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# Уникальная возможность отображения списка полных трехсторонних графиков

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**Аннотация**

Учитывая список $L(v)$ для каждой вершины $v$, мы говорим, что граф $G$ является $L$-раскрашиваемым, если существует правильная раскраска вершины G, где каждая вершина $v$ берет свой цвет из $L(v)$. Граф является однозначно раскрашиваемым списком $k$, если существует присвоение списка $L$ такое, что $|L(v)| = k$ для каждой вершины $v$, и граф имеет ровно одну раскраску $L$ с этими списками. Если граф $G$ не является однозначно раскрашиваемым списком $k$, мы также говорим, что $G$ обладает свойством $M(k)$. Наименьшее целое число $k$, такое, что $G$ обладает свойством $M(k)$, называется $m$-числом $G$, обозначаемым $m(G)$. В этой статье сначала мы охарактеризуем свойство полных трехсторонних графов, когда это однозначно $k$-список раскрашиваемых графов, наконец, мы докажем, что $m(K_{2,2,m}) = m(K_{2,3,n}) = m(K_{2,4,p}) = m(K_{3,3,3}) = 4$ за каждые $m \geqslant 9, n \geqslant 5, p \geqslant 4$.

*Ключевые слова:* Раскраска вершин (раскраска), раскраска списка, однозначно раскрашиваемый список графов, полный r-частичный граф.

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# Uniquely list colorability of complete tripartite graphs

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**Abstract**

Given a list $L(v)$ for each vertex $v$, we say that the graph $G$ is $L$-colorable if there is a proper vertex coloring of G where each vertex $v$ takes its color from $L(v)$. The graph is uniquely $k$-list colorable if there is a list assignment $L$ such that $|L(v)| = k$ for every vertex $v$ and the graph has exactly one $L$-coloring with these lists. If a graph $G$ is not uniquely $k$-list colorable, we also say that $G$ has property $M(k)$. The least integer $k$ such that $G$ has the property $M(k)$ is called the $m$-number of $G$, denoted by $m(G)$. In this paper, first we characterize about the property of the complete tripartite graphs when it is uniquely $k$-list colorable graphs, finally we shall prove that $m(K_{2,2,m}) = m(K_{2,3,n}) = m(K_{2,4,p}) = m(K_{3,3,3}) = 4$ for every $m \geqslant 9, n \geqslant 5, p \geqslant 4$.

*Keywords:* Vertex coloring (coloring), list coloring, uniquely list colorable graph, complete r-partite graph.

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## 1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If $G$ is a graph, then $V(G)$ and $E(G)$ (or $V$ and $E$ in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denoted by $N_W(S)$. The subgraph of $G$ induced by $W \subseteq V(G)$ is denoted by $G[W]$. The empty and complete graphs of order $n$ are denoted by $O_n$ and $K_n$, respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [2].

A graph $G = (V, E)$ is called *r-partite graph* if $V$ admits a partition into $r$ classes $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that the subgraphs of $G$ induced by $V_i$, $i = 1, \ldots, r$, is empty, if $r = 2$ then $G$ is called *bipartite graph*, if $r = 3$ then $G$ is called *tripartite graph*. An $r$-partite graph in which every two vertices from different partition classes are adjacent is called complete $r$-partite graph and is denoted by $K_{|V_1|,|V_2|,\ldots,|V_r|}$ . The complete $r$-partite graph $K_{|V_1|,|V_2|,\ldots,|V_r|}$ with $|V_1| = |V_2| = \ldots = |V_r| = s$ is denoted by $K_s^r$.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Their *union* $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their *join* defined is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining $V_1$ with $V_2$.

Let $G = (V, E)$ be a graph and $\lambda$ is a positive integer.

A $\lambda$-*coloring* of $G$ is a mapping $f : V(G) \to \{1, 2, \ldots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer $\lambda$ such that $G$ has a $\lambda$-coloring is called the *chromatic number* of $G$ and is denoted by $\chi(G)$. We say that a graph $G$ is *n-chromatic* if $n = \chi(G)$.

Let $(L(v))_{v\in V}$ be a family of sets. We call a coloring $f$ of $G$ with $f(v) \in L(v)$ for all $v \in V$ is a *list coloring from the lists* $L(v)$. We will refer to such a coloring as an $L$-coloring. The graph $G$ is called $\lambda$-*list-colorable*, or $\lambda$-*choosable*, if for every family $(L(v))_{v\in V}$ with $|L(v)| = \lambda$ for all $v$, there is a coloring of $G$ from the lists $L(v)$. The smallest positive integer $\lambda$ such that $G$ has a $\lambda$-choosable is called the *list-chromatic number*, or *choice number* of $G$ and is denoted by $ch(G)$. The idea of list colorings of graphs is due independently to V. G. Vizing [14] and to P. Erdös, A. L. Rubin, and H. Taylor [7].

