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О w-сверхразрешимости конечной группы, факторизуемой взаимно перестановочными подгруппами[1]

Н. В. Артеменко, А. А. Трофимук

Артеменко Наталья Витальевна — Брестский государственный университет им. А. С. Пушкина (г. Брест, Беларусь).
e-mail: artemenkonatasha@outlook.com
Трофимук Александр Александрович — кандидат физико-математических наук, Брестский государственный университет им. А. С. Пушкина (г. Брест, Беларусь).
e-mail: alexander.trofimuk@gmail.com

Аннотация

Подгруппы A и B называются взаимно перестановочными, если A перестановочна с каждой подгруппой из B, а B перестановочна с каждой подгруппой из A. В статье получены достаточные условия w-сверхразрешимости группы $G = AB$, факторизуемой взаимно перестановочными сомножителями A и B. Кроме того, установлено строение w-сверхразрешимого корадикала такой группы.

Ключевые слова: конечная группа, w-сверхразрешимая группа, взаимно перестановочные подгруппы, $\mathfrak{F}$-корадикал.

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On the w-supersolubility of a finite group factorized by mutually permutable subgroups

N. V. Artemenko, A. A. Trofimuk

Artemenko Natalia Vital'evna — Brest State A. S. Pushkin University (Brest, Belarus).
Trofimuk Alexander Alexandrovich — candidate of physical and mathematical sciences, Brest State A. S. Pushkin University (Brest, Belarus).
e-mail: alexander.trofimuk@gmail.com

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Abstract

The subgroups A and B of a group G are called mutually permutable if A permutes with all subgroups of B and B permutes with all subgroups of A. The sufficient conditions of w-supersolubility of a group $G = AB$ that is factorized by two mutually permutable w-supersoluble subgroups A and B were obtained. Besides we found the construction of w-supersoluble residual of such group.

Keywords: finite group, w-supersoluble group, mutually permutable subgroups, $\mathfrak{F}$-residual.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [1, 2].

A. F. Vasil’ev, T. I. Vasil’eva and V. N. Tyutyanov in [3] proposed the following definition. A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G, if either $H = G$, or there is a chain subgroups

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = G, \ |H_i : H_{i-1}| \in \mathbb{P}, \ \forall i.$$

Besides, a group G is called w-*supersoluble* [3] (widely supersoluble), if every Sylow subgroup of G is $\mathbb{P}$-subnormal in G. Denote by w$\mathfrak{U}$ the class of all w-supersoluble groups. Note that $\mathfrak{U} \subset$ w$\mathfrak{U}$. Here $\mathfrak{U}$ is the class of all supersoluble groups. In [3, Theorem 2.7, Proposition 2.8] proved that w$\mathfrak{U}$ is a subgroup-closed saturated formation and every group from w$\mathfrak{U}$ has an ordered Sylow tower of supersoluble type. By [4, Theorem 1], $G \in$ w$\mathfrak{U}$ if and only if every metanilpotent (biprimary) subgroup of G is supersoluble.

In monograph [6, p. 149] presented the following definition: two subgroups A and B of a group G are said to be *mutually permutable* if A permutes with all subgroups of B and B permutes with all subgroups of A. Asaad and Shaalan established the supersolubility of a group $G = AB$ with mutually permutable subgroups A and B provided that B is nilpotent [7, Theorem 3.2] and in the case that the derived subgroup G' is nilpotent [7, Theorem 3.8].

In the present work, the structure of the w-supersoluble residual of the group $G = AB$ with mutually permutable w-supersoluble subgroups A and B is established. Besides we obtained some sufficient conditions for w-supersolubility of such groups.

2. Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel. A group whose chief factors have prime orders is called *supersoluble*.

Denote by $O_p(G)$, $F(G)$ and $\Phi(G)$ the greatest normal p-subgroup of G, the Fitting and Frattini subgroups of G respectively. We use E_{p^t} to denote an elementary abelian group of order p^t and Z_m to denote a cyclic group of order m. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$.

The monographs [1, 9] contain the necessary information of the theory of formations.

