичный диск, область Зигеля.

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Carleman's formula for the matrix domains of Siegel

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Abstract

The domain of Siegel first type is not a bounded domain, but Carleman's formulas for it play an important role in the further presentation. In this paper, the Carleman formula for the Siegel domain is found.

Keywords: Classical domains, Carleman's formula, Shilov boundary, Cauchy kernel, matrix unit disc, Siegel domain.

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1. Introduction, preliminaries and problem statement

Integral representations of holomorphic functions play an important role in the classical theory of functions of one complex variable and in multidimensional complex analysis. They solve the classical problem of restoring at points in the domain D a holomorphic function that behaves quite well when approaching the ∂D boundary, by its values on ∂D or on the $S-$Shilov boundary. Along with this classical problem, we can naturally consider the following: to restore a holomorphic function in D by its values on a certain set $M \subset \partial D$ that does not contain S. Of course, M must be the uniqueness set for the class of holomorphic functions under consideration.

The first result in the direction of solving such a problem was obtained by T. Carleman in 1926 for the domain $D \subset \mathbb{C}$ of one special form [1]. His idea of introducing a «quenching» function into the Cauchy integral formula was developed by G. M. Goluzin and V. I. Krylov in 1933 in relation to simply connected flat domains [2]. Their method provided for the construction of some auxiliary holomorphic function depending on the set M, which was possible for simply connected domains $D \subset \mathbb{C}$, but, generally speaking, it is no longer possible for multi-connected regions in $\mathbb{C}$ or for domains in $\mathbb{C}^n$, $n > 1$.

In 1935, E. Cartan proved that there are only six possible types of irreducible, homogeneous, bounded, symmetric domains. Of these, $\Re_I, \Re_{II}, \Re_{III}$ and $\Re_{IV}$ are called classical domains (see [3]):

$$\Re_I = \left\{ Z \in \mathbb{C}\left[m \times k\right] : I^{(m)} - Z\overline{Z}' > 0 \right\},$$

$$\Re_{II} = \left\{ Z \in \mathbb{C}\left[m \times m\right] : I^{(m)} - Z\overline{Z} > 0, \quad \forall Z' = Z \right\},$$

$$\Re_{III} = \left\{ Z \in \mathbb{C}\left[m \times m\right] : I^{(m)} + Z\overline{Z} > 0, \quad \forall Z' = -Z \right\},$$

$$\Re_{IV} = \left\{ z \in \mathbb{C}^n : \left|zz'\right|^2 + 1 - 2\overline{z}z' > 0, \quad \left|zz'\right| < 1 \right\}.$$

The dimensions of these regions are $mk, m(m+1)/2, m(m-1)/2, n$, respectively.

All these domains are biholomorphically nonequivalent, so the complex analysis for them is constructed differently.

For symmetric domains, the Carleman formulas are obtained in [4]. The proofs from [4] are not transferred to Siegel domains, since there is no point in the Siegel domain through which a sufficiently powerful family of complex lines can be drawn that intersect the backbone of D along a curve (in symmetric domains, such a point is point 0). Therefore, in the work [5], the value of the holomorphic function is restored not in the domain, but on the backbone D.

Currently, the study of integral representations of holomorphic functions and their applications in matrix balls associated with the classical domains mentioned above has become one of the topical issues. In [6], holomorphic automorphisms for a matrix ball of the first type are described. Integral formulas for the matrix ball of the second type were obtained by G. Khudayberganov and

Z. Matyakubov [7], [8], and the third type were studied by G. Khudayberganov, U. Rakhmonov and integral formulas [9], [10] were found.

In [11], the volumes of a matrix ball of the third type and a generalized Lie ball are calculated. The full volumes of these regions are necessary to find the kernels of integral formulas for these regions (Bergman, Cauchy-Seguet, Poisson kernels, etc. (see [8], [12])). In [13], holomorphic and pluriharmonic functions for classical domains of the first Cartan type were determined, and the Laplace and Hua Lo-Ken operators were also studied. Moreover, a relationship was established between these operators.