Let $G$ be a graph with $n$ vertices and suppose that for each vertex $v$ in $G$, there exists a list of $k$ colors $L(v)$, such that there exists a unique $L$-coloring for $G$, then $G$ is called a *uniquely k-list colorable graph* or a U$k$LC graph for short. If a graph $G$ is not uniquely $k$-list colorable, we also say that $G$ has property $M(k)$. So $G$ has the property $M(k)$ if and only if for any collection of lists assigned to its vertices, each of size $k$, either there is no list coloring for $G$ or there exist at least two list colorings. The least integer $k$ such that $G$ has the property $M(k)$ is called the *m-number* of $G$, denoted by $m(G)$. The idea of uniquely colorable graph was introduced independently by Dinitz and Martin [6] and by Mahmoodian and Mahdian [10].

For example, one can easily see that the graph $K_{1,1,2}$ has the property M(3) and it is U2LC, so $m(K_{1,1,2}) = 3$.

The list coloring model can be used in the channel assignment. The fixed channel allocation scheme leads to low channel utilization across the whole channel. It requires a more effective channel assignment and management policy, which allows unused parts of channel to become available temporarily for other usages so that the scarcity of the channel can be largely mitigated [15]. It is a discrete optimization problem. A model for channel availability observed by the secondary users is introduced in [15]. The research of list coloring consists of two parts: the choosability and the unique list colorability. In [9], we characterized uniquely list colorability of the graph $G = K_2^m + K_n$.

In this paper, first we characterize about the property of the complete tripartite graphs when it is uniquely $k$-list colorable graphs (Section 2), finally we shall prove that $m(K_{2,2,m}) = m(K_{2,3,n}) = m(K_{2,4,p}) = m(K_{3,3,3}) = 4$ for every $m \geqslant 9, n \geqslant 5, p \geqslant 4$ (Section 3).

## 2. Property of the complete tripartite graphs when it is $k$-list colorable

We need the following Lemmas 1–6 to prove our results.

Lemma 1 ([10]). *Each UkLC graph is also a $U(k-1)LC$ graph.*

Lemma 2 ([10]). *The graph $G$ is UkLC if and only if $k < m(G)$.*

Lemma 3 ([10]). *A connected graph $G$ has the property $M(2)$ if and only if every block of $G$ is either a cycle, a complete graph, or a complete bipartite graph.*

Lemma 4 ([10]). *For every graph $G$ we have $m(G) \leqslant E(\overline{G})| + 2$.*

Lemma 5 ([10]). *Every UkLC graph has at least $3k-2$ vertices.*

Lemma 6 ([10]). *A connected graph $G$ has the property $M(2)$ if and only if every block of $G$ is either a cycle, a complete graph, or a complete bipartite graph.*

Theorem 1. *Let $G = K_{m,n,p}$ be a UkLC graph with $k \geqslant 2$. Then*
*(i)* $\max\{m,n,p\} \geqslant 2$;
*(ii) If $k \geqslant 3$ then* $\min\{m,n,p\} \geqslant 2$;
*(iii)* $k < \frac{m^2+n^2+p^2-(m+n+p)+4}{2}$;
*(iv)* $k \leqslant \left\lfloor \frac{m+n+p+2}{3} \right\rfloor$.

Proof. (i) For suppose on the contrary that $\max\{m,n,p\} = 1$. Then $m = n = p = 1$, so $G$ is a complete graph $K_3$. By Lemma 3, $G$ has the property $M(2)$, a contradiction.

(ii) For suppose on the contrary that $\min\{m,n,p\} = 1$. Without loss of generality, we may assume that $\min\{m,n,p\} = m = 1$. Let $V(G) = V_1 \cup V_2 \cup V_3$ is a partition of $V(G)$ such that $|V_1| = m, |V_2| = n, |V_3| = p$, $V_1 = \{a\}$ and for every $i = 1,2,3$ the subgraphs of $G$ induced by $V_i$, is empty graph.