A class group $\mathfrak{F}$ is called a formation if the following statements is true:

(1) if $G \in \mathfrak{F}$ and $N \lhd G$, then $G/N \in \mathfrak{F}$.

(2) if $G/N_1 \in \mathfrak{F}$ and $G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$.

A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. A formation $\mathfrak{F}$ is called *hereditary* if, together with each group, $\mathfrak{F}$ contains all its subgroups. The formations of all nilpotent, abelian and groups with abelian Sylow subgroups are denoted by $\mathfrak{N}$, $\mathfrak{A}$ and $\mathcal{A}$, respectively.

Let $\mathfrak{F}$ be a formation. Recall that the $\mathfrak{F}$-residual of G, that is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$. We define $\mathfrak{X}\mathfrak{Y} = \{G \in \mathfrak{E} \mid G^{\mathfrak{Y}} \in \mathfrak{X}\}$ and call $\mathfrak{X}\mathfrak{Y}$ the *formation product* of $\mathfrak{X}$ and $\mathfrak{H}$. Here $\mathfrak{E}$ is the class of all finite groups.

If H is a subgroup of G, then $H_G = \bigcap_{x \in G} H^x$ is called *the core* of H in G. If a group G contains a maximal subgroup M with trivial core, then G is said to be *primitive* and M is its *primitivator*.

A simple check proves the following lemma.

Lemma 8. *Let $\mathfrak{F}$ be a saturated formation and G be a group. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all non-trivial normal subgroups N of G. Then G is a primitive group.*

Lemma 9 ([2, Theorem II.3.2]). *Let G be a soluble primitive group and M is a primitivator of G. Then the following statements hold:*

(1) $\Phi(G) = 1$*;*

(2) $F(G) = C_G(F(G)) = O_p(G)$ *and* $F(G)$ *is an elementary abelian subgroup of order* p^n *for some prime* p *and some positive integer* n*;*

(3) G *contains a unique minimal normal subgroup* N *and moreover,* $N = F(G)$*;*

(4) $G = F(G) \rtimes M$ *and* $O_p(M) = 1$.

Lemma 10 ([5, Proposition 2.2.8, Proposition 2.2.11]). *Let $\mathfrak{F}$ and $\mathfrak{H}$ be formations, K be normal in G. Then:*

(1) $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$*;*

(2) $G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$*;*

(3) *if* $\mathfrak{H} \subseteq \mathfrak{F}$*, then* $G^{\mathfrak{F}} \leqslant G^{\mathfrak{H}}$.

Lemma 11. *If $Y \leq X$ and $\mathfrak{F}$ is a hereditary formation, then $Y^{\mathfrak{F}} \leq X^{\mathfrak{F}}$.*

Proof. Since $X^{\mathfrak{F}}$ is normal in X, it follows that $YX^{\mathfrak{F}}$ is a subgroup of X. Then

$$Y/Y \cap X^{\mathfrak{F}} \simeq YX^{\mathfrak{F}}/X^{\mathfrak{F}} \in \mathfrak{F},$$

because $\mathfrak{F}$ is hereditary. Hence $Y^{\mathfrak{F}} \leq Y \cap X^{\mathfrak{F}} \leq X^{\mathfrak{F}}$. □

Lemma 12. *Let $G = AB$ be the mutually permutable product of* w*-supersoluble subgroups A and B. If N is a minimal normal subgroup of G, then both AN and BN are* w*-supersoluble.*

Proof. By [6, Theorem 4.1.15, Lemma 4.3.3(4)], G is soluble and $\{N \cap A, N \cap B\} = \{1, N\}$. If $N \leq A \cap B$, then $AN = A \in \mathrm{w}\mathfrak{U}$ and $BN = B \in \mathrm{w}\mathfrak{U}$. If $A \cap N = 1 = B \cap N$, then by [6, Lemma 4.3.9], $|N|$ is prime. Then $AN/N \simeq A \in \mathrm{w}\mathfrak{U}$ and by [10, Lemma 2.16], $AN \in \mathrm{w}\mathfrak{U}$. Similarly $BN \in \mathrm{w}\mathfrak{U}$

