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Оценка меры для p-адических диофантовых приближений

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Аннотация

В статье получена количественная оценка меры множества p-адических чисел, для которых неравенство $|P(x)|_p < Q^{-w}$ при $w > 3n/2 + 2$ имеет решение в целочисленных полиномах P степени n и высоты $H(P)$, не превышающей $Q \in \mathbb{N}$.

Ключевые слова: метрические диофантовы приближения, p-адические числа, теорема Спринджука.

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Measure estimate for p-adic Diophantine approximation

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Abstract

A quantitative estimate for the measure of the set of p-adic numbers for which the inequality $|P(x)|_p < Q^{-w}$ for $w > 3n/2 + 2$ has a solution in integral polynomials P of degree n and of height $H(P)$ at most $Q \in \mathbb{N}$, is established.

Keywords: Metric Diophantine approximation, p-adic numbers, Sprindzuk theorem.

Bibliography: 16 titles.

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1. Introduction

Let $n \in \mathbb{N}$, and $p \in \mathbb{N}$ be a prime number. Given a polynomial $P(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{Z}[t]$ with $a_n \neq 0$, $\deg P = n$ is the degree of P and $H(P)$ is the height of P, i.e. $H(P) = \max_{0 \leqslant i \leqslant n} |a_i|$. Denote by $\mathcal{P}_{\leqslant n}$ the class of integer polynomials P of degree at most n and $\mathcal{P}_n$ the class of integer polynomials P of degree n. Throughout this paper $\mathbb{Q}_p$ denotes the p-adic field with p-adic metric $|\cdot|_p$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leqslant 1\}$ denotes the p-adic integers. A ball $K(a;r)$ in $\mathbb{Q}_p$ is defined as

$$K(a;r) = \{x \in \mathbb{Q}_p : |x-a|_p \leqslant r\}.$$

It has diameter $diam(K(a;r)) = r$ and measure $\mu(K(a;r)) = r$, where μ is the unique Haar measure on the locally compact abelian group $\mathbb{Q}_p$ such that $\mu(\mathbb{Z}_p) = 1$. Let $\mathbb{Q}_p^*$ be the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers. In what follows the Vinogradov symbols $\ll$ and $\gg$ will be used to avoid specifying unimportant constants ($f \ll g$ means that there exists a constant c such that $f \leqslant cg$ with a similar definition for $f \gg g$); if $f \ll g$ and $f \gg g$ then we write $f \asymp g$.

Given any $w \in \mathbb{R}^+$, denote by $L_n(w)$ the set of $x \in K$ for which the inequality

$$|P(x)|_p < H(P)^{-w} \tag{1}$$

has infinitely many solutions in polynomials $P \in \mathcal{P}_{\leqslant n}$. Regarding the set $L_n(w)$ Sprindzuk proved the following statement [16].

Theorem 1. *Let $w > n+1$. Then $\mu(L_n(w)) = 0$.*

There are several generalizations of Sprindzuk's result by replacing the RHS in (1) with a monotonically/non-monotonically decreasing function Ψ, see [2], [5]; by replacing the LHS in (1) with non-degenerate curves/non-degenerate manifolds in higher dimensions, see [1], [14], [15]; by considering simultaneous approximation, see [6],[7],[13]; by considering the inhomogeneous Diophantine approximation in the LHS in (1), see [3], [4].

Given a natural number $Q > 1$, consider the class of integral polynomials

$$\mathcal{P}_n(Q) = \{P \in \mathcal{P}_n : H(P) \leqslant Q\}.$$

Let $K = K(0;p^n) \subset \mathbb{Q}_p$ be a ball of diameter p^n centered at 0. For $w \in \mathbb{R}^+$ denote by $L_n(Q,w)$ the set of $x \in K$ for which the inequality

$$|P(x)|_p < Q^{-w} \tag{2}$$

has a solution in polynomials $P \in \mathcal{P}_n(Q)$. One of the consequences of the Sprindzuk's result is that $\mu(L_n(Q,w)) \to 0$ for $w > n+1$ as $Q \to \infty$. The main goal of this paper is to obtain a quantitative estimate for the Haar measure of the set $L_n(Q,w)$. The first significant contribution to obtaining an effective estimate for the set $L_n(Q,w)$ was made in [10], and it was shown that $\mu(L_n(Q,w)) \ll Q^{-(w-n-1)/n}\mu(K)$ for $w > n+1$.

We now state the main result of this paper, which consists in improving the known measure estimate for the set $L_n(Q,w)$ in the case when $w > 3n/2 + 2$.

Theorem 2. *Let $n \in \mathbb{N}$, $w \in \mathbb{R}^+$ with $w > 3n/2 + 2$. Then, for any positive real number ϵ and sufficiently large Q, we have*

$$\mu(L_n(Q,w)) \ll Q^{-(w-2)/n + n\epsilon}\mu(K)$$

where the constant implied by the Vinogradov symbol depends on n, w, p, ϵ and K.

Moreover, we expect that the following result to be true.

Conjecture 1. *Let $n \in \mathbb{N}$, $w \in \mathbb{R}^+$ with $w > n+1$. Then, for sufficiently large Q, we have*

$$\mu(L_n(Q,w)) \ll \begin{cases} Q^{-(w-n-1)}\mu(K) & for \quad n+1 < w \leqslant n+2, \\ Q^{-(w-2)/n}\mu(K) & for \quad w > n+2 \end{cases}$$

where the constant implied by the Vinogradov symbol depends on n, w, p and K.

In relation to a quantitative estimate for an analogue of the set $L_n(Q,w)$ in $\mathbb{R}$ see [9], [11].

2. Proof of Theorem 2

2.1. Outline of the proof

For the polynomials P of degree $n = \deg P \geqslant 2$ we proceed the proof of Theorem by induction with the following induction hypothesis.

Induction Hypothesis 1. *For any positive real number ϵ, there exist a constant $f(n,p,K,r,\epsilon)$ depending on n, p, K, r and ϵ such that for every $1 \leqslant m \leqslant n-1$ one has*

$$\mu(x \in K : \exists P \in \mathcal{P}_m(Q) \ s.t. \ |P(x)|_p < Q^{-r}, \ r > m+1) < f(n,p,K,r,\epsilon)Q^{\frac{-(r-2)}{m}+m\epsilon}\mu(K)$$

for Q sufficiently large.

The base case for $m = 1$ follows from the following result which was proved in [10].

Lemma 1. *Let $K_0 = K_0(0;p^r) \subset \mathbb{Q}_p$ be a ball of diameter p^r centered at 0 with $r \geqslant 0$. Define $J(Q)$ to be the set of points $x \in K_0$ for which the inequality $|P_1(x)|_p = |ax+b|_p < Q^{-w}$ holds with $w > 2$ for some $P \in \mathcal{P}_1(Q)$. Then $\mu(J(Q)) \ll Q^{2-w}\mu(K_0)$ for sufficiently large Q.*

Next, the proof is divided into two cases: irreducible and reducible polynomials.

2.2. Irreducible polynomials

In this section, we consider only irreducible polynomials P from the class of polynomials $\mathcal{P}_n(Q)$. Denote by $L_n^{IRR}(Q,w)$ a set of points $x \in K$ such that there exists an irreducible polynomial $P \in \mathcal{P}_n(Q)$ satisfying the inequality $|P(x)|_p < Q^{-w}$.

2.2.1. Preliminaries

We will consider only *leading* polynomials, that is those polynomials $P \in \mathcal{P}_n$ which satisfy

$$|a_n(P)| \gg H(P), \ \ |a_n|_p > p^{-n}. \tag{3}$$

It was shown in [16] that if polynomial P does not satisfy the first inequality in (3) then a transformation $S(t) = P(t+m)$ for some $0 \leqslant m \leqslant n$ can be performed followed by an inversion (if necessary) to obtain $T(t) = t^n S(\frac{1}{t})$. Thus, this new polynomial $T(t) = \sum_{i=0}^n b_i t^i \in \mathbb{Z}[t]$ satisfies $|b_n| \gg H(T) \asymp H(P)$. These transformations preserve measures (up to a constant) of sets which satisfy inequality of the form (2).

Consider irreducible polynomials $P \in \mathcal{P}_n$ satisfying (3). Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the polynomial P in $\mathbb{Q}_p^*$. Define the sets

$$S_P(\alpha_i) = \{x \in \mathbb{Q}_p : \ |x-\alpha_i|_p = \min_{1\leqslant m\leqslant n}|x-\alpha_m|_p\}, \quad 1 \leqslant i \leqslant n.$$

Further assume without loss of generality that $i = 1$.

The resultant of $P(x) = a_n\prod_{i=1}^n(x-\alpha_i)$ and $Q(x) = b_m\prod_{j=1}^m(x-\beta_j)$ is defined as $R(P,Q) = a_n^m b_m^n \prod_{1\leqslant i\leqslant n}\prod_{1\leqslant j\leqslant m}(\alpha_i - \beta_j)$, and the discriminant of P is defined as $D(P) = a_n^{2n-2}\prod_{1\leqslant i<j\leqslant n}(\alpha_i-\alpha_j)^2$.