Since $G$ is a U$k$LC graph, there exists a list of $k$ colors $L(v)$ for each vertex $v$, such that there exists a unique $L$-coloring $f$ for $G$. Set graph $H = G - V_1$, it is not difficult to see that $H$ is complete bipartite graph $K_{n,p}$. We assign the following lists $L'(v)$ for the vertices $v$ of $H$:

If $f(a) \in L(v)$ then $L'(v) = L(v) \setminus \{f(a)\}$.

If $f(a) \notin L(v)$ then $L'(v) = L(v) \setminus \{b\}$, where $b \in L(v)$ and $b \neq f(v)$.

It is clear that $|L'(v)| = k - 1 \geqslant 2$ for every $v \in V(H)$. By Lemma 3, $H$ has the property $M(2)$. So by Lemma 1, $H$ has the property $M(k-1)$. It follows that with lists $L'(v)$, there exists at least two list colorings for the vertices $v$ of $H$. So it is not difficult to see that with lists $L(v)$, there exists at least two list colorings for the vertices $v$ of $G$, a contradiction.

(iii) It is not difficult to see that $|E(\overline{G})| = \frac{m^2+n^2+p^2-(m+n+p)}{2}$. By Lemma 4, we have

$$m(G) \leqslant |E(\overline{G})| + 2 = \frac{m^2+n^2+p^2-(m+n+p)+4}{2}.$$

By Lemma 2, we have $k < \frac{m^2+n^2+p^2-(m+n+p)+4}{2}$.

(iv) Assertion (iii) follows immediately from Lemma 5.

Let $G = K_{m,n,p}$ be a U$k$LC graph with $V(G) = V_1 \cup V_2 \cup V_3, G[V_1] = O_m, G[V_2] = O_n, G[V_3] = O_p, 2 \leqslant m \leqslant n \leqslant p, k \geqslant 3$. Set

$$V_1 = \{u_1, u_2, \ldots, u_m\}, V_2 = \{v_1, v_2, \ldots, v_n\}, V_3 = \{w_1, w_2, \ldots, w_p\}.$$

Suppose that, for the given $k$-list assignment $L$:

$L_{u_i} = \{a_{i,1}, a_{i,2}, \ldots, a_{i,k}\}$ for every $i = 1, \ldots, m$,

$L_{v_i} = \{b_{i,1}, b_{i,2}, \ldots, b_{i,k}\}$ for every $i = 1, \ldots, n$,

$L_{w_i} = \{c_{i,1}, c_{i,2}, \ldots, c_{i,k}\}$ for every $i = 1, \ldots, p$,

there is a unique $k$-list color $f$:

$f(u_i) = a_{i,1}$ for every $i = 1, \ldots, m$,

$f(v_i) = b_{i,1}$ for every $i = 1, \ldots, n$,

$f(w_i) = c_{i,1}$ for every $i = 1, \ldots, p$.

Theorem 2. *(i)* $a_{i,1} \neq b_{j,1}$ *for every* $i = 1, \ldots, m, j = 1, \ldots, n$*;*

*(ii)* $a_{i,1} \neq c_{j,1}$ *for every* $i = 1, \ldots, m, j = 1, \ldots, p$*;*

*(iii)* $b_{i,1} \neq c_{j,1}$ *for every* $i = 1, \ldots, n, j = 1, \ldots, p$*;*

*(iv)* $a_{i,1} \notin \{a_{j,2}, a_{j,3}, \ldots, a_{j,k}\}$ *for every* $i, j = 1, 2, \ldots, m$*;*

*(v)* $b_{i,1} \notin \{b_{j,2}, b_{j,3}, \ldots, b_{j,k}\}$ *for every* $i, j = 1, 2, \ldots, n$*;*

*(vi)* $c_{i,1} \notin \{c_{j,2}, c_{j,3}, \ldots, c_{j,k}\}$ *for every* $i, j = 1, 2, \ldots, p$.

Proof. (i) Since $G = K_{m,n,p}$ is a complete tripartite graph, $u_i$ is adjacent to $v_j$ for every $i = 1, \ldots, m, j = 1, \ldots, n$. So it is not difficult to see that $a_{i,1} = f(u_i) \neq f(v_j) = b_{j,1}$ for every $i = 1, \ldots, m, j = 1, \ldots, n$.

(ii) Similar proofs (i).

(iii) Similar proofs (i).