In this paper [20], prove a criterion for plurisubharmonic functions in terms of integral mean by complex ellipsoids. Moreover, by using the criterion, prove an analogue of Blaschke–Privalov theorem for plurisubharmonic functions. In [21] is discussed the problem of the holomorphic extendability of a function to a matrix ball, given on a piece of its skeleton. For this purpose complete orthonormal systems in the matrix ball are used. In this paper [22] is to find optimal estimates for the Bergman kernels for the classical domains. In this paper [23], the automorphisms of the matrix ball associated with the classical domains of the second type are described, and also the properties of the second type matrix ball $B^{(2)}_{m,n}$ are studied.

In this paper, the Carleman formula for the Siegel domain is found.

Consider the space m^2 of complex variables, denoted by $\mathbb{C}^{m^2}$. In some questions, it is convenient to represent the points Z of this space in the form of square $[m\times m]$ - matrices, i.e. in the form of $Z=(z_{ij})^m_{i,j=1}$. With this representation of points, the space $\mathbb{C}^{m^2}$ will be denoted by $\mathbb{C}[m\times m]$.

The domain of Siegel first type is not a bounded domain, but Carleman's formulas for it play an important role in the further presentation. In this paper, we consider Carleman formulas in Siegel matrix domains.

Let $\Re_{II}$ a classical domain of the second type according to the classification of E. Cartan, defined as a set

$$\Re_{II}=\left\{Z\in\mathbb{C}[m\times m]:\ I-Z\overline{Z}>0\right\},$$

where Z of a symmetric matrix of order m ($I-$ unit $[m\times m]-$matrix). The boundary $\Re_{II}$ consists of a set

$$\partial\Re_{II}=\left\{Z\in\mathbb{C}[m\times m]:\ \det(I-Z\overline{Z})=0,\quad I-Z\overline{Z}\geqslant 0\right\},$$

that is, from the set of matrices Z, for which the matrix $I-Z\overline{Z}$ is a nonnegatively definite, but not positively definite Hermitian matrix (its eigenvalues are nonnegative and at least one of them is zero). On the border there is a set of

$$S_{II}=\left\{Z\in\mathbb{C}[m\times m]:\ Z\overline{Z}=I\right\},$$

which is called the skeleton of $\Re_{II}$ (note that S_{II} is the Shilov boundary for $\Re_{II}$ (see [14, page 95])). It is clear that S_{II} is the set of all unitary $[m\times m]-$ matrices (the set of unitary matrices of order m is usually referred to the $U(m)$). It should be noted that the set of matrices

$$\left\{Z:\det(I-Z\overline{Z})=0\right\}$$

contains a limited component distinguished by the condition $I-Z\overline{Z}\geqslant 0$, and an unlimited for $I-Z\overline{Z}\leqslant 0$. These components intersect in the skeleton S_{II}.

Let the set be $M\subset S_{II}$ and $\mu(M)>0$, where μ is the normalized Lebesgue measure by S_{II}.

We parametrize S_{II} in the following way: $U=e^{i\phi}u$, $0\leqslant\phi\leqslant 2\pi$, $u\in SU(m)$, where $SU(m)$ is a group of special unitary matrices, i.e. $\det u=1$. Since $\det U=e^{im\phi}\det u=e^{im\phi}$, the set $\{U:U=\lambda u,\ |\lambda|=1\}$, $u\in SU(m)$ intersects the set of elements of the group $SU(m)$ at exactly m roots of unity $e^{im\phi}=1$.