A number of lemmas for later use are now given.

Lemma 2. *Let $P = \sum_{i=0}^{n} a_i t^i$ be a polynomial with rational integer coefficients. If $|a_n|_p > c_1$, where c_1 is a constant depending only on n, then*

$$|\alpha_i|_p \leqslant \max(1/c_1, 1)$$

for every root α_i, $i = 1, 2, \ldots, n$, of P.

This is Lemma 2 of [3].

Suppose that $P \in \mathcal{P}_n$ satisfies (3). From Lemma 2 therefore

$$|\alpha_i|_p < p^n, \quad i = 1, 2, \ldots, n. \tag{4}$$

Lemma 3 ([16, 3]). *Let $x \in S_P(\alpha_1)$. Then*

$$|x - \alpha_1|_p \leqslant |P(x)|_p |P'(x)|_p^{-1} \text{ for } P'(x) \neq 0,$$

$$|x - \alpha_1|_p \leqslant |P(x)|_p |P'(\alpha_1)|_p^{-1} \text{ for } P'(\alpha_1) \neq 0,$$

and

$$|x - \alpha_1|_p \leqslant \min_{2 \leqslant j \leqslant n} (|P(x)|_p |P'(\alpha_1)|_p^{-1} \prod_{k=2}^{j} |\alpha_1 - \alpha_k|_p)^{\frac{1}{j}} \text{ for } P'(\alpha_1) \neq 0.$$

Lemma 4. *Let $K \subset \mathbb{Q}_p$ be a ball and $B \subset K$ be a measurable set satisfying $\mu(B) \geqslant \geqslant m^{-1}\mu(K) > 0$, $m \in \mathbb{N}$. Assume that for all $x \in B$ we have $|P(x)|_p < H(P)^{-a}$, where $a > 0$ and* $\deg P \leqslant n$. *Then for all $x \in K$ we have*

$$|P(x)|_p < (pm(n+1))^{n+1} H(P)^{-a}.$$

This is Lemma 5 of [8].

Lemma 5 ([3]). *Fix $\theta > 0$ and $Q > Q_0(\theta)$. Suppose that $\eta \in \mathbb{R}^+$ and let $P_1, P_2 \in \mathcal{P}_n(Q)$. Further suppose that P_1, P_2 have no roots in common. Let J denote a ball with diameter $Q^{-\eta}$. If there exists real number $\tau > 0$ such that for all $x \in J$*

$$\max(|P_1(x)|_p, |P_2(x)|_p) < Q^{-\tau},$$

then

$$\tau + 2\max(\tau - \eta, 0) < 2n + \theta.$$

This lemma will be used repeatedly throughout the proof to obtain contradictions.

Lemma 6 ([10]). *Let $P \in \mathcal{P}_n(Q)$ be an irreducible polynomial satisfying (3). Then*

$$|P'(\alpha_1)|_p \gg Q^{-n+1}$$

for sufficiently large Q.

In what follows it is often necessary to compare the value of the derivative of P at the root α_1 with the derivative of P at $x \in S_P(\alpha_1)$. The following lemma gives a general result.

Lemma 7 ([10]). *Let $w_1, w_2 \in \mathbb{R}$ and $w_1 \geqslant 2w_2$. Let $x \in S_P(\alpha_1) \cap K$ for some $P \in \mathcal{P}_n(Q)$ and suppose that $|P(x)|_p < Q^{-w_1}$. If $|P'(x)|_p > p^{(n-1)^2/2} Q^{-w_2}$ then $|P'(\alpha_1)|_p = |P'(x)|_p$. On other hand, if $|P'(x)|_p \leqslant p^{(n-1)^2/2} Q^{-w_2}$ then $|P'(\alpha_1)|_p < p^{(n-1)^2/2} Q^{-w_2}$.*

2.2.2. Partitioning the roots

Depending on the size of w we consider the partitions of the set $\mathcal{A}(P)$ of roots of $P \in \mathcal{P}_n(Q)$. These partitions are depended of the values of the derivatives of the polynomials at the roots, and are based on Lemma 6 and Lemma 7, and are detailed below:

$$\begin{array}{llclll}
T_{0,1}: & Q^{-\frac{1}{2}} & < & |P'(\alpha)|_p & \leqslant & p^{n(n-1)} \quad for \quad w > 3n/2+2,\\
T_{n-1,1}: & Q^{-1-\frac{(n-2)(w-2)}{2n}} & < & |P'(\alpha)|_p & \leqslant & Q^{-\frac{1}{2}} \quad for \quad 3n/2+2 < w < 2n+2,\\
T_{n-1,2}: & C(n,p)Q^{-n+1} & < & |P'(\alpha)|_p & \leqslant & Q^{-\frac{1}{2}} \quad for \quad w \geqslant 2n+2,\\
T_{n,1}: & p^{(n-1)^2/2}Q^{-\frac{w}{2}} & < & |P'(\alpha)|_p & \leqslant & Q^{-1-\frac{(n-2)(w-2)}{2n}} \quad for \quad 3n/2+2 < w \leqslant 2n-2,\\
T_{n,2}: & C(n,p)Q^{-n+1} & < & |P'(\alpha)|_p & \leqslant & Q^{-1-\frac{(n-2)(w-2)}{2n}} \quad for \quad \max(3n/2+2, 2n-2) < w < 2n+2,\\
T_{n+1,1}: & & & |P'(\alpha)|_p & \leqslant & p^{(n-1)^2/2}Q^{-\frac{w}{2}} \quad for \quad 3n/2+2 < w \leqslant 2n-2,
\end{array}$$

where $C(n,p)$ arises from Lemma 6, i.e. $|P'(\alpha)|_p > C(n,p)Q^{-n+1}$.

Therefore, for each range of w there are the following subdivisions:

$$\begin{array}{lll}
for & 3n/2+2 < w \leqslant 2n-2: & \alpha \in T_{0,1} \cup T_{n-1,1} \cup T_{n,1} \cup T_{n+1,1},\\
for & \max(3n/2+2, 2n-2) < w < 2n+2: & \alpha \in T_{0,1} \cup T_{n-1,1} \cup T_{n,2},\\
for & w \geqslant 2n+2: & \alpha \in T_{0,1} \cup T_{n-1,2}.
\end{array}$$

Let $\sigma(P)$ denote the set of points for which (2) and $|P'(x)|_p > p^{(n-1)^2/2}Q^{-\frac{w}{2}}$ hold for a fixed polynomial $P \in \mathcal{P}_n(Q)$. By Lemma 3, one has the equality $|P'(\alpha)|_p = |P'(x)|_p$ for $x \in S_P(\alpha)$ and $\alpha \in T_{0,1} \cup T_{n-1,i} \cup T_{n,i}$, $i = 1, 2$, and the set $\sigma(P) \cap S_P(\alpha)$ is contained in $\sigma(P, \alpha)$ which is defined by the inequality:

$$\sigma(P,\alpha) := \{x \in K \cap S_P(\alpha) : |x-\alpha|_p < Q^{-w}|P'(\alpha)|_p^{-1}\}. \tag{5}$$

For $i \in \{0, n-1, n, n+1\}$ and $j \in \{1, 2\}$ define the set $L_{n,i,j}(Q, w)$ of $x \in K \cap S_P(\alpha)$ for which the system

$$|P(x)|_p < Q^{-w}, \quad \alpha \in T_{i,j}$$

has a solution $P \in \mathcal{P}_n(Q)$.

There follow two auxiliary results and several subsections depending on the sizes of derivatives at certain roots.

2.2.3. Auxiliary results

The key ingredients of the proof of the theorem for the cases of large, middle and small derivatives are the following results.

Define the set $L_n(Q, w, d)$ of $x \in K \cap S_P(\alpha)$ for which the system

$$|P(x)|_p < Q^{-w}, \quad |P'(\alpha)|_p > cQ^d \tag{6}$$

has a solution $P \in \mathcal{P}_n(Q)$.