(iv) If $i = j$, then it is obvious that the conclusion is true. If $i \neq j$, then we suppose that there exists $i_0, j_0$ such that $i_0, j_0 = 1, \ldots, m; i_0 \neq j_0$ and $a_{i_0,1} \in \{a_{j_0,2}, a_{j_0,3}, \ldots, a_{j_0,k}\}$. It is clear that $a_{i_0,1} \neq a_{j_0,1}$. Let $f'$ be the coloring of $G$ such that

(a) $f'(u_{j_0}) = a_{i_0,1}$;

(b) $f'(u_i) = a_{i,1}$ for every $i \in \{1, \ldots, m\}, i \neq j_0$;

(c) $f'(v_i) = b_{i,1}$ for every $i = 1, \ldots, n$;

(d) $f'(w_i) = c_{i,1}$ for every $i = 1, \ldots, p$.

Then $f'$ is a $k$-list coloring for $G$ and $f' \neq f$, a contradiction.

(v) Similar proofs (iii).

(vi) Similar proofs (iii).

Set $\overline{f(v)} = L(v) \setminus \{f(v)\}$ for every $v \in V(G)$.

Theorem 3. *(i)* $|f(V_i)| > k-2$ *for every* $i = 1, 2, 3$*;*

*(ii)* $\cup_{v\in V_i}\overline{f(v)} \subseteq f(V_j \cup V_t)$ *for every* $i, j, t \in \{1, 2, 3\}$ *and* $i, j, t$ *are doubles a distinction;*

*(iii)* $\cup_{v\in V(G)}\overline{f(v)} \subseteq f(V(G))$*;*

*(iv) There exists* $v \in V_i \cup V_j$ *such that* $\overline{f(v)} \subseteq f(V_t)$ *for every* $i, j, t \in \{1, 2, 3\}$ *and* $i, j, t$ *are doubles a distinction.*

Proof. (i) For suppose on the contrary that $|f(V_1)| = t \leqslant k-2$,. Set $H = G - V_1$, it is not difficult to see that $H$ is complete bipartite graph $K_{n,p}$. We assign the following lists $L'(v)$ for the vertices $v$ of $H$:

If $f(V_1) \subseteq L(v)$ then $L'(v) = L(v) \setminus f(V_1)$.

If there exists $A \subseteq f(V_1)$ such that $A \cap L(v) = \emptyset$, then

$$L'(v) = L(v) \setminus \{d_1, d_2, \ldots, d_{t-|A|}, e_1, e_2, \ldots, e_{|A|}\},$$

where

$$d_1, d_2, \ldots, d_{t-|A|} \in L(v) \setminus A, e_1, e_2, \ldots, e_{|A|} \in L(v)$$

and $f(v) \notin \{e_1, e_2, \ldots, e_{|A|}\}$.
It is clear that $|L'(v)| = k - t \geqslant 2$ for every $v \in V(H)$. By Lemma 3, $H$ has the property $M(2)$. So by Lemma 1, $H$ has the property $M(k-t)$. It follows that with lists $L'(v)$, there exist at least two list colorings for the vertices $v$ of $H$. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices $v$ of $G$, a contradiction. Thus, $|f(V_1)| > k-2$.

By the same method of proof as above, we can also prove that $|f(V_2)| > k-2$ and $|f(V_3)| > k-2$.

(ii) For suppose on the contrary that $\cup_{v\in V_1}\overline{f(v)} \nsubseteq f(V_2 \cup V_3)$. Then there exists $i_0, j_0$ such that $a_{i_0,j_0} \notin f(V_2 \cup V_3)$ with $1 \leqslant i_0 \leqslant m, 2 \leqslant j_0 \leqslant k$. Let $f'$ be the coloring of $G$ such that

(a) $f'(u_{i_0}) = a_{i_0,j_0}$;
(b) $f'(u_i) = a_{i,1}$ for every $i \in \{1, \ldots, m\}, i \neq i_0$;
(c) $f'(v_i) = b_{i,1}$ for every $i = 1, \ldots, n$;
(d) $f'(w_i) = c_{i,1}$ for every $i = 1, \ldots, p$.

Then $f'$ is a $k$-list coloring for $G$ and $f' \neq f$, a contradiction. Thus,

$$\cup_{v\in V_1}\overline{f(v)} \subseteq f(V_2 \cup V_3).$$

By the same method of proof as above, we can also prove that $\cup_{v\in V_2}\overline{f(v)} \subseteq f(V_1 \cup V_3)$ and $\cup_{v\in V_3}\overline{f(v)} \subseteq f(V_1 \cup V_2)$.

(iii) For suppose on the contrary that $\cup_{v\in V(G)}\overline{f(v)} \nsubseteq f(V(G)$. Without loss of generality, we may assume that there exists $i_0, j_0$ such that $a_{i_0,j_0} \notin f(V(G))$ with $1 \leqslant i_0 \leqslant m, 2 \leqslant j_0 \leqslant k$.