Lemma 1 (see. [14]). *Haar's measure $d\mu$ of the manifold S_{II} can be written as $d\mu=h(u)d\varphi$ $d\mu_0(u)$, where $d\mu_0$ is a normalized Lebesgue measure on $SU(m)$, and h is a smooth positive function on $SU(m)$.*

We introduce the set

$$M_{0,u} = \left\{U : U \in M,\, U = \lambda u,\ \lambda = e^{i\varphi}\,,\ 0 \leqslant \varphi \leqslant 2\pi\right\}\,,\, u \in SU(m),$$

$$M_0^{'} = \{u :\ u \in SU(m)\,,\, m_1 M_{0,u} > 0\}\,.$$

where m_1 is Lebesgue measure. By Fubini's theorem $\mu_0(M_0^{'}) > 0$. We denote

$$\psi_0(U) = \frac{1}{2\pi i}\int\limits_{M_{0,u}} \frac{\eta+\lambda}{\eta-\lambda}\frac{d\eta}{\eta}\quad,\quad \varphi_0 = \exp\psi_0\ .$$

Hardy class $H^1(\Re_{II})$ consists of all functions f that are holomorphic in the domain $\Re_{II}$ for which

$$\|f\|_{H^1} = \sup_{0<r<1}\int\limits_{S_{II}} |f\,(rZ)|\, d\mu < \infty,$$

here $rZ = (rz_{11},\, rz_{12}, ...,\, rz_{mm})$.

Lemma 2 (see. [14]). Let $f \in H^1(\Re_{II})$. Then the following formula is valid

$$f(0) = \frac{m}{\int\limits_{M_0^{'}} d\mu_1}\lim_{j\to\infty}\int\limits_{M} f(U)\left[\frac{\varphi_0(U)}{\varphi_0(0)}\right]^j d\mu_U.$$

Let $\varphi_A(Z)$ is an automorphism of $\Re_{II}$, which transforms point A to 0 (see [3]). We denote

$$\mu_A(K) = \mu_1({\varphi_A}^{-1})(K)),$$

$$M_{A,\omega} = \{U : U \in M,\, U = {\varphi_A}^{-1}\left(\lambda{\varphi_A}^{-1}(\omega)\right)\},\ |\lambda| = 1,\ \omega \in S_A = \varphi_A(SU(m)),$$

$$M_A^{'} = \{\omega :\ \omega \in S_A\,,\, m_1 M_{A,\omega} > 0\},$$

$$\psi_A(U) = \frac{1}{2\pi i}\int\limits_{M_{A,\omega}} \frac{\eta+\lambda}{\eta-\lambda}\frac{d\eta}{\eta}\quad,\quad \varphi_A = \exp\psi_A,$$

(ψ_A depends on U, because λ and ω are functions U).

Theorem 1 (see. [4]). *Let $f \in H^1(\Re_{II})$. Then for any point $A \in \Re_{II}$ the following Carleman's formula holds*

$$f(A) = \frac{m}{\int\limits_{M_A^{'}} d\mu_A}\lim_{j\to\infty}\int\limits_{M} f(U)\left[\frac{\varphi_A(U)}{\varphi_A(A)}\right]^j H(A,\overline{U})d\mu_U,$$

where $H(A,\overline{U})-$ is Cauchy kernel for the classical domain of the second type.

2. Main results

Matrix upper half-plane is the domain (Siegel)

$$D_{II} = \{W \in \mathbb{C}[m\times m] :\ \operatorname{Im} W > 0\}$$

where $W = \|w_{jk}\|$, $(j,k = 1, ..., m)$-symmetric matrix of order m, the elements of which are complex numbers of the $\mathbb{C}$, where $\operatorname{Im} W$ is defined as

$$\operatorname{Im} W = \frac{1}{2i}(W - \overline{W}).$$

Obviously, the matrix ImW is Hermitian: its elements $h_{jk} = \frac{1}{2i}(w_{jk} - \overline{w}_{jk})$ satisfy the conditions $\overline{h}_{jk} = h_{kj}$, and in particular, $h_{jj} = \operatorname{Im} w_{jj}$ are real. The inequality $H > 0$ for Hermitian matrix H means that it is positive definite, i.e. all its eigenvalues are positive.

On ∂D_{II} we define a set

$$\Gamma_{II} = \{W \in \mathbb{C}[m \times m] : \ \operatorname{Im} W = 0\},$$

which is called a skeleton of the domain D_{II}. It consists of all Hermitian matrices of order m.