PROPOSITION 1. *Let $d \leqslant 0$. If there exist an integer number $k \in [0, n-1]$ and a real number v satisfying*

$$\max(-2d, 2+\frac{k}{n}(w-2)) \leqslant v \leqslant w-n+k-\frac{w-2}{n} \tag{7}$$

then

$$\mu(L_n(Q,w,d)) \leqslant \begin{cases} 2\cdot 3^{n-k}c_0^{-1}Q^{-(w-2)/n}\mu(K) + kf(k,p,K,w,\epsilon)2^{-\frac{w-2}{n}+k\epsilon}Q^{-\frac{w-2}{n}+k\epsilon}\mu(K) \\ if\, 1\leqslant k\leqslant n-1, \\ 2\cdot 3^{n}c_0^{-1}Q^{-(w-2)/n}\mu(K) \\ if\, k=0 \end{cases}$$

for $w > 3n/2+2$ *and sufficiently large* Q. *Here*

$$c_0 = \min(cp^n, \min_{2\leqslant j\leqslant n} c^{\frac{j}{j-1}}p^{\frac{j(n-1)-n^2}{j-1}}, 2^{-k-1-v}(p(k+1))^{-k-1}).$$

Proof. Let c_0 be a constant to be chosen later. For a polynomial $P\in\mathcal{P}_n(Q)$ with α satisfying (6) define the ball

$$\sigma_0(P,\alpha) = \{x\in K\cap S_P(\alpha):\ |x-\alpha|_p < c_0Q^{-v}|P'(\alpha)|_p^{-1}\}.$$

Denote by $R_0(P)$ the set of roots of the polynomial P satisfying the condition $|P'(\alpha)|_p > cQ^d$. Let $\sigma_0(P) = \cup_{\alpha\in R_0(P)}\sigma_0(P,\alpha)$. From (5) and (6) it follows that $\mu(\sigma_0(P))\leqslant\mu(K)$ and $\sigma(P)\subseteq\sigma_0(P)$ for $-d\leqslant v < w$, and $c_0\leqslant cp^n$ and $Q>Q_0$. Also, $\mu(\sigma(P))\leqslant c_0^{-1}Q^{v-w}\mu(\sigma_0(P))$.

Fix k, $k\in[0,n-1]$. For each $(n-k)$-tuple $\mathbf{b}_k = (b_n,\ldots,b_{k+1})\in\mathbb{Z}^{n-k}$ such that $|b_i|\leqslant Q$ for $i=k+1,\ldots,n$ define the following subclass of $\mathcal{P}_n(Q)$

$$\mathcal{P}_n(Q,\mathbf{b}_k) = \{P = \sum_{i=0}^{n} a_ix^i \in\mathcal{P}_n(Q):\ a_i = b_i\ if\ k+1\leqslant i\leqslant n\}.$$

Note that $\mathcal{P}_n(Q) = \cup_{\mathbf{b}_k}\mathcal{P}_n(Q,\mathbf{b}_k)$ and the number of different vectors $\mathbf{b}_k$ does not exceed $(2Q+1)^{n-k}$.

The balls $\sigma_0(P,\alpha)$ are divided into essential and inessential domains for $P\in\mathcal{P}_n(Q,\mathbf{b}_k)$. First, the essential balls $\sigma_0(P,\alpha)$ are considered. By definition

$$\sum_{P\in\mathcal{P}_n(Q,\mathbf{b}_k)}\ \sum_{\alpha\in R_0(P),\ \sigma_0(P,\alpha)\, essential}\mu(\sigma_0(P,\alpha))\leqslant 2\mu(K).$$

Using this and the fact that the number of different vectors $\mathbf{b}_k$ does not exceed $(2Q+1)^{n-k}$, it follows that

$$\sum_{\mathbf{b}_k}\ \sum_{P\in\mathcal{P}_n(Q,\mathbf{b}_k)}\mu(\sigma(P))\leqslant 2\cdot 3^{n-k}c_0^{-1}Q^{v-w+n-k}\leqslant 2\cdot 3^{n-k}c_0^{-1}Q^{-(w-2)/n}\mu(K) \tag{8}$$

for $v\leqslant w-n+k-(w-2)/n$ and $Q>Q_0$.

We now turn to the inessential balls. Suppose that $\sigma_0(P,\alpha)$ is innessential so that there exists $\bar P\in\mathcal{P}_n(Q,\mathbf{b}_k)$, $P\neq\bar P$ such that $\mu(\sigma_0(P,\bar P)) = \mu(\sigma_0(P,\alpha_1)\cap\sigma_0(\bar P))\geqslant\mu(\sigma_0(P,\alpha))/2$. It can be readily verified that on $\sigma_0(P,\bar P)$

$$\begin{aligned} |(j!)^{-1}P^{(j)}(\alpha)(x-\alpha_1)^j|_p &< p^{n^2-j(n-1)}c_0^jc^{-j}Q^{(-d-v)j} \\ &\leqslant c_0Q^{-v},\ 2\leqslant j\leqslant n, \end{aligned}$$

for $v\geqslant -2d$, $d\leqslant 0$, $c_0\leqslant\min_{2\leqslant j\leqslant n}c^{\frac{j}{j-1}}p^{\frac{j(n-1)-n^2}{j-1}}$. Thus, using the Taylor expansion of P in the neighbourhood of α, it is easy to obtain $|P(x)|_p < c_0Q^{-v}$, $x\in\sigma_0(P,\bar P)$. Similar estimate holds for $\bar P$ on $\sigma_0(P,\bar P)$.

Put $R(t) = P(t) - \bar{P}(t)$ so that $\deg R = n_R \leqslant k$ and $H(R) \leqslant 2Q$. Then, by Lemma 4, in $\sigma_0(P, \alpha_1)$ we have

$$\begin{array}{rcl} |R(x)|_p & < & (2p(k+1))^{k+1} c_0 Q^{-v} \\ & = & 2^{k+1+v}(p(k+1))^{k+1} c_0 Q_1^{-v} \end{array}$$

for $Q_1 = 2Q$. Choose $c_0 \leqslant 2^{-k-1-v}(p(k+1))^{-k-1}$. Then $|R(x)|_p < Q_1^{-v}$ in $\sigma_0(P, \alpha_1)$.

First, consider the case when $1 \leqslant n_R \leqslant k \leqslant n-1$. Applying the Induction Hypothesis 1 to polynomials $R \in \mathcal{P}_{n_R}(Q_1)$, we obtain

$$\mu(L_{n_R}(Q_1, v)) \quad \leqslant \quad f(n_R, p, K, v, \epsilon) Q_1^{-(v-2)/n_R + n_R \epsilon} \mu(K)$$

for $v > 3n_R/2 + 2$.

Second, consider the case when $n_R \leqslant k = 0$. From $|R(x)|_p < Q_1^{-v}$ and $|R(x)|_p \geqslant |R(x)|^{-1} \geqslant$ $\geqslant (2Q)^{-1}$ since $R(x) = a_0'$ and $1 \leqslant |a_0'| \leqslant 2Q$, we get a contradiction in $(2Q)^{-1} \leqslant |R(x)|_p < Q_1^{-v} =$ $= (2Q)^{-v}$ for $v \geqslant 1$.

Therefore, the measure of the set of x belonging to inessential balls does not exceed

$$\begin{array}{rcl} \mu(\cup_{n_R=0}^{k} L_{n_R}(Q_1, v)) & \leqslant & \sum_{n_R=1}^{k} \mu(L_{n_R}(Q_1, v)) \\ & < & \sum_{n_R=1}^{k} f(n_k, p, K, v, \epsilon) Q_1^{-\frac{v-2}{n_R} + n_R \epsilon} \mu(K) \\ & \leqslant & k f(k, p, K, v, \epsilon) 2^{-\frac{v-2}{k} + k\epsilon} Q^{-\frac{v-2}{k} + k\epsilon} \mu(K) \\ & \leqslant & k f(k, p, K, w, \epsilon) 2^{-\frac{w-2}{n} + k\epsilon} Q^{-\frac{w-2}{n} + k\epsilon} \mu(K) \end{array}$$

for $v \geqslant 2 + \frac{k}{n}(w-2)$ and $w \geqslant 3n/2 + 2$.

This, together with (8) gives

$$\begin{array}{rcl} \mu(L_n(Q, w, d)) & \leqslant & \mu(\cup_{\mathbf{b}_k} \mathcal{P}_n(Q, \mathbf{b}_k)) \\ & \leqslant & \left\{ \begin{array}{l} 2 \cdot 3^{n-k} c_0^{-1} Q^{-(w-2)/n} \mu(K) + k f(k, p, K, w, \epsilon) 2^{-\frac{w-2}{n} + k\epsilon} Q^{-\frac{w-2}{n} + k\epsilon} \mu(K) \\ for\ 1 \leqslant k \leqslant n-1, \\ 2 \cdot 3^n c_0^{-1} Q^{-(w-2)/n} \mu(K) for\ k = 0. \end{array} \right. \end{array}$$

□

Now we are going to prove the second result. Let $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}_{>0}$ for $i = 1, 2$. Define the set $L_n(Q, w, a_1, a_2)$ of $x \in K \cap S_P(\alpha)$ for which the system

$$|P(x)|_p \leqslant Q^{-w},\ b_1 Q^{a_1} < |P'(\alpha)|_p \leqslant b_2 Q^{a_2},\ a_1 \leqslant a_2,\ \ w > 3n/2 + 2,\ w \geqslant -2a_1 \tag{9}$$

has a solution $P \in \mathcal{P}_n(Q)$. Let $\epsilon_0 < \epsilon$ be a sufficiently small positive real number.

Proposition 2. *If there exists real number u satisfying*

$$\max(0, -a_2 + \epsilon_0, 2n + 3a_2 + \theta + 3\epsilon_0) \leqslant u \leqslant w + a_1 - (w-2)/n + \epsilon \tag{10}$$

for some values of $\theta > 0$, then

$$\mu(L_n(Q, w, a_1, a_2)) \ll Q^{-(w-2)/n + \epsilon} \mu(K)$$

for sufficiently large Q.