Let $f'$ be the coloring of $G$ such that

(a) $f'(u_{i_0}) = a_{i_0,j_0}$;
(b) $f'(u_i) = a_{i,1}$ for every $i \in \{1, \ldots, m\}, i \neq i_0$;
(c) $f'(v_i) = b_{i,1}$ for every $i = 1, \ldots, n$;
(d) $f'(w_i) = c_{i,1}$ for every $i = 1, \ldots, p$.

Then $f'$ is a $k$-list coloring for $G$ and $f' \neq f$, a contradiction.

(iv) For suppose on the contrary that $\overline{f(v)} \nsubseteq f(V_1)$ for every $v \in V_2 \cup V_3$, then $|\overline{f(v)} \setminus f(V_1)| \geqslant 1$ for every $v \in V_2 \cup V_3$. So $|L(v) \setminus f(V_1)| \geqslant 2$ for every $v \in V_2 \cup V_3$. Set graph

$$H = G - V_1 = G[V_2 \cup V_3] = K_{n,p}.$$

Let $L'(v) \subseteq L(v) \setminus f(V_1)$ such that $|L'(v)| = 2$ for every $v \in V_2 \cup V_3$. By Lemma 3, $H$ has the property $M(2)$, it follows that with lists $L'(v)$, there exist at least two list colorings for the vertices $v$ for every $v \in V_2 \cup V_3$. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices $v$ of $G$, a contradiction. Thus, there exists $v \in V_2 \cup V_3$ such that $\overline{f(v)} \subseteq f(V_1)$.

By the same method of proof as above, we can also prove that there exists $v \in V_1 \cup V_3$ such that $\overline{f(v)} \subseteq f(V_2)$ and there exists $v \in V_1 \cup V_2$ such that $\overline{f(v)} \subseteq f(V_3)$.

## 3. On property $M(4)$ of some complete tripartite graphs

Set the complete tripartite graph $G = K_{m,n,p}$. Let $V(G) = V_1 \cup V_2 \cup V_3$ is a partition of $V(G)$ such that $V_1 = \{u_1, u_2, \ldots, u_m\}$, $V_2 = \{v_1, v_2, \ldots, v_n\}$, $V_3 = \{w_1, w_2, \ldots, w_p\}$ and for every $i = 1, 2, 3$ the subgraphs of $G$ induced by $V_i$, is empty graph.

Lemma 7. *$m(K_{2,2,p}) = 3$ if $1 \leqslant p \leqslant 2$.*

Proof. By Lemma 3, $G$ is U2LC. Suppose that $G$ is U3LC. By Lemma 5, $|V(G)| \geqslant 7$, a contradiction. So $m(G) = 3$.

Lemma 8 ([10]). *$m(K_{2,2,3}) = m(K_{2,3,3}) = 3$.*

Lemma 9 ([17]). *$m(K_{2,2,p}) = 3$ if $4 \leqslant p \leqslant 8$.*

Lemma 10. *$m(K_{2,2,p}) = 3$ if $1 \leqslant p \leqslant 8$.*

Proof. It follows from Lemma 7, Lemma 8 and Lemma 9.

Lemma 11 ([18]). *The graph $K_{2,3,4}$ has the property M(3).*

Lemma 12. *$m(K_{2,3,4}) = 3$.*

Proof. It follows from Lemma 3 and Lemma 11.

Theorem 4. *$G = K_{m,n,p}$ is U3LC if one of the following conditions occurs.*
*(i) $m \geqslant 2, n \geqslant 2$ and $p \geqslant 9$;*
*(ii) $m \geqslant 2, n \geqslant 3$ and $p \geqslant 5$;*
*(iii) $m \geqslant 2, n \geqslant 4$ and $p \geqslant 4$;*
*(iv) $m, n, p \geqslant 3$.*