Similarly to Lemma 10.1 and Theorem 10.3 from [16], the following is proved

Lemma 3. *Transform $W = \Phi(Z)$ (Cayley transform), where*

$$W = \Phi(Z) = i(I + Z)(I - Z)^{-1}, \tag{1}$$

is a biholomorphic maps $\Re_{II}$ to D_{II}, while S_{II} goes to Γ_{II}.

Proof. First, we prove that for $Z \in \Re_{II}$, the matrix $I - Z$ is non-degenerate. First of all, we note that the condition $I - Z\overline{Z} > 0$ implies $Z \neq 0$.

Let $w - m$ be a dimensional column vector and

$$(I - Z)w = 0.$$

Then $w = Zw$, from where $w^* = w^*\overline{Z}$. Further,

$$w^*w = w^*\overline{Z}Zw \ \text{ and } \ w^*(I - \overline{Z}Z)w = 0.$$

Since the conditions

$$I - Z\overline{Z} > 0 \quad \text{and} \ \ I - \overline{Z}Z > 0$$

are equivalent (see [3]), then the left part of the last equality for $w \neq 0$ is positive definite. Hence, $w = 0$. This means the non-degeneracy of the matrix $I - Z$, i.e. the existence of $(I - Z)^{-1}$ and the holomorphy of the map Φ in the domain $\Re_{II}$.

Next, from (1) we find:

$$\operatorname{Im} W = \frac{1}{2i}(W - W^*) = \frac{1}{2i}\left[i(I + Z)(I - Z)^{-1} + i(I - \overline{Z})^{-1}(I + \overline{Z})\right] =$$

$$= \frac{1}{2}(I - \overline{Z})^{-1}\left[(I - \overline{Z})(I + Z) + (I + \overline{Z})(I - Z)\right](I - Z)^{-1} =$$

$$= \frac{1}{2}(I - \overline{Z})^{-1}\left[I + Z - \overline{Z} - \overline{Z}Z + I - Z + \overline{Z} - \overline{Z}Z\right](I - Z)^{-1} =$$

$$= (I - \overline{Z})^{-1}\left[I - \overline{Z}Z\right](I - Z)^{-1}.$$

It can be seen from this that for non-degeneracy $I - Z$, Hermitian matrices

$$\operatorname{Im} W \ \text{ and } \ I - \overline{Z}Z$$

simultaneously positively defined. We proved that F maps $\Re_{II}$ to D_{II}.

From (1) we can find the inverse mapping $Z = \Phi^{-1}(W)$:

$$Z = (W + iI)^{-1}(W - iI). \tag{2}$$

For $W \in D_{II}$, the matrix $W + iI$ is non-degenerate (this is proved as above). Therefore, the mapping Φ^{-1} is holomorphic in D_{II}. It can also be seen from (1) that for $W \in D_{II}$, the matrix $I - Z$ is non-degenerate, therefore, $I - Z\overline{Z} > 0$, i.e. Φ^{-1} maps D_{II} to $\Re_{II}$.

So, the mapping Φ biholomorphically maps $\Re_{II}$ to D_{II}, and it is clear from (2) that it converts S_{II} to Γ_{II}.

The lemma is proved.

Using the transformation of the classical domain of the second type Φ and the automorphism Φ_A that converts the point $A \in \Re_{II}$ to 0 (0−zero matrix of order m) (see [3]), we define the following transformation

$$\Psi_B = \Phi \circ \Phi_A \circ \Phi^{-1},\ B = \Phi(A)$$

which is an automorphism of the domain D_{II}, transforming point B of D_{II} to the point iI.