Proof. Divide the ball K into smaller balls K_i with diameter $\mu(K_i) = Q^{-u}$. It is clear that $\mu(K_i) \leqslant \mu(K)$ for $u \geqslant 0$, and $\mu(\sigma(P, \alpha)) \leqslant \mu(K_i)$ for $w \geqslant -2a_1$, $u < w + a_1$ and sufficiently large Q. We say that a polynomial P belongs to K_i if there exists $x \in K_i$ such that (9) holds.

If there is at most one irreducible polynomial $P \in \mathcal{P}_n(Q)$ that belongs to every ball K_i then by (5) the measure of these x, that satisfy (9), does not exceed

$$n b_1^{-1} Q^{-w-a_1+u} \mu(K) \leqslant n b_1^{-1} Q^{-(w-2)/n + \epsilon} \mu(K) \tag{11}$$

for $u \leqslant w + a_1 - (w-2)/n + \epsilon$.

If at least two irreducible polynomials P_i of the form $P_i(t) = s_i P(t)$ for the same P, $s_i \in \mathbb{Z}$, belong to the ball K_i then the measure in this case coincides (up to a constant) with the measure in (11).

Now show that the assumption that at least two irreducible polynomials P_1 and P_2, $P_1 \neq P_2$, without common roots belong to the ball K_i will lead to a contradiction. Using Taylor series expansion, it can be readily verified that on K_i,

$$\begin{array}{rcl} |P_i(x)|_p & \leqslant & \max(b_2 Q^{-u+a_2}, \max_{2\leqslant j\leqslant n} p^{n(n-j)+j} Q^{-uj}) \\ & = & b_2 Q^{-u+a_2} \leqslant Q^{-u+a_2+\epsilon_0} \end{array}$$

for $i = 1, 2$, $a_2 \leqslant 0$, $u \geqslant -a_2 + \epsilon_0$ and sufficiently large Q. We now use Lemma 5 with $\tau = u - a_2 - \epsilon_0$ and $\eta = u$. Then,

$$\tau + 2\max(\tau - \eta, 0) = u - 3a_2 - 3\epsilon_0.$$

From Lemma 5

$$u - 3a_2 - 3\epsilon_0 < 2n + \theta$$

for all $\theta > 0$ which is a contradiction if $u \geqslant 2n + 3a_2 + 3\epsilon_0 + \theta$ for some values of θ. □

2.2.4. Large derivative

This section deals with the case when the derivatives of the polynomials at the roots are large.

Proposition 3. *For sufficiently large Q, $n \geqslant 1$ and $w \geqslant n + 2$*

$$\mu(L_{n,0,1}(Q, w)) \ll Q^{-(w-2)/n} \mu(K).$$

Proof. Take $d = -1/2$ and $c = 1$ in Proposition 1. Then $L_{n,0,1}(Q, w) \subseteq L_n(Q, w, -1/2)$. In this case we choose $k = 0$ and $v = 2$. It is easy to check that the conditions (7) are satisfied for $w \geqslant n + 2$, and

$$\mu(L_{n,0,1}(Q, w)) \leqslant 2 \cdot 3^n c_0^{-1} Q^{-(w-2)/n} \mu(K)$$

for sufficiently large Q, where $c_0 = \min(\min_{2\leqslant j\leqslant n} p^{\frac{j(n-1)-n^2}{j-1}}, 2^{-3}p^{-1})$. □

2.2.5. Special cases $n = 2$ and $n = 3$ for a non-large derivative

This section deals with special cases when the derivatives of the quadratic and cubic polynomials at the roots are taking non-large values.

Case: $n = 2$

Note that the set $L_2^{IRR}(Q, w) \setminus L_{2,0,1}(Q, w)$ is defined as the set of $x \in K \cap S_P(\alpha)$ for which

$$|P(x)|_p < Q^{-w}, \;\; C(2,p)Q^{-1} < |P'(\alpha)|_p \leqslant Q^{1/2}$$

hold for some $P \in \mathcal{P}_2(Q)$. To find the estimate of the measure for the last set we will use Proposition 2. Take $a_1 = -1$, $a_2 = -1/2$, $b_1 = C(2,p)$, $b_2 = 1$ in the Proposition 2. In this case we choose $u = w/2 + \epsilon$. It easy to check that the conditions (10) are satisfied for $w > 5$, $\epsilon_0 < \epsilon/4$, $\theta \leqslant \epsilon/4$, and $\mu(L_2^{IRR}(Q, w) \setminus L_{2,0,1}(Q, w)) \ll Q^{-(w-2)/2+\epsilon} \mu(K)$ for sufficiently large Q.

Case: $n = 3$

Let $L_3^{IRR}(Q, w) \setminus L_{3,0,1}(Q, w) = L_3'(Q, w) \cup L_3''(Q, w)$, where the set $L_3'(Q, w)$ is defined as the set of $x \in K \cap S_P(\alpha)$ satisfying $|P(x)|_p < Q^{-w}$, $\; Q^{-1} < |P'(\alpha)|_p \leqslant Q^{-1/2}$ for some $P \in \mathcal{P}_3(Q)$; and the set $L_3''(Q, w)$ consists of $x \in K \cap S_P(\alpha)$ for which $|P(x)|_p < Q^{-w}$, $\; C(3,p)Q^{-2} < |P'(\alpha)|_p \leqslant Q^{-1}$ hold for some $P \in \mathcal{P}_3(Q)$.

To estimate the measure of the set $L_3'(Q,w)$ we will use Proposition 1. Take $d=-1$ and $c=1$ in Proposition 1. Then $L_3'(Q,w)\subseteq L_3(Q,w,-1)$. In this case we choose $k=0$ and $v=2$. It is easy to check that the conditions (7) are satisfied for $w\geqslant 6.5$, and

$$\mu(L_3'(Q,w))\leqslant 54p^5Q^{-(w-2)/3}\mu(K)$$

for sufficiently large Q.

To estimate the measure of the set $L_3''(Q,w)$ we will use Proposition 2. Take $a_1=-2$, $a_2=-1$, $b_1=C(3,p)$, $b_2=1$ in the Proposition 2. In this case we choose $u=2w/3-4/3+\epsilon$. It easy to check that the conditions (10) are satisfied for $w>13/2$, $\epsilon_0<\epsilon/4$, $\theta\leqslant\epsilon/4$, and $\mu(L_3''(Q,w))\ll Q^{-(w-2)/3+\epsilon}\mu(K)$ for sufficiently large Q.

Thus, $\mu(L_3^{IRR}(Q,w)\setminus L_{3,0,1}(Q,w))\ll Q^{-(w-2)/3+\epsilon}\mu(K)$ for $w>13/2$ and sufficiently large Q.

From now on $n\geqslant 4$.

2.2.6. Middle value derivative

This section deals with the case when the derivatives of the polynomials at the roots are taking middle values.

Case 1: $w\geqslant 2n+2$

Proposition 4. *Let $n\geqslant 2$ and $w\geqslant 2n+2$. For sufficiently large Q*

$$\mu(L_{n,n-1,2}(Q,w))\ll Q^{-(w-2)/n+(n-2)\epsilon}\mu(K).$$

Proof. Take $d=-n+1$ and $c=C(n,p)$ in Proposition 1. Then $L_{n,n-1,2}(Q,w)\subseteq L_n(Q,w,-n+1)$. In this case we choose $k=n-2$ and $v=\frac{(w-2)(n-1)}{n}$. It is easy to check that the conditions (7) are satisfied for $n\geqslant 2$ and $w\geqslant 2n+2$, and

$$\mu(L_{n,n-1,2}(Q,w))\leqslant 2\cdot 3^2c_0^{-1}Q^{-(w-2)/n}\mu(K)+(n-2)f(n-2,p,K,w,\epsilon)2^{-\frac{w-2}{n}+(n-2)\epsilon}Q^{-\frac{w-2}{n}+(n-2)\epsilon}\mu(K)$$

for sufficiently large Q, where

$$c_0=\min(C(n,p)p^n,\min\nolimits_{2\leqslant j\leqslant n}(C(n,p))^{\frac{j}{j-1}}p^{\frac{j(n-1)-n^2}{j-1}},2^{-n+1-(w-2)(n-1)/n}(p(n-1))^{-n+1}).$$

□

Case 2: $w<2n+2$

Define the set $L'_{n,n-1,1}(Q,w)$ of $x\in K\cap S_P(\alpha)$ for which the system

$$Q^{-1-\frac{(n-2)(w-2)}{2n}}<|P'(\alpha)|_p\leqslant Q^{-1-\frac{(n-3)(w-2)}{2n}},\ \ 3n/2+2<w<2n+2 \tag{12}$$

has a solution $P\in\mathcal{P}_n(Q)$.