Proof. (i) We assign the following lists for the vertices of $G$: $L(u_1) = \{1, 2, 6\}$, $L(u_2) = L(u_3) = \ldots = L(u_m) = \{3, 4, 5\}$;
$L(v_1) = \{1, 3, 6\}$, $L(v_2) = L(v_3) = \ldots = L(v_n) = \{2, 4, 6\}$;
$L(w_1) = \{1, 4, 5\}$, $L(w_2) = \{1, 3, 6\}$, $L(w_3) = \{1, 4, 6\}$, $L(w_4) = \{1, 5, 6\}$, $L(w_5) = \{2, 3, 4\}$, $L(w_6) = \{2, 3, 5\}$, $L(w_7) = \{2, 3, 6\}$, $L(w_8) = \{2, 4, 6\}$, $L(w_9) = L(w_{10}) = \ldots = L(w_p) = \{2, 5, 6\}$.
A unique coloring $f$ of $G$ exists from the assigned lists: $f(u_1) = 6, f(u_2) = f(u_3) = \ldots = f(u_m) = 5$;
$f(v_1) = 3$, $f(v_2) = f(v_3) = \ldots = f(v_n) = 4$;
$f(w_1) = f(w_2) = f(w_3) = f(w_4) = 1$, $f(w_5) = f(w_6) = \ldots = f(w_p) = 2$.
(ii) We assign the following lists for the vertices of $G$: $L(u_1) = \{1, 3, 6\}$, $L(u_2) = L(u_3) = \ldots = L(u_m) = \{2, 4, 5\}$;
$L(v_1) = \{1, 2, 3\}$, $L(v_2) = L(v_3) = \ldots = L(v_n) = \{2, 4, 5\}$;
$L(w_1) = \{1, 3, 5\}$, $L(w_2) = \{1, 4, 5\}$, $L(w_3) = \{1, 4, 6\}$, $L(w_4) = \{2, 3, 4\}$, $L(w_5) = L(w_6) = \ldots = L(w_p) = \{2, 5, 6\}$.
A unique coloring $f$ of $G$ exists from the assigned lists: $f(u_1) = 6, f(u_2) = f(u_3) = \ldots = f(u_m) = 5$;
$f(v_1) = 3$, $f(v_2) = f(v_3) = \ldots = f(v_n) = 4$;
$f(w_1) = f(w_2) = f(w_3) = 1$, $f(w_4) = f(w_5) = \ldots = f(w_p) = 2$.
(iii) We assign the following lists for the vertices of $G$: $L(u_1) = \{1, 3, 5\}$, $L(u_2) = L(u_3) = \ldots = L(u_m) = \{2, 4, 6\}$;
$L(v_1) = \{1, 2, 3\}$, $L(v_2) = \{1, 3, 5\}$, $L(v_3) = \{1, 2, 4\}$, $L(v_4) = L(v_5) = \ldots = L(v_n) = \{2, 4, 6\}$;
$L(w_1) = \{1, 4, 5\}$, $L(w_2) = \{1, 3, 6\}$, $L(w_3) = \{2, 3, 4\}$, $L(w_4) = L(w_5) = \ldots = L(w_p) = 2, 5, 6$.
A unique coloring $f$ of $G$ exists from the assigned lists: $f(u_1) = 5$, $f(u_2) = f(u_3) = \ldots = f(u_m) = 6$;
$f(v_1) = f(v_2) = 3$, $f(v_3) = f(v_4) = \ldots = f(v_n) = 4$;

$f(w_1) = f(w_2) = 1$, $f(w_3) = f(w_4) = \ldots = f(w_p) = 2$.

(iv) We assign the following lists for the vertices of $G$: $L(u_1) = \{1, 4, 6\}$, $L(u_2) = \{2, 3, 6\}$, $L(u_3) = L(u_4) = \ldots = L(u_m) = \{2, 4, 5\}$;

$L(v_1) = \{2, 3, 6\}$, $L(v_2) = \{1, 2, 4\}$, $L(v_3) = L(v_4) = \ldots = L(v_n) = \{4, 5, 6\}$;

$L(w_1) = \{2, 3, 5\}$, $L(w_2) = \{2, 4, 6\}$, $L(w_3) = L(w_4) = \ldots = L(w_p) = \{3, 4, 6\}$.

A unique coloring $f$ of $G$ exists from the assigned lists: $f(u_1)=1, f(u_2)=f(u_3)=\ldots=f(u_m)=2$;

$f(v_1) = 3$, $f(v_2) = f(v_3) = \ldots = f(v_n) = 4$;

$f(w_1) = 5$, $f(w_2) = f(w_3) = \ldots = f(w_p) = 6$.

Corollary. (i) $G = K_{2,2,p}$ is U3LC if and only if $p \geqslant 9$;

(ii) $G = K_{2,3,p}$ is U3LC if and only if $p \geqslant 5$;

(iii) $G = K_{2,4,p}$ is U3LC if and only if $p \geqslant 4$;

(iv) $G = K_{3,3,p}$ is U3LC if and only if $p \geqslant 3$.