Let $\dot{U}$ element of the volume S_{II}, a $\dot{V}$ element of the volume in Γ_{II}. In [3] is proved the following relation between $\dot{U}$ and $\dot{V}$ under the mapping Φ:

$$\dot{U} = 2^{\frac{m(m+1)}{2}} (\det(V^2 + I))^{-\frac{m+1}{2}} \dot{V}, \tag{3}$$

where $V \in \Gamma_{II}$. Since $V^* = V$ and

$$\det(V^2 + I) = \det(V - iI) \det(V + iI) =$$

$$= \overline{\det(V + iI)} \det(V + iI) = |\det(V + iI)|^2,$$

(3) can be written as

$$\dot{U} = 2^{\frac{m(m+1)}{2}} |\det(V + iI)|^{-(m+1)} \dot{V}. \tag{4}$$

To consider multidimensional analogues of Carleman formulas, it is desirable to expand the class of functions for which these formulas are true in the matrix upper half-plane D_{II} (Siegel domain).

The class of holomorphic functions in D_{II} we denote as $A(D_{II})$. Let $f \in A(D_{II})$ and

$$\frac{f\,(W)}{\det^2 (W + iI)} \in H^1 (\Gamma_{II}), \tag{5}$$

i.e. by definition, there is such a $C_1 > 0$, what, the relation is fulfilled

$$\lim_{r \to 1-0} \int\limits_{\Gamma_{II}} \left| \frac{f\,(rW)}{\det^2 (rW + iI)} \right| d\mu_V =$$

$$= \int\limits_{\Gamma_{II}} \left| \frac{f\,(V)}{\det^2 (V + iI)} \right| d\mu_V < C_1 < +\infty. \tag{6}$$

Now, using the mapping (1) and the relations (3) and (4), we get the following equalities:

$$\int\limits_{S_{II}} \left| f \left(i\,(I + U)\,(I - U)^{-1} \right) \right| d\mu_U =$$

$$= 2^{\frac{m(m+1)}{2}} \int\limits_{\Gamma_{II}} \left| f\,(V)\, |\det(V + iI)|^{-(m+1)} \right| d\mu_V =$$

$$= 2^{\frac{m(m+1)}{2}} \int\limits_{\Gamma_{II}} \left| \frac{f\,(V)}{\det^2 (V + iI)} \cdot |\det(V + iI)|^{-m+1} \right| d\mu_V. \tag{7}$$

By the relation (6) the integral of the fraction of the first the factor is bounded and since there is a $V + iI$-non-degenerate matrix under integral, i.e. (7) available then converges and integral (7) is bounded.

It follows from this that there is $C_2 > 0$,

$$\int\limits_{S_{II}} \left| f\left(i\,(I+U)\,(I-U)^{-1}\right)\right| d\mu_U < C_2,$$

i.e.

$$f\left(i\,(I+Z)\,(I-Z)^{-1}\right) \in H^1\left(S_{II}\right). \tag{8}$$

Therefore, for (8) is true if and only if (5) is true. Note that the scalar version of the above relation is presented in [15, p. 147].

Theorem 2. *If the function $f \in A(D_{II})$ satisfies the condition (5) and the set $\tilde{M} \in \partial D_{II}$ has positive Lebesgue measure, then the following Carleman's formula is true*

$$f(W) = \frac{\det^{\frac{m+1}{2}}(W+iI)}{i^{\left(\frac{m+1}{2}\right)^2}}\times$$

$$\times \lim_{j\to\infty}\int\limits_{\tilde{M}} f(V)\left[\frac{\tilde{\varphi}(V)}{\tilde{\varphi}(W)}\right]^j \frac{d\mu_V}{\det^{\frac{m+1}{2}}(\overline{V}-W)\det^{\frac{m+1}{2}}(V+iI)}, \tag{9}$$

where the limit is uniform on compact subsets ∂D_{II}, and $V \in \tilde{M}$.

Proof. Let $F(Z) = f(i(I+Z)(I-Z)^{-1})$, then $F(Z) \in H^1(\Re_{II})$ and the Carleman formula is valid

$$F(Z) = \lim_{j\to\infty}\int\limits_{M} F(U)\left[\frac{\varphi(U)}{\varphi(Z)}\right]^j \frac{d\mu_U}{\det^{\frac{m+1}{2}}(I-ZU^*)},$$

where $M-$the image of $\tilde{M}$ when mapping $Z = (W+iI)^{-1}(W-iI)$ of the Siegel domain to a classical domain of the second type.