Define the set $L''_{n,n-1,1}(Q,w)$ of $x\in K\cap S_P(\alpha)$ for which the system

$$Q^{-1-\frac{(n-3)(w-2)}{2n}}<|P'(\alpha)|_p\leqslant Q^{-\frac{1}{2}},\ \ 3n/2+2<w<2n+2 \tag{13}$$

has a solution $P\in\mathcal{P}_n(Q)$. Then $L_{n,n-1,1}(Q,w)=L'_{n,n-1,1}(Q,w)\cup L''_{n,n-1,1}(Q,w)$.

Proposition 5. *Let $n\geqslant 4$. For sufficiently large Q*

$$\mu(L'_{n,n-1,1}(Q,w))\ll Q^{-(w-2)/n+\epsilon}\mu(K).$$

Proof. Take $a_1 = -1 - \frac{(n-2)(w-2)}{2n}$, $a_2 = -1 - \frac{(n-3)(w-2)}{2n}$, $b_1 = b_2 = 1$ in the Proposition 2.

First, we deal with the case $n \geqslant 5$. In this case we choose $u = 1 + \frac{(n-3)(w-2)}{2n} + \frac{1}{n} + \epsilon_0$. It easy to check that the conditions (10) are satisfied for $n \geqslant 5$, $w > 3n/2 + 2$, $\epsilon_0 < \min(\epsilon, \frac{1}{2n})$, $\theta \leqslant n - 5 + \frac{1}{n} - 2\epsilon_0$, and in this case $\mu(L'_{n,n-1,1}(Q,w)) \ll Q^{-(w-2)/n+\epsilon}\mu(K)$ for sufficiently large Q.

Second, we consider the case when $n = 4$. In this case we choose $u = 2n - 3 - \frac{3(n-3)(w-2)}{2n} + \frac{1}{n} + \theta$. Choose $\epsilon_0 < \min(\epsilon, \frac{1}{3n})$. It easy to check that the conditions (10) are satisfied for $n = 4$, $3n/2+2 < w < 2n+2$, and $0 < \theta \leqslant n - \frac{11}{4} - \frac{1}{n} + \epsilon$, and in this case $\mu(L'_{n,n-1,1}(Q,w)) \ll Q^{-(w-2)/n+\epsilon} \cdot$ $\cdot\, \mu(K)$ for sufficiently large Q. □

Proposition 6. *For sufficiently large Q*

$$\mu(L''_{n,n-1,1}(Q,w)) \ll Q^{-(w-2)/n+(n-3)\epsilon}\mu(K).$$

Proof. Take $d = -1 - \frac{(n-3)(w-2)}{2n}$ and $c = 1$ in Proposition 1. Then $L''_{n,n-1,1}(Q,w) \subseteq$ $\subseteq L_n(Q, w, -1 - \frac{(n-3)(w-2)}{2n})$. In this case we choose $k = n-3$ and $v = 2 + \frac{(w-2)(n-3)}{n}$. It is easy to check that the conditions (7) are satisfied for $w \geqslant 3n/2 + 2$, and

$$\mu(L''_{n,n-1,1}(Q,w)) \leqslant 2\cdot 3^3 c_0^{-1} Q^{-(w-2)/n}\mu(K) + (n-3) f(n-3,p,K,w,\epsilon) 2^{-\frac{w-2}{n}+(n-3)\epsilon} Q^{-\frac{w-2}{n}+(n-3)\epsilon}\mu(K)$$

for sufficiently large Q, where

$$c_0 = \min(p^n, \min_{2\leqslant j\leqslant n} p^{\frac{j(n-1)-n^2}{j-1}}, 2^{-n-(w-2)(n-3)/n}(p(n-2))^{-n+2}).$$

□

2.2.7. Small derivative

This section deals with the case when the derivatives of the polynomials at the roots are small.

Proposition 7. *Let $n \geqslant 3$. For sufficiently large Q*

$$\mu(L_{n,n,1}(Q,w)) \ll Q^{-(w-2)/n+\epsilon}\mu(K).$$

Proof. Take $a_1 = -w/2$, $a_2 = -1 - (n-2)(w-2)/(2n)$, $b_1 = p^{(n-1)^2/2}$, $b_2 = 1$ in the Proposition 2. In this case we choose $u = 1 + \frac{(n-2)(w-2)}{2n} + \epsilon_0$. It easy to check that the conditions (10) are satisfied for $n \geqslant 3$, $w > 3n/2 + 2$, $\epsilon_0 < \min(\epsilon, \frac{1}{3})$, $\theta \leqslant n - 2 - 2\epsilon_0$, and $\mu(L_{n,n,1}(Q,w)) \ll Q^{-(w-2)/n+\epsilon}\mu(K)$ for sufficiently large Q. □

Proposition 8. *Let $n \geqslant 3$. For sufficiently large Q*

$$\mu(L_{n,n,2}(Q,w)) \ll Q^{-(w-2)/n+\epsilon}\mu(K).$$

Proof. Take $a_1 = -n+1$, $a_2 = -1 - (n-2)(w-2)/(2n)$, $b_1 = C(n,p)$, $b_2 = 1$ in the Proposition 2. In this case we choose $u = 1 + \frac{(n-2)(w-2)}{2n} + \epsilon_0$. It easy to check that the conditions (10) are satisfied for $n \geqslant 3$, $w > \max(3n/2+2, 2n-2)$, $\epsilon_0 < \min(\epsilon, \frac{1}{3})$, $\theta \leqslant n - 2 - 2\epsilon_0$, and $\mu(L_{n,n,2}(Q,w)) \ll Q^{-(w-2)/n+\epsilon}\mu(K)$ for sufficiently large Q. □

2.2.8. Very small derivative

In this section, we consider the case when the derivative is very small. Recall that $L_{n,n+1,1}(Q,w)$ is set of $x \in K \cap S_P(\alpha_1)$ with $\alpha_1 \in T_{n+1,1}$ for which the system

$$|P(x)|_p < Q^{-w}, \quad |P'(\alpha_1)| \leqslant p^{(n-1)^2/2} Q^{-w/2} \tag{14}$$

has a solution in polynomials $P \in \mathcal{P}_n(Q)$. Define by $\sigma^*(P)$ the set of solutions of the system (14) for a fixed polynomial $P \in \mathcal{P}_n(Q)$.

Let α_1 be any root of a reducible polynomial $P \in \mathcal{P}_n(Q)$. Reorder the other roots of P so that

$$|\alpha_1 - \alpha_2|_p \quad \leqslant \quad |\alpha_1 - \alpha_3|_p \quad \leqslant \ldots \leqslant \quad |\alpha_1 - \alpha_n|_p.$$

For the polynomial P define the real numbers ρ_j by

$$|\alpha_1 - \alpha_j|_p = Q^{-\rho_j}, \quad 2 \leqslant j \leqslant n, \quad \rho_2 \geqslant \rho_3 \geqslant \ldots \geqslant \rho_n. \tag{15}$$

Let $0 < \epsilon_1 < \frac{1}{n^2}$ be sufficiently small, and $T = [\epsilon_1^{-1}] + 1$. Also, define the integers l_j, $2 \leqslant j \leqslant n$, by the relations

$$\frac{l_j - 1}{T} \leqslant \rho_j < \frac{l_j}{T}, \qquad l_2 \geqslant l_3 \geqslant \ldots \geqslant l_n \geqslant 0. \tag{16}$$

Finally, define the numbers q_i by $q_i = \frac{l_{i+1} + \ldots + l_n}{T}$ $(1 \leqslant i \leqslant n-1)$. Now for every polynomial P we define a vector $\mathbf{l} = (l_2, \ldots, l_n)$. The number of different vectors $\mathbf{l}$ is a constant depending on n, p and ϵ_1. Let $\mathcal{P}_n(Q, \mathbf{l})$ be the class of irreducible polynomials $P \in \mathcal{P}_n(Q)$ satisfying (3) and corresponding to a vector $\mathbf{l}$.

For $k \in \mathbb{N} \cup \{0\}$, let $\mathcal{P}_n(Q, \mathbf{l}, k)$ denote the subclass of $\mathcal{P}_n(Q, \mathbf{l})$ given by

$$\mathcal{P}_n(Q, \mathbf{l}, k) = \{P \in \mathcal{P}_n(Q, \mathbf{l}) : \ Q^{k\epsilon_1} \leqslant H(P) < Q^{(k+1)\epsilon_1}\}.$$

Then we have $\mathcal{P}_n(Q) = \cup_{\mathbf{l}} \cup_{k=0}^{T-1} \mathcal{P}_n(Q, \mathbf{l}, k)$. For $P \in \mathcal{P}_n(Q, \mathbf{l}, k)$ satisfying (3) we have the following estimates

$$|P'(\alpha_1)|_p > p^{-n} Q^{-q_1} \ \ \textit{and} \ \ |P^{(l)}(\alpha_1)|_p \leqslant Q^{-q_l + (n-l)\epsilon_1}, \quad 1 \leqslant l \leqslant n-1, \tag{17}$$

which come from (15)–(16) and

$$P^{(l)}(\alpha_1) = l! a_n(P) \sum_{(j_1, j_2, \ldots, j_{n-l}) \subset (2,3,\ldots,n), \ j_s \neq j_k, \ P(\alpha_{j_s}) = 0} \ \prod_{s=1}^{n-l} (\alpha_1 - \alpha_{j_s}).$$

Also, by (14) and (17), we get $p^{-n} Q^{-q_1} < |P'(\alpha_1)|_p \leqslant p^{(n-1)^2/2} Q^{-w/2}$, which implies that

$$q_1 \geqslant w/2 \tag{18}$$

for sufficiently large Q.