If $N \leq A$ and $B \cap N = 1$, then $AN = A \in \mathrm{w}\mathfrak{U}$. Let N be non-cyclic. Then $N \leq C_G(B)$ by [6, Lemma 4.3.3(5)]. Since $BN = B \times N$, we have $BN \in \mathrm{w}\mathfrak{U}$ by [3, Theorem 2.7]. Hence N is cyclic. Since $BN/N \simeq B \in \mathrm{w}\mathfrak{U}$, by [10, Lemma 2.16], $BN \in \mathrm{w}\mathfrak{U}$. Similarly for $N \leq B$ and $A \cap N = 1$. □

3. New sign of w-supersolubility

We recall that two subgroups A and B of a group G are said to be *mutually sn-permutable* if A permutes with all subnormal subgroups of B and B permutes with all subnormal subgroups of A. Some sufficient conditions for w-supersolubility of the group $G = AB$ with mutually sn-permutable w-supersoluble subgroups A and B were obtained in [11]. In particular, they proved that G is w-supersoluble, if $(|A/A^{\mathcal{A}}|, |B/B^{\mathcal{A}}|) = 1$. Besides, if A is nilpotent and B is w-supersoluble, then G

may be not w-supersoluble, see [11, Example 1]. However, this statement is true for the product of such mutually permutable subgroups A and B.

Theorem 1. *Let $G = AB$ be the mutually permutable product of* w*-supersoluble subgroups A and B. Then G is* w*-supersoluble in each of the following cases:*
(1) *B is nilpotent;*
(2) $(|G : AF(G)|, |G : BF(G)|) = 1$;
(3) *B is normal in G.*

Proof. We prove all three statements at the same time using induction on the order of G.

Since by [3, Proposition 2.8], every w-supersoluble group has an ordered Sylow tower of supersoluble type, then by [7, Corollary 3.6], G has an ordered Sylow tower of supersoluble type. Hence G is soluble.

If N is a non-trivial normal subgroup of G, then AN/N and BN/N are mutually permutable by [6, Lemma 4.1.10], $AN/N \simeq A/A \cap N$ and $BN/N \simeq B/B \cap N$ are w-supersoluble by [3, Theorem 2.7] ($BN/N \simeq B/B \cap N$ is nilpotent, $BN/N \simeq B/B \cap N$ is siding group).

Since $F(G)N/N \leq F(G/N)$, then $F(G)AN/N \leq F(G/N)AN/N$ and $|G/N : F(G/N)AN/N|$ divides $|G : F(G)A|$. Similarly $|G/N : F(G/N)BN/N|$ divides $|G : F(G)B|$.

Hence $|G/N : F(G/N)AN/N|$ and $|G/N : F(G/N)BN/N|$ are coprime.

By induction $G/N = (AN/N)(BN/N)$ is w-supersoluble and G is primitive by Lemma 8. Hence by Lemma 9, $\Phi(G) = 1$, $N = C_G(N) = F(G) = O_p(G)$ is a unique minimal normal subgroup of G. Besides $G = N \rtimes M$, $N = P$ is a Sylow p-subgroup of G for the greatest $p \in \pi(G)$. If N is cyclic, then $G \in \mathfrak{U} \subseteq \mathrm{w}\mathfrak{U}$. Hence N is non-cyclic.

1. By [6, Lemma 4.3.3(4)], $\{N \cap A, N \cap B\} = \{1, N\}$. Case $A \cap N = 1 = B \cap N$ is false, because N is the Sylow subgroup of G.

If $N \leq B$, then $N = B$, because $N = C_G(N)$ and N is the Sylow p-subgroup of G. By Lemma 12, $G = AB = AN$ is w-supersoluble.

If $N \nleq B$, then $B \cap N = 1$ and $N \leq A$. By [6, Lemma 4.3.3(5)], $N \leq C_G(B)$. Hence $B \leq C_G(N) = N$ and $G = AB = AN = A$, a contradiction.