Proof. (i) It follows from Lemma 10 and (i) of Theorem 3.

(ii) It follows from (ii) of Theorem 1, Lemma 8 and Lemma 12.

(iii) It follows from (ii) of Theorem 1, Lemma 10 and Lemma 12.

(iv) It follows from (ii) of Theorem 1 and Lemma 8.

Lemma 13. *The graph $G = K_{2,n,p}$ has the property $M(4)$.*

Proof. For suppose on the contrary that $G$ is U4LC. Then for each vertex $v$ in $G$, there exists a list of 4 colors $L(v)$, such that there exists a unique $L$-coloring for $G$. By (i) of Theorem 3 we have $2 = |V_1| \geqslant |f(V_1)| > 4 - 2 = 2$, contradiction. Thus, $G = K_{2,n,p}$ has the property M(4).

The join of $O_m$ and $K_n$, $O_m + K_n = S(m, n)$, is called a complete split graph.

Lemma 14 ([10]). *For every $n \geqslant 2$, we have $m(S(3, n)) = 3$.*

Theorem 5. *(i) $m(K_{2,2,p}) = 4$ if and only if $p \geqslant 9$;*
*(ii) $m(K_{2,3,p}) = 4$ if and only if $p \geqslant 5$;*
*(iii) $m(K_{2,4,p}) = 4$ if and only if $p \geqslant 4$;*
*(iv) $m(K_{3,3,3}) = 4$.*

Proof. (i) It follows from (i) of Theorem 4 and Lemma 13.

(ii) It follows from (ii) of Theorem 4 and Lemma 13.

(iii) It follows from (iii) of Theorem 4 and Lemma 13.

(iv) For suppose on the contrary that $G = K_{3,3,3}$ is U4LC. Then for each vertex $v$ in $G$, there exists a list of 4 colors $L(v)$, such that there exists a unique $L$-coloring for $G$. By (i) of Theorem 3, $|f(V_1)|, |f(V_2)| > 4-2 = 2$, it follows that $|f(V_1)| = |f(V_2)| = 3$. So $f(u_i) \neq f(u_j)$ and $f(v_i) \neq f(v_j)$ for every $i, j = 1, 2, 3, i \neq j$. Set graph $G' = (V', E')$ with $V' = V(G)$,

$$E' = E(G) \cup \{u_i u_j | i, j = 1, 2, \ldots, m; i \neq j\} \cup \{v_i v_j | i, j = 1, 2, \ldots, n; i \neq j\}.$$

It is clear that $G'$ is complete split graph $S(3, 6)$. By Lemma 14, $G'$ has the property $M(3)$. By Lemma 1, $G'$ has the property $M(4)$, so with lists $L(v)$, there exist at least two list colorings for the vertices $v$ of $G'$. Since $V(G) = V(G')$, it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices $v$ of $G$, a contradiction. Thus, $G$ has the property $M(4)$. By (iv) of Theorem 4, we have $m(K_{3,3,3}) = 4$.

## СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. M. Behzad, *Graphs and thei chromatic number*, Doctoral Thesis (Michigan State University), 1965.

2. M. Behzad and G. Chartrand, *Introduction to the theory of graphs*, Allyn and Bacon, Boston, 1971.

3. M. Behzad, G. Chartrand and J. Cooper, *The coloring numbers of complete graphs*, J. London Math. Soc. № 42 (1967), 226 – 228.

4. J.A. Bondy and U.S.R. Murty, *Graph theory with applications*, MacMillan, 1976.

5. R. Diestel, *Graph Theory*, Springer – Verlag New Youk, 2000.

6. J.H. Dinitz and W.J. Martin, The stipulation polynomial of a uniquely list colorable graph, *Austran. J. Combin.* № 11 (1995) 105–115.

7. P. Erdös, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proceedings of west coast conference on combinatorics, graph theory, and computing*, number 26 in Congr. Numer., pages 125–157, Arcata, CA, September 1979.

8. M. Ghebleh and E.S. Mahmoodian, On uniquely list colorable graphs, *Ars Combin.* № 59 (2001) 307–318.

9. Le Xuan Hung, Colorings of the graph $K_2^m + K_n$, *Journal of Siberian Federal University. Mathematics & Physics*, to appear.

10. M. Mahdian and E.S. Mahmoodian, A characterization of uniquely 2-list colorable graphs, *Ars Combin.* № 51 (1999) 295–305.