We say that $\mathbf{l} \in G_-$ if the following condition $l_2/T + q_1 \leqslant n + n^2\epsilon_1$ holds. Similarly, we say that $\mathbf{l} \in G_+$ if the condition $l_2/T + q_1 > n + n^2\epsilon_1$ holds. Then the set $L_{n,n+1,1}(Q,w)$ can be written as

$$L_{n,n+1,1}(Q,w) = L^-_{n,n+1,1}(Q,w) \cup L^+_{n,n+1,1}(Q,w),$$

where $L^{\mp}_{n,n+1,1}(Q,w) = \cup_{\mathbf{l} \in G_{\mp}} \cup_{k=0}^{T-1} \cup_{P \in \mathcal{P}_n(Q,\mathbf{l},k)} \sigma^*(P)$.

To establish this case we need to consider the following two propositions.

Proposition 9. *For sufficiently large Q, we have*

$$\mu(L^-_{n,n+1,1}(Q,w)) \ll Q^{-(w-2)/n} \mu(K).$$

Proof. Divide the ball K into smaller balls K_j with $\mu(K_j) = Q^{-r}$ with $r = n - q_1 + n^2\epsilon_1$. Lemma 5 will now be used to show there cannot exist two irreducible polynomials P_1 and P_2 without common roots which satisfy (14). To show this, suppose that $P_1, P_2 \in \mathcal{P}_n(Q, \mathbf{l}, k)$ belong to I_j, $P_1 \neq P_2$. Develop P_1 as a Taylor series expansion in the neighbourhood K_j of α_1 to obtain

$$\begin{aligned} |P_1(x)|_p &\leqslant \max\nolimits_{1\leqslant j\leqslant n} |(j!)^{-1}P^{(j)}(\alpha_1)(x-\alpha_1)^j|_p \\ &\leqslant \max(Q^{-q_1+(n-1)\epsilon_1-r}, \max\nolimits_{2\leqslant j\leqslant n} p^j Q^{-q_j+(n-j)\epsilon_1-jr}) \\ &\leqslant Q^{q_1+(n-1)\epsilon_1-r} = Q^{-n-\epsilon_1(n^2-n+1)} \end{aligned}$$

for $l_2/T + q_1 \leqslant n + n^2\epsilon$ and sufficiently large Q. Obviously, the same estimate holds for P_2 on K_j. Thus, there exist two polynomials P_1 and P_2 of height at most $Q_4 = Q^{(k+1)\epsilon_1}$ which satisfy $|P_i(x)|_p < Q_4^{\frac{-n-\epsilon_1(n^2-n+1)}{(k+1)\epsilon_1}}$ on a ball with diameter $Q_4^{\frac{-n+q_1-n^2\epsilon_1}{(k+1)\epsilon_1}}$. Then Lemma 5 can be used with $\tau = \frac{n+\epsilon_1(n^2-n+1)}{(k+1)\epsilon_1}$ and $\eta = \frac{n-q_1+n^2\epsilon_1}{(k+1)\epsilon_1}$. Putting these together gives that

$$\begin{aligned} \tau + 2\max(\tau-\eta, 0) &= \frac{n+2q_1+(n^2-3n+3)\epsilon_1}{(k+1)\epsilon_1} \\ &>^{(18)} \frac{n+w+(n^2-3n+3)\epsilon_1}{(k+1)\epsilon_1} \\ &>^{w>\frac{3}{2}n+2} \frac{5n/2+2+(n^2-3n+3)\epsilon_1}{(k+1)\epsilon_1} \\ &\geqslant^{1\leqslant k+1\leqslant 1+[\epsilon_1^{-1}]\leqslant 1+\epsilon_1^{-1}} \frac{5n/2+2+(n^2-3n+3)\epsilon_1}{1+\epsilon_1} \\ &= 5n/2+2+(n^2-11n/2+1)\epsilon_1(1+\epsilon_1)^{-1} \\ &>^{\epsilon_1>0,\, n\geqslant 6} 5n/2+2. \end{aligned}$$

From Lemma 5 this implies that $5n/2+2 < 2n+\theta$ for all $\theta > 0$, and it is not difficult to check that this is a contradiction for $\theta < \frac{n}{2}+2$. Therefore, at most one polynomial $P \in \mathcal{P}_n(Q, \mathbf{l}, k)$ belongs to each K_j. Thus, the number of polynomials $P \in \mathcal{P}_n(Q, \mathbf{l}, k)$ is $Q^r\mu(K)$. By applying Lemma 7 and the inequalities (17) and (2), we obtain for $P \in \mathcal{P}_n(Q, \mathbf{l}, k)$,

$$|x-\alpha_1|_p \leqslant |P(x)|_p |P(\alpha_1)|_p^{-1} < p^n Q^{-w+q_1};$$

the latter set is containing the set $\sigma^*(P) \cap S_P(\alpha_1)$. Thus, the measure of the set $L^-_{n,n+1,1}(Q, w)$ for $P \in \mathcal{P}_n(Q, \mathbf{l}, k)$, is

$$\ll Q^{n-w+n^2\epsilon_1}\mu(K).$$

Summing the last estimate over k and $\mathbf{l}$, we obtain that

$$\begin{aligned} \mu(L^-_{n,n+1,1}(Q, w)) &\ll \sum\nolimits_{\mathbf{l}} \sum\nolimits_{k=0}^{[\epsilon_1^{-1}]} Q^{n-w+n^2\epsilon_1}\mu(K) \\ &\ll Q^{-(w-2)/n}\mu(K) \end{aligned}$$

for $w > \frac{3}{2}n+2$, $n \geqslant 1$ and sufficiently large Q.

Proposition 10. *For sufficiently large Q, we have*

$$\mu(L^+_{n,n+1,1}(Q, w)) \ll Q^{-(w-2)/n}\mu(K)$$

where the constant implied by the Vinogradov symbol depends on n, p, ϵ_1 and K.

Proof. Expressing the discriminant $D(P)$ of an irreducible polynomial $P \in \mathcal{P}_n(Q, \mathbf{l})$ in the form $|D(P)|_p = |a_n^{2n-2}(P)|_p \prod_{1\leqslant i<j\leqslant n} |\alpha_i - \alpha_j|_p^2$ and using (16), $|a_n|_p \leqslant 1$, and $|D(P)| \ll Q^{2n-2}$, we obtain

$$\sum_{j=2}^{n} (j-1)l_j/T \leqslant n-1 \tag{19}$$

for sufficiently large Q. Using (19) and the definitions of q_i and the set $L^+_{n,n+1,1}(Q,w)$, we get

$$\frac{n+n^2\epsilon_1}{2}+\frac{3q_2}{2}<\frac{q_1+l_2T^{-1}}{2}+\frac{3q_2}{2}\leqslant(l_2/T+q_2/2)+\frac{3q_2}{2}=l_2/T+2q_2\leqslant\sum_{j=2}^{n}(j-1)l_j/T\leqslant n-1 \quad (20)$$

By (20) and using the definitions of q_i, we obtain

$$2l_3/T+q_2\leqslant 3q_2<n-2-n^2\epsilon_1, \quad (21)$$

which implies $l_3/T<(n-2-q_2-n^2\epsilon_1)/2$. Therefore, by (20)

$$l_3/T<(n-q_2+n^2\epsilon_1)/2<l_2/T. \quad (22)$$

Next we show that there is no pair P_1, P_2 of different polynomials in the set $P\in\mathcal{P}_n(Q,\mathbf{l},k)$ with roots α_1, β_1 respectively, satisfying (22) and the inequality

$$|\alpha_1-\beta_1|_p\leqslant Q^{(q_2-n-(n^2-2)\epsilon_1)/2}, \quad (23)$$

where $P_1(\alpha_i)=0$ and $P_2(\beta_i)=0$ for $1\leqslant i\leqslant n$. Assume that there exists such a pair of polynomials. Then, by (22) and (23), we have

$$\begin{array}{rcl}|\alpha_i-\beta_j|_p & \leqslant & \max(|\alpha_i-\alpha_1|_p,|\alpha_1-\beta_1|_p,|\beta_1-\beta_j|_p)\\ & \leqslant & \max(Q^{-l_{\max(i,j)}/T+\epsilon_1},Q^{(q_2-n-(n^2-2)\epsilon_1)/2})\\ & \leqslant & \begin{cases}Q^{(q_2-n-(n^2-2)\epsilon_1)/2} & for \quad \max(i,j)\leqslant 2,\\ Q^{-l_{\max(i,j)}/T+\epsilon_1} & for \quad \max(i,j)\geqslant 3.\end{cases}\end{array}$$