2. By Lemma 12, $AF(G)$ and $BF(G)$ are w-supersoluble. Since P is normal in G and G is soluble, it follows that there exists a chain of subgroup with prime indices between P and G. Therefore, P is $\mathbb{P}$-subnormal in G. Let Q be a Sylow q-subgroup of G, $q \neq p$. By hypothesis, q is not divide $|G : F(G)A|$ or $|G : F(G)A|$. Hence $Q^x \leq F(G)A$ or $Q^y \leq F(G)B$ for some $x, y \in G$. Let $Q^x \leq F(G)A$. Then Q^x is $\mathbb{P}$-subnormal in $F(G)A$. Since B is soluble, B has a series $1 = B_0 < B_1 < \ldots < B_k < B_{k+1} < \ldots < B_n = B$ such that $|B_{k+1} : B_k|$ is prime. Because A and B are mutually permutable, we have $F(G)AB_k$ is subgroup for all k. Hence in a chain of subgroups

$$F(G)A \leq F(G)AB_1 \leq \ldots \leq F(G)AB_k \leq F(G)AB_{k+1} \leq \ldots \leq F(G)AB_n = G$$

the indices $|F(G)AB_{k+1} : F(G)AB_k| = |B_{k+1} : B_k|/|B_{k+1} \cap F(G)A) : B_k \cap F(G)A|$ divide the prime numbers. Hence $F(G)A$ is $\mathbb{P}$-subnormal in G. By [3, Lemma 1.4 (5,8)], Q is $\mathbb{P}$-subnormal in G and hence G is w-supersoluble, a contradiction.

3. Since N is minimal normal in G, it follows that $N \leq B$. By [6, Lemma 4.3.3(4, 5)], $A \cap N = 1$ and $N \leq C_G(A)$. Hence $G = AB = NB = B \in \mathrm{w}\mathfrak{U}$, because $A \leq C_G(N) = N$.

Let N be a minimal normal subgroup of G such that $N \leq B'$. If N is not contained in M, then $G = N \rtimes M$ and $|N|$ is prime. By [10, Lemma 2.16], G is w-supersoluble. Suppose that N is contained in M and N_1 is a subgroup of N of prime order such that N_1 is normal in M. Then N_1 is normal in B and therefore is normal in G. By [10, Lemma 2.16], G is w-supersoluble. The theorem is proved. □

4. The w-supersoluble residual of the group $G = AB$ with mutually permutable w-supersoluble subgroups A and B

In [12, Theorem 2.1], V.S. Monakhov proved for a group $G = AB$ with mutually permutable supersoluble subgroups A and B that $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$. The w-supersoluble version of this result is presented in Theorem 2.

In [13], the authors proved that if G is the mutually permutable product of the w-supersoluble subgroups A and B and $G^{\mathcal{A}}$ is nilpotent, then G is w-supersoluble. In first part of Theorem 2 we gave a new proof of this result without theory of formation function.

THEOREM 2. *Let $G = AB$ be the mutually permutable product of* w*-supersoluble subgroups A and B. Then $G^{\mathrm{w}\mathfrak{U}} = (G^{\mathcal{A}})^{\mathfrak{N}}$.*

PROOF.

Suppose that $G^{\mathcal{A}}$ is nilpotent. Then $(G^{\mathcal{A}})^{\mathfrak{N}} = 1$. Next we check that G is w-supersoluble.

We use induction on the order of G. Let N be a non-trivial normal subgroup of G. Since by Lemma 10,

$$(G/N)^{\mathcal{A}} = G^{\mathcal{A}}N/N \simeq G^{\mathcal{A}}/G^{\mathcal{A}} \cap N,$$

it follows that $(G/N)^{\mathcal{A}}$ is nilpotent and by induction, G/N is w-supersoluble.

Let $W = G^{\mathcal{A}}$. Since $\mathrm{w}\mathfrak{U}$ is saturated by [3, Theorem 2.7], we have that G is primitive by Lemma 8. Hence by Lemma 9, $\Phi(G) = 1$, $N = C_G(N) = F(G) = O_p(G) = W$ is a unique minimal normal subgroup of G. Besides, $G = N \rtimes M$, $M \in \mathcal{A}$, $N = P$ is a Sylow p-subgroup of G, where p is a greatest prime in $\pi(G)$. If N is cyclic, then $G \in \mathrm{w}\mathfrak{U}$ by [10, Lemma 2.16]. Hence N is non-cyclic.