11. R.C. Read, An introduction to chromatic polynomials, *J. Combin. Theory* № 4 (1968) 52–71.

12. Ngo Dac Tan and Le Xuan Hung, On colorings of split graphs, *Acta Mathematica Vietnammica*, Vol. 31, № 3, 2006, pp. 195 – 204.

13. V.G. Vizing, On an estimate of the chromatic class of a $p$-graph, *Discret. Analiz.* № 3 (1964), pp. 23–30. (In Russian)

14. V. G. Vizing. Coloring the vertices of a graph in prescribed colors. In *Diskret. Analiz*, № 29 in Metody Diskret. Anal. v Teorii Kodov i Shem, pp. 3–10, 1976.

15. W. Wang and X. Liu, List-coloring based channel allocation for open-spectrum wireless networks, in *Proceedings of the IEEE International Conference on Vehicular Technology (VTC '05)*, 2005, pp. 690 – 694.

16. R.J. Wilson, *Introduction to graph theory*, Longman group ltd, London, (1975).

17. Yancai Zhao and Erfang Shan, On characterization of uniquely 3-list colorable complete multipartite graphs, *Discussiones Mathematicae Graph Theory*, № 30, 2010, pp. 105–114.

18. Y.Q. Zhao, W.J. He, Y.F. Shen and Y.N. Wang, Note on characterization of uniquely 3-list colorable complete multipartite graphs, in: *Discrete Geometry, Combinatorics and Graph Theory*, LNCS 4381 (Springer, Berlin, 2007, pp. 278–287.


## REFERENCES


1. M. Behzad, 1965, “Graphs and thei chromatic number”, Doctoral Thesis (Michigan State University).

2. M. Behzad and G. Chartrand, 1971, "Introduction to the theory of graphs", Allyn and Bacon, Boston.

3. M. Behzad, G. Chartrand and J. Cooper, 1967, "The coloring numbers of complete graphs", J. London Math. Soc. № 42, pp. 226 – 228.

4. J.A. Bondy and U.S.R. Murty, 1976, "Graph theory with applications", MacMillan.

5. R. Diestel, 2000, "Graph Theory", Springer – Verlag New York.

6. J.H. Dinitz and W.J. Martin, 1995, "The stipulation polynomial of a uniquely list colorable graph", *Austran. J. Combin.* № 11, pp. 105–115.

7. P. Erdös, A. L. Rubin, and H. Taylor, 1979, "Choosability in graphs". In *Proceedings of west coast conference on combinatorics, graph theory, and computing*, number 26 in Congr. Numer., pp. 125–157, Arcata, CA.

8. M. Ghebleh and E.S. Mahmoodian, 2001, "On uniquely list colorable graphs", *Ars Combin.* № 59, pp. 307–318.

9. Le Xuan Hung, "Colorings of the graph $K_2^m + K_n$", *Journal of Siberian Federal University. Mathematics & Physics*, to appear.

10. M. Mahdian and E.S. Mahmoodian, 1999, "A characterization of uniquely 2-list colorable graphs", *Ars Combin.* № 51, pp. 295–305.

11. R.C. Read, 1968, "An introduction to chromatic polynomials", *J. Combin. Theory*, № 4, pp. 52–71.

12. Ngo Dac Tan and Le Xuan Hung, 2006, "On colorings of split graphs", *Acta Mathematica Vietnammica*, Volume 31, № 3, pp. 195 – 204.

13. V.G. Vizing, 1964, "On an estimate of the chromatic class of a $p$-graph", *Discret. Analiz.* № 3, pp. 23–30. (In Russian).

14. V. G. Vizing ,1976, "Coloring the vertices of a graph in prescribed colors". In *Diskret. Analiz*, number 29 in Metody Diskret. Anal. v Teorii Kodov i Shem, pp. 3–10.

15. W. Wang and X. Liu, 2005, "List-coloring based channel allocation for open-spectrum wireless networks
, in *Proceedings of the IEEE International Conference on Vehicular Technology (VTC '05)*, pp. 690 – 694.

16. R.J. Wilson, 1975, *Introduction to graph theory*, Longman group ltd, London.

17. Yancai Zhao and Erfang Shan, 2010, "On characterization of uniquely 3-list colorable complete multipartite graphs", *Discussiones Mathematicae Graph Theory*, № 30, pp. 105–114.

18. Y.Q. Zhao, W.J. He, Y.F. Shen and Y.N. Wang, 2007, "Note on characterization of uniquely 3-list colorable complete multipartite graphs", in: *Discrete Geometry, Combinatorics and Graph Theory*, LNCS 4381 (Springer, Berlin, pp. 278–287.

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