Considering the resultant $R(P_1,P_2)$ of the polynomials P_1 and P_2, we obtain

$$\begin{array}{rcl}|R(P_1,P_2)|_p & = & |a_n(P_1)^n|_p|a_n(P_2)^n|_p\prod_{1\leqslant i,j\leqslant n}|\alpha_i-\beta_j|_p\\ & \leqslant & Q^{2q_2-2n-2(n^2-2)\epsilon_1}\prod_{\max(i,j)\geqslant 3}Q^{-l_{\max(i,j)}/T+\epsilon_1}.\end{array}$$

But since $\sum_{\max(i,j)\geqslant 3}l_{\max(i,j)}/T=\sum_{j=3}^{n}(2j-1)l_j/T\geqslant 5q_2$ it follows that

$$|R(P_1,P_2)|_p\leqslant Q^{-2n-3q_2-n^2\epsilon_1}.$$

Since the polynomials P_1 and P_2 are irreducible then $|R(P_1,P_2)|\ll Q^{2n(k+1)\epsilon_1}$ and $|R(P_1,P_2)|_p\gg$ $\gg Q^{-2n(k+1)\epsilon_1}$. Thus, the inequality (23) leads to a contradiction for $n\geqslant 3$ and sufficiently large Q. Therefore we conclude that a ball $K(\alpha_1;r)$ with its centre at the point α_1, $P_1(\alpha_1)=0$, and with diameter r satisfying $p^{-r_0}\leqslant r<p^{-r_0+1}$ (with $r_0\in\mathbb{Z}$) and not exceeding $cQ^{(q_2-n-(n^2-2)\epsilon_1)/2}$, cannot contain a root β_1 of any polynomial $P_2\in\mathcal{P}_n(Q,\mathbf{l},k)$ (with $\mathbf{l}\in G_+$) other than P_1. We cover each of the numbers α_1 under consideration by the ball $K(\alpha_1;r)$. Thus, we see that these balls are mutually disjoint and have the diameter $\gg Q^{(q_2-n-(n^2-2)\epsilon_1)/2}$. Therefore the number of polynomials $P_1\in\mathcal{P}_n(Q,\mathbf{l},k)$ with $\mathbf{l}\in G_+$ is $\ll Q^{(-q_2+n+(n^2-2)\epsilon_1)/2}$.

By applying Lemma 3, the inequalities (17) and $|P(x)|_p<Q^{-w}$, we obtain for $P\in\mathcal{P}_n(Q,\mathbf{l},k)$,

$$|x-\alpha_1|_p\leqslant(|P(x)|_p|\alpha_1-\alpha_2|_p/|P'(\alpha_1)|_p)^{1/2}\ll Q^{(-w+q_2)/2}.$$

Thus, the measure of the set $L^+_{n,n+1,1}(Q,w)$ for $P\in\mathcal{P}_n(Q,\mathbf{l},k)$ with at least one root satisfying (23), will be $\ll Q^{(n-w+(n^2-2)\epsilon_1)/2}$. Summing the last estimate over k and $\mathbf{l}$, we obtain that

$$\begin{array}{rcl}\mu(L^+_{n,n+1,1}(Q,w)) & \ll & \sum_{\mathbf{l}}\sum_{k=0}^{T-1}Q^{(n-w+(n^2-2)\epsilon_1)/2}\\ & \ll & Q^{-(w-2)/n}\mu(K)\end{array}$$

for $w>3n/2+2$, $n\geqslant 4$ and $\epsilon_1<1/n^2$.

2.3. Reducible polynomials

Denote by $L_n^{RED}(Q,w)$ a set of points $x\in K$ such that there exists a reducible polynomial $P\in\mathcal{P}_n(Q)$ satisfying the inequality $|P(x)|_p<Q^{-w}$. Let $P\in\mathcal{P}_n(Q)$ be a reducible polynomial of the form

$$P(x)=P_1(x)P_2(x),\ \ \deg P_1=n_1,\ \ \deg P_2=n-n_1,\ \ 1\leqslant n_1\leqslant n-1,$$

and the inequality $|P(x)|_p<Q^{-w}$ holds for $x\in K$. For a fixed P by $\lambda(P)$ denote the set of $x\in K$ satisfying $|P(x)|_p<Q^{-w}$.

By Gelfond's lemma [12],

$$2^{-n}H(P_1)H(P_2)\leqslant H(P)\leqslant 2^nH(P_1)H(P_2).$$

By definition of height, we have $H(P_i)\geqslant 1$ so that $H(P_i)\leqslant 2^nQ$ for $i=1,2$.

Define $L_{n,1}^{RED}(Q,w)\subset L_n^{RED}(Q,w)$ ($L_{n,2}^{RED}(Q,w)$ respectively) to be the set of points $x\in K$ for which the inequality $|P(x)|_p<Q^{-w}$ holds for some reducible polynomial $P\in\mathcal{P}_n(Q)$ of the form $P(x)=P_1(x)P_2(x)$ with $1\leqslant H(P_1)<Q$ ($Q\leqslant H(P_1)\leqslant 2^nQ$ respectively).

We need to consider two cases.

2.3.1. Case 1: $1\leqslant H(P_1)<Q$.

Let $\beta\in(0,1)$ be a sufficiently small positive real number such that $\frac{1}{\beta}\in\mathbb{N}$ and it satisfies the condition that will be specified later. Let the height of P_1 be bounded as follows: $Q^{m\beta}\leqslant H(P_1)<Q^{(m+1)\beta}$ where $0\leqslant m\leqslant\frac{1}{\beta}-1$. Then the height of P_2 satisfies $H(P_2)\leqslant 2^nQ^{1-m\beta}$.

There exists $a\in\mathbb{R}$ such that

$$\mu\left(x\in\lambda(P):\ |P_1(x)|_p<(2p(n_1+1))^{-n_1-1}Q^{-a}\right)=\mu(\lambda(P))/2. \tag{24}$$

Then for the complement to (24) we have

$$\mu\left(x\in\lambda(P):\ |P_1(x)|_p\geqslant(2p(n_1+1))^{-n_1-1}Q^{-a}\right)=\mu(\lambda(P))/2$$

or

$$\mu\left(x\in\lambda(P):\ |P_2(x)|_p<(2p(n_1+1))^{n_1+1}Q^{-w+a}\right)=\mu(\lambda(P))/2. \tag{25}$$

In the next step of the proof we will use the Lemma 4. By applying Lemma 4 and the estimates (24), (25), we have

$$|P_1(x)|_p<Q^{-a},\quad Q^{m\beta}\leqslant H(P_1)<Q^{(m+1)\beta}, \tag{26}$$

$$|P_2(x)|_p<(2p)^{n+2}(n_1+1)^{n_1+1}(n-n_1+1)^{n-n_1+1}Q^{-w+a},\ \ H(P_2)\leqslant 2^nQ^{1-m\beta} \tag{27}$$

for all $x\in\lambda(P)$.

Denote by $M_{n_1,m}^1(Q)$ a set of points $x\in K$ such that there exists a polynomial $P_1\in\mathcal{P}_{n_1}(Q^{(m+1)\beta})\setminus\mathcal{P}_{n_1}(Q^{m\beta})$ satisfying the inequality (26) for $a\geqslant 2(m+1)\beta+n_1(w-2)/n-d_m\epsilon$ and $M_{n_1,m}^2(Q)$ a set of points $x\in K$ such that there exists a polynomial $P_2\in\mathcal{P}_{n-n_1}(2^nQ^{1-m\beta})$ satisfying the inequality (27) for $a<2(m+1)\beta+n_1(w-2)/n-d_m\epsilon$. Here $d_m=0$ for $m\geqslant 2$, and $d_m=n_1/2$ for $m=0,1$.

Let us estimate the measure of the set $M_{n_1,m}^1(Q)$. For convenience we put $Q_2=Q^{(m+1)\beta}$ and $w_1=\frac{2(m+1)\beta+n_1(w-2)/n-d_m\epsilon}{(m+1)\beta}$. Clearly $M_{n_1,m}^1(Q)\subset L_{n_1}(Q_2,w_1)$. By the Induction Hypothesis 1 the set $L_{n_1}(Q_2,w_1)$ has measure at most

$$\begin{aligned} f(n_1,p,K,w_1,\epsilon)Q_2^{\frac{-(w_1-2)}{n_1}+n_1\epsilon}\mu(K) &= f(n_1,p,K,w_1,\epsilon)\left(Q^{(m+1)\beta}\right)^{\frac{-(w_1-2)}{n_1}+n_1\epsilon}\mu(K)\\ &= f(n_1,p,K,w_1,\epsilon)Q^{-(w-2)/n+d_m\epsilon/n_1+n_1\epsilon(m+1)\beta}\mu(K)\end{aligned}$$

for $w > \frac{3}{2}n((m+1)\beta + \frac{2d_m\epsilon}{3n_1}) + 2$, and sufficiently large Q.