Next, consider the inverse mapping on (1)

$$Z = (W+iI)^{-1}(W-iI), \quad U = (V+iI)^{-1}(V-iI),$$

and make the following calculations:

$$I - ZU^* = I - (W+iI)^{-1}(W-iI)(\overline{V}+iI)(\overline{V}-iI)^{-1} =$$

$$= (W+iI)^{-1}\left[(W+iI)(\overline{V}-iI) - (W-iI)(\overline{V}+iI)\right](\overline{V}-iI)^{-1} =$$

$$= (W+iI)^{-1}\left[W\overline{V} - iW + i\overline{V} + I - W\overline{V} - iW - i\overline{V} - I\right](\overline{V}-iI)^{-1} =$$

$$= 2i(W+iI)^{-1}\left[\overline{V}-W\right](\overline{V}-iI)^{-1},$$

and the condition (3) holds

$$d\mu_U = 2^{\frac{m(m+1)}{2}}|\det(V+iI)|^{-(m+1)}d\mu_V.$$

Calculations show that

$$\frac{d\mu_U}{\det^{\frac{m+1}{2}}(I-ZU^*)} = \frac{\det^{\frac{m+1}{2}}(W+iI)\det^{\frac{m+1}{2}}(\overline{V}-iI)}{(2i)^{\frac{m(m+1)}{2}}\det^{\frac{m+1}{2}}(\overline{V}-W)}\cdot\frac{2^{\frac{m(m+1)}{2}}d\mu_V}{\left|\det^{\frac{m+1}{2}}(V+iI)\right|^2} =$$

$$= \frac{\det^{\frac{m+1}{2}}(W+iI)}{i^{\frac{m(m+1)}{2}}\det^{\frac{m+1}{2}}(\overline{V}-W)\det^{\frac{m+1}{2}}(V+iI)}d\mu_V.$$

Next, φ plays the role of $\tilde{\varphi}$ for the set M. By the theorem of M. A. Lavrentiev (see [17]) M is a set of also positive Lebesgue measure, such that the harmonic measure M passes into the harmonic measure $\tilde{M}$, therefore, φ will pass into $\tilde{\varphi}$, and we come to the formula (9).

The theorem is proved.

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. T.Carleman, *Les fonctions quasi analytiques*, Paris: Gauthier-Villars (1926), pp. 3–6.

2. G. M. Golusin, W. J. Krylow, *Verallgemeinerung einer Formel von Carleman und ihre Anwendung zur analytischen Fortsetzung*, Mat. Sb., 40:2 (1933), 144–149.

3. Хуа Л. К. *Гармонический анализ функций многих комплексных переменных в классических областях // М.: ИЛ*, 1959. 163 с.

4. А. М. Кытманов, Т. Н. Никитина, *Аналоги формулы Карлемана для классических областей*, Матем. заметки, 45:3 (1989), 87–93.

5. А. М. Кытманов, Т. Н. Никитина, *Многомерные формулы Карлемана в областях Зигеля*, Изв. вузов. Матем., 1990, 3, 44–49.

6. С. Косбергенов, *О формуле Карлемана для матричного шара*, Изв. вузов. Матем., 1999, 1, 76–79.

7. G.Khudayberganov, U.S.Rakhmonov, Z.Q.Matyakubov, *Integral formulas for some matrix domains*, Contemporary Mathematics, AMS, Volume 662, pp. 89-95.(2016).

8. G.Khudayberganov, U.S.Rakhmonov, *The Bergman and Cauchy-Szego kernels for matrix ball of the second type*, Journal of Siberian Federal University. Mathematics and Physics 7:3, pp. 305-310.(2014).

9. G.Khudayberganov, B.P.Otemuratov, U.S.Rakhmonov, *Boundary Morera theorem for the matrix ball of the third type*, Journal of Siberian Federal University. Mathematics and Physics, 11:1, 40-45.(2018).