By [6, Lemma 4.3.3(4)], $\{N \cap A, N \cap B\} = \{1, N\}$. If $A \cap N = 1 = B \cap N$, then we have a contradiction, since N is a Sylow subgroup of G. If $N \leq A$ and $B \cap N = 1$, then by [6, Lemma 4.3.3(5)], $N \leq C_G(B)$. Hence $B \leq C_G(N) = N$ and $G = AB = AN = A$, a contradiction. Similarly for $N \leq B$ and $A \cap N = 1$.

Hence in the future we consider that $N \leq A \cap B$. Let T be an arbitrary $\{q, r\}$-subgroup of G. By [6, Theorem 1.1.19], there are Hall $\{q, r\}$-subgroups $G_{\{q,r\}}$, $A_{\{q,r\}}$, $B_{\{q,r\}}$ in G, A and B respectively such that $G_{\{q,r\}} = A_{\{q,r\}}B_{\{q,r\}}$. Since A and B are mutually permutable, $A_{\{q,r\}}$ and $B_{\{q,r\}}$ are mutually permutable [6, Lemma 4.1.21]. If $G_{\{q,r\}}$ is a proper subgroup of G, then by Lemma 11, $G^{\mathcal{A}}_{\{q,r\}} \leq G^{\mathcal{A}}$ and $G_{\{q,r\}} \in \mathrm{w}\mathfrak{U}$ by induction. Then by [4, Theorem 1], T is supersoluble and $G \in \mathrm{w}\mathfrak{U}$, a contadiction.

Hence G is biprimary and M is an abelian Sylow q-subgroup for some $q \in \pi(G)$. By [14, Lemma I.1.3], M is cyclic. By Dedekind's identity, $M = (A \cap M)(B \cap M)$. Then $M = A \cap M$ or $M = B \cap M$. Suppose that $M = A \cap M$. Then $M \leq A$. Since $N \leq A$, we have that $G = NM \leq A$ and $G = A \in \mathrm{w}\mathfrak{U}$.

If G is w-supersoluble, then $G^{\mathrm{w}\mathfrak{U}} = 1$ and $G^{\mathcal{A}}$ is nilpotent by [3, Theorem 2.13]. Consequently $G^{\mathrm{w}\mathfrak{U}} = 1 = (G^{\mathcal{A}})^{\mathfrak{N}}$ and the statement is true.

Further, we assume that G is non-w-supersoluble and $G^{\mathcal{A}}$ is non-nilpotent. Since $\mathrm{w}\mathfrak{U} \subseteq \mathfrak{N}\mathcal{A}$, it follows that

$$G^{(\mathfrak{N}\mathcal{A})} = (G^{\mathcal{A}})^{\mathfrak{N}} \leqslant G^{\mathrm{w}\mathfrak{U}}$$

by Lemma 10 (2-3). Next we check the converse inclusion. For this we prove that $G/(G^{\mathcal{A}})^{\mathfrak{N}}$ is w-supersoluble. By Lemma 10 (1),

$$(G/(G^{\mathcal{A}})^{\mathfrak{N}})^{\mathcal{A}} = G^{\mathcal{A}}(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}} = G^{\mathcal{A}}/(G^{\mathcal{A}})^{\mathfrak{N}}$$

and $(G/(G^{\mathcal{A}})^{\mathfrak{N}})^{\mathcal{A}}$ is nilpotent. The quotients

$$G/(G^{\mathcal{A}})^{\mathfrak{N}} = (A(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}})(B(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}},$$

$$A(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}} \simeq A/A \cap (G^{\mathcal{A}})^{\mathfrak{N}},$$
$$B(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}} \simeq B/B \cap (G^{\mathcal{A}})^{\mathfrak{N}},$$

hence the subgroups $A(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}}$ and $B(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}}$ are w-supersoluble by [3, Theorem 2.7] and are mutually permutable by [6, Lemma 4.1.10]. As shown above, $G/(G^{\mathcal{A}})^{\mathfrak{N}}$ is w-supersoluble and $G^{\mathrm{w}\mathfrak{U}} \leq (G^{\mathcal{A}})^{\mathfrak{N}}$. $\square$

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