Therefore, for $w > 3n/2 + 2$, $1 \leqslant n_1 \leqslant n-1$, and $Q > Q_0$, we have

$$L_{n_1}(Q_2, w_1) \leqslant \begin{cases} f(n,p,K,w,\epsilon)Q^{-(w-2)/n+(n-1)\epsilon}\mu(K) \; for \; 2 \leqslant m \leqslant 1/\beta - 1, \; d_m = 0, \\ f(n,p,K,w,\epsilon)Q^{-(w-2)/n+(n-1/2)\epsilon}\mu(K) \\ for \; m = 0,1, \; d_m = n_1/2, \; \beta \leqslant (3-\epsilon)/6. \end{cases}$$

Now let us estimate the measure of the set $M^2_{n_1,m}(Q)$. We set $Q_3 = 2^n Q^{1-m\beta}$ and $w_2 = \frac{w-2(m+1)\beta - n_1(w-2)/n + d_m\epsilon - \beta/2}{1-m\beta}$. In the view of the definition of the set $M^2_{n_1,m}(Q)$, we get

$$|P_2(x)| < Q_3^{-w_2}, \quad H(P_2) \leqslant Q_3$$

for $Q > Q_0$. Therefore, $M^2_{n_1,m}(Q) \subseteq L_{n-n_1}(Q_3, w_2)$. Then by Induction Hypothesis 1, we obtain

$$\begin{aligned} \mu(L_{n-n_1}(Q_3,w_2)) &< f(n-n_1,p,K,w_2,\epsilon)Q_3^{-\frac{w_2-2}{n-n_1}+(n-n_1)\epsilon}\mu(K) \\ &= f(n-n_1,p,K,w_2,\epsilon)2^{-\frac{w-2}{1-m\beta}+\frac{5n\beta-2nd_m\epsilon}{2(n-n_1)(1-m\beta)}+n(n-n_1)\epsilon} \cdot \\ &\quad \cdot Q^{-\frac{w-2}{n}+\frac{5\beta-2d_m\epsilon}{2(n-n_1)}+(n-n_1)(1-m\beta)\epsilon}\mu(K) \end{aligned}$$

for $w > 2 + \frac{3n}{2}(1 - m\beta + \frac{5\beta - 2d_m\epsilon}{3(n-n_1)})$ and sufficiently large Q. Thus, for $w > 3n/2+2$, $1 \leqslant n_1 \leqslant n-1$, and $Q > Q_0$, we have

$$L_{n-n_1}(Q_3,w_2) \leqslant \begin{cases} f(n-n_1,p,K,w_2,\epsilon)2^{-(w-2)+n(n-1+\frac{1}{2(n-1)})\epsilon}Q^{-(w-2)/n+(n-1+\frac{1}{2(n-1)})\epsilon}\mu(K) \\ for \; 2 \leqslant m \leqslant \frac{1}{\beta} - 1, \; d_m = 0, \; \beta \leqslant \epsilon/5; \\ f(n,p,K,w,\epsilon)2^{-(w-2)+n(n-1)\epsilon}Q^{-(w-2)/n+(n-1)\epsilon}\mu(K) \\ for \; m = 0,1, \; d_m = n_1/2, \; \beta \leqslant \epsilon/5. \end{cases}$$

Combining the conditions imposed on the values of β, we obtain

$$0 < \beta \leqslant \min\{\epsilon/5, (3-\epsilon)/6\}.$$

Note that $L^{RED}_{n,1}(Q,w) \subset \cup_{n_1=1}^{n-1} \cup_{0 \leqslant m \leqslant \frac{1}{\beta}-1} \left(M^1_{n_1,m}(Q) \cup M^2_{n_1,m}(Q)\right)$. Adding up the measures over all cases gives that

$$\mu(L^{RED}_{n,1}(Q,w)) \ll Q^{-\frac{w-2}{n}+(n-\frac{1}{2})\epsilon}\mu(K)$$

for sufficiently large Q.

2.3.2. Case 2: $Q \leqslant H(P_1) \leqslant 2^n Q$.

We proceed as in Case 1. The height of P_2 satisfies $H(P_2) \leqslant 2^n$ and further we proceed as in Case 1. There exists $a \in \mathbb{R}$ such that

$$\mu\left(x \in \lambda(P): \; |P_1(x)|_p < 2^{-na}(2p(n_1+1))^{-n_1-1}Q^{-a}\right) = \mu(\lambda(P))/2. \tag{28}$$

Then using Lemma 4 and the estimate (28), we get

$$|P_1(x)|_p < 2^{-na}Q^{-a}, \quad Q \leqslant H(P_1) \leqslant 2^n Q, \tag{29}$$

$$|P_2(x)|_p < 2^{n+2+na}p^{n+2}(n_1+1)^{n_1+1}(n-n_1+1)^{n-n_1+1}Q^{-w+a}, \;\; H(P_2) \leqslant 2^n \tag{30}$$

for all $x \in \lambda(P)$.

Denote by $M^3_{n_1}(Q)$ a set of points $x \in K$ such that there exists a polynomial $P_1 \in \mathcal{P}_{n_1}(2^n Q) \setminus \setminus \mathcal{P}_{n_1}(Q)$ satisfying the inequality (29) for $a \geqslant 2 + n_1(w-2)/n$ and $M^4_{n_1}(Q)$ a set of points $x \in K$ such that there exists a polynomial $P_2 \in \mathcal{P}_{n-n_1}(2^n)$ satisfying the inequality (30) for $a < 2+n_1(w-2)/n$. Thus

$$L^{RED}_{n,2}(Q,w) \subset \cup_{n_1=1}^{n-1} \left(M^3_{n_1}(Q) \cup M^4_{n_1}(Q)\right).$$

Clearly,

$$M^3_{n_1}(Q) \subset L_{n_1}\left(2^nQ, 2+\frac{n_1(w-2)}{n}\right).$$

By the Induction Hypothesis 1 the set $L_{n_1}\left(2^nQ, 2+\frac{n_1(w-2)}{n}\right)$ has measure at most

$$f(n_1,p,K,w,\epsilon)\,(2^nQ)^{-(w-2)/n+n_1\epsilon}\,\mu(K) \leqslant 2^{-(w-2)+n(n-1)\epsilon} f(n_1,p,K,w,\epsilon) Q^{-(w-2)/n+(n-1)\epsilon}\mu(K)$$

for $w > \frac{3}{2}n+2$, $n_1 \leqslant n-1$ and sufficiently large Q.

To find the measure of the set $M^4_{n_1}(Q)$ we will use direct calculations. Denote by α_1 a zero of $P_2 \in \mathcal{P}_{n-n_1}(2^n)$, and assume that α_1 is such that $|x-\alpha_1|_p$ is minimal. From the identity $|P_2(x)|_p = |a_{n-n_1}|_p|x-\alpha_1|_p \ldots |x-\alpha_{n-n_1}|_p$ it follows that $|x-\alpha_1|_p \leqslant \left(\frac{|P_2(x)|_p}{|a_{n-n_1}|_p}\right)^{\frac{1}{n-n_1}}$. This means that

$$M^4_{n_1}(Q) \subseteq \cup_{P_2\in\mathcal{P}_{n-n_1}(2^n)} \cup_{\alpha_1\in\mathcal{A}(P_2)} \sigma_0(P_2,\alpha_1)$$

where

$$\sigma_0(P_2,\alpha_1) := \{x \in K \,:\, |x-\alpha_1|_p \leqslant (|P_2(x)|_p|a_{n-n_1}(P_2)|_p^{-1})^{\frac{1}{n-n_1}}\}.$$

This, together with the estimates (30), gives

$$\sigma_0(P_2,\alpha_1) \subset \{x\in K \,:\, |x-\alpha_1| \leqslant (2^{4n+2+n_1(w-2)}p^{n+2}(n_1+1)^{n_1+1}(n-n_1+1)^{n-n_1+1})^{\frac{1}{n-n_1}}Q^{-\frac{w-2}{n}}\}$$

for $P_2 \in \mathcal{P}_{n-n_1}(2^n)$, where $|a_{n-n_1}(P_2)| \leqslant 2^n$, $|a_{n-n_1}(P_2)|_p \geqslant |a_{n-n_1}(P_2)|^{-1} \geqslant 2^{-n}$. The number of different polynomials $P_2 \in \mathcal{P}_{n-n_1}(2^n)$ does not exceed $(2^{n+1}+1)^{n-n_1+1}$. Thus,

$$\mu(M^4_{n_1}(Q)) \;\leqslant\; \sum_{P_2\in\mathcal{P}_{n-n_1}(2^n)} \sum_{\alpha_1\in\mathcal{A}(P_2)} \mu(\sigma_0(P_2,\alpha_1))) \ll Q^{-\frac{w-2}{n}}\mu(K).$$

Therefore, since $L^{RED}_n(Q,w) = L^{RED}_{n,1}(Q,w) \cup L^{RED}_{n,2}(Q,w)$ we have $\mu(L^{RED}_n(Q,w)) \ll$
$\ll Q^{-\frac{w-2}{n}+(n-\frac{1}{2})\epsilon}\mu(K)$ for $w > \frac{3}{2}n+2$ and sufficiently large Q.

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