10. G.Khudayberganov, U.S.Rakhmonov, *Carleman Formula for Matrix Ball of the Third Type*, Algebra, Complex Analysis, and Pluripotential Theory. USUZCAMP 2017. Springer Proceedings in Mathematics and Statistics vol. 264, pp. 101-108, Springer, Cham.(2017).

11. U. S. Rakhmonov, J. Sh. Abdullayev, *On volumes of matrix ball of third type and generalized Lie balls*, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 29:4 (2019), 548–557

12. G. Khudayberganov, J. Abdullayev, *Relationship between the Kernels Bergman and Cauchy-Szegő in the domains* $\tau^+(n-1)$ *and* $\Re^n_{IV}$, Journal of Siberian Federal University. Mathematics & Physics, **13:5**, 559-567(2020).

13. G.Khudayberganov, A.M.Khalknazarov, J.Sh.Abdullayev, *Laplace and Hua Luogeng operators*, Russian Math. (Iz. VUZ), 64:3 (2020), 66–71.

14. Л. А. Айзенберг, *Формулы Карлемана в комплексном анализе*, Новосибирск: Наука. 1990. - 248 с.

15. П. Кусис, *Введение в теорию пространств* H^p . М.: Мир. 1984. 368 с.

16. Худайберганов, А. М. Кытманов, Б. А. Шаимкулов, *Анализ в матричных областях*, Монография. Красноярск: Сибирский федеральный ун-т, 2017. – 292 с.

17. И. И. Привалов, *Граничные свойства аналитических функций*, М.: Гостехиздат, 1950. – 338 с.

18. Jonibek Sh. Abdullayev. *An analogue of Bremermann's theorem on finding the Bergman kernel for the Cartesian product of the classical domains* $\Re_I(m,k)$ *and* $\Re_{II}(n)$, Bul. Acad. Stiinte Repub. Mold. Mat., 2020, no. 3, 88–96.

19. G.Khudayberganov, J.Sh.Abdullayev. *The boundary Morera theorem for domain* $\tau^+(n-1)$, Ufimsk. Mat. Zh., 13:3 (2021), pp. 196–210.

20. K.Rakhimov, Sh.Shopulatov. *A mean value criterion for plurisubharmonic functions*, Complex Variables and Elliptic Equations, (2021) DOI:10.1080/17476933.2021.1954623.

21. G.Khudayberganov, J.Sh.Abdullayev. *Holomorphic continuation into a matrix ball of functions defined on a piece of its skeleton,* Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki, 2021, vol. 31, issue 2, pp. 296–310.

22. J.Sh.Abdullayev. *Estimates the Bergman kernel for classical domains E. Cartan's*, Chebyshevskii sbornik, 2021, vol. 22, no. 3, pp. 21–32.

23. Uktam S. Rakhmonov, Jonibek Sh. Abdullayev, *On properties of the second type matrix ball* $B_{m,n}^{(2)}$ *from space* $\mathbb{C}^n[m \times m]$, J. Sib. Fed. Univ. Math. Phys., 15:3 (2022), pp. 329–342.

REFERENCES

1. T.Carleman, *Les fonctions quasi analytiques*, Paris: Gauthier-Villars (1926), pp. 3–6.

2. G. M. Golusin, W. J. Krylow, *Verallgemeinerung einer Formel von Carleman und ihre Anwendung zur analytischen Fortsetzung*, Mat. Sb., 40:2 (1933), 144–149.

3. Hua Luogeng, *Harmonic analysis of functions of several complex variables, in classical domains*, Moscow, IL, 1963 (in Russian).

4. A.M.Kytmanov, T.N.Nikitina, *Analogs of Carleman's formula for classical domains*, Math. Notes, 45:3 (1989), 243–248.

5. A.M.Kytmanov, T.N.Nikitina, *Multidimensional Carleman formulas in Siegel domain*, Soviet Math. (Iz. VUZ), 34:3 (1990), 50–56.

6. S.Kosbergenov, *On the Carleman formula for a matrix ball*, Russian Math. (Iz. VUZ), 43:1 (1999), 72–75.

7. G.Khudayberganov, U.S.Rakhmonov, Z.Q.Matyakubov, *Integral formulas for some matrix domains*, Contemporary Mathematics, AMS, Volume 662, pp. 89-95.(2016).

8. G.Khudayberganov, U.S.Rakhmonov, *The Bergman and Cauchy-Szego kernels for matrix ball of the second type*, Journal of Siberian Federal University. Mathematics and Physics 7:3, pp. 305-310.(2014).

9. G.Khudayberganov, B.P.Otemuratov, U.S.Rakhmonov, *Boundary Morera theorem for the matrix ball of the third type*, Journal of Siberian Federal University. Mathematics and Physics, 11:1, 40-45.(2018).

10. G.Khudayberganov, U.S.Rakhmonov, *Carleman Formula for Matrix Ball of the Third Type*, Algebra, Complex Analysis, and Pluripotential Theory. USUZCAMP 2017. Springer Proceedings in Mathematics and Statistics vol. 264, pp. 101-108, Springer, Cham.(2017).

11. U. S. Rakhmonov, J. Sh. Abdullayev, *On volumes of matrix ball of third type and generalized Lie balls*, Вестн. Удмуртск. ун-та. Матем. Мех. Компьют. науки, 29:4 (2019), 548–557

12. G. Khudayberganov, J. Abdullayev, *Relationship between the Kernels Bergman and Cauchy-Szegő in the domains* $\tau^+(n-1)$ *and* $\Re^n_{IV}$, Journal of Siberian Federal University. Mathematics & Physics, **13:5**, 559-567(2020).

13. G.Khudayberganov, A.M.Khalknazarov, J.Sh.Abdullayev, *Laplace and Hua Luogeng operators*, Russian Math. (Iz. VUZ), 64:3 (2020), 66–71.

14. L. A. Aizenberg, *Carleman Formulas in complex analysis*, Novosibirsk, Nauka, 1990 (in Russian).

15. P. Kusis, *Introduction to the theory of spaces* H^p . M.: Мир. 1984. 368 с.

16. G. Khudayberganov, A. M. Kytmanov, B. A. Shaimkulov, *Analysis in matrix domains*, Monograph. Krasnoyarsk, Siberian Federal University, 2017 (in Russian).

17. I. I. Privalov, *Boundary properties of analytic functions*, M.: Gostekhizdat, 1950. – 338 c.

18. Jonibek Sh. Abdullayev. *An analogue of Bremermann's theorem on finding the Bergman kernel for the Cartesian product of the classical domains* $\Re_I(m,k)$ *and* $\Re_{II}(n)$, Bul. Acad. Stiinte Repub. Mold. Mat., 2020, no. 3, 88–96.

19. G. Khudayberganov, J. Sh. Abdullayev. *The boundary Morera theorem for domain* $\tau^+(n-1)$, Ufimsk. Mat. Zh., 13:3 (2021), pp. 196–210.

20. K. Rakhimov, Sh. Shopulatov. *A mean value criterion for plurisubharmonic functions*, Complex Variables and Elliptic Equations, (2021) DOI:10.1080/17476933.2021.1954623.

21. G. Khudayberganov, J. Sh. Abdullayev. *Holomorphic continuation into a matrix ball of functions defined on a piece of its skeleton,* Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki, 2021, vol. 31, issue 2, pp. 296–310.

22. J. Sh. Abdullayev. *Estimates the Bergman kernel for classical domains E. Cartan's*, Chebyshevskii sbornik, 2021, vol. 22, no. 3, pp. 21–32.

23. Uktam S. Rakhmonov, Jonibek Sh. Abdullayev, (2022), *On properties of the second type matrix ball* $B^{(2)}_{m,n}$ *from space* $\mathbb{C}^n[m\times m]$, J. Sib. Fed. Univ. Math. Phys., 15:3, pp. 329–342.

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