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Обобщение некоторых интегральных неравенств для оператора Римана — Лиувилля

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Аннотация

Неравенство Чебышева является одним из самых важных неравенств в математике. Оно играет важную роль в теории вероятности, а также тесно связано с неравенством Маркова в анализе.

В [6, 7], используя интегральный оператор Римана — Лиувилля I^{α}, авторы установили и доказали некоторые новые интегральные неравенства для чебышевского функционала

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx.$$

В данной работе рассматриваются некоторые обобщения интегральных неравенств чебышевского типа, где используются дробные интегралы Римана — Лиувилля в соответствии с другой функцией.

Ключевые слова: Дробные интегралы, неравенства Чебышева, дробный оператор Римана — Лиувилля, обобщения.

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Generalizations of some integral inequalities for Riemann–Liouville operator

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Abstract

The Chebyshev inquality is one of important inequalities in mathematics. It's a necessary tool in probability theory. The item of Chebyshev's inequality may also refer to Markov's inequality in the context of analysis.

In[6, 7], using the usual Riemann–Liouville fractional integral operator I^{α}, were established and proved some new integral inequalities for the Chebyshev fonctional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx.$$

In this work, we give some generalizations of Chebyshev-type integral inequalities by using Riemann—Liouville fractional integrals of function with respect to another function.

Keywords: Fractional integral, Chebyshev's inequality, Riemann—Liouville Fractional operator, generalizations.

Bibliography: 8 titles.

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1. Introduction

Many Integral inequalities of various types have been presented in the literature.
See [4,5], among them, we choose to recall the following Chebyshev Inequality

$$T(f,g) \geq 0, \tag{1}$$

where

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx \tag{2}$$

f,g are two integrable functions, synchronous on $[a,b]$ (i.e.$(f(x) - f(y))(g(x) - g(y)) \geq 0$), for $x, y \in [a,b]$.

In this section, we give some definitions and theorems.

In [8] the following definition was given.

Definition 1. Let $f \in L_1([0,\infty))$. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ is defined as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \, \alpha > 0, \quad I^0 f(x) = f(x), \tag{3}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-t} t^{\alpha-1} dt$.

Definition 2. For $(a,b) \subset \mathbb{R}$ we shall denote by $L_p(a,b)(1 \le p \le \infty)$ the set of functions f real-value Lebesgue measurable on (a,b), such that

$$\|f\|_{L_p} = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} < \infty, \tag{4}$$

and for the case $p = \infty$

$$\|f\|_{L_p} = ess \sup_{a \le t \le b} |f(t)| < \infty.$$

Definition 3. Let $f \in L_1([0,\infty[)$ and $h(x)$ be an increasing and positive function on $[0,\infty[$ having a continuous derivative $h'(x)$ on $[0,\infty[, h(0) = 0$. The space $X_h^p(0,\infty)(1 \le p < \infty)$ of those real-valued Lebesgue measurable functions f on $[0,\infty[$ for which

$$\|f\|_{\mathbf{X}_h^p} = \left(\int_0^{\infty} |f(t)|^p h^{'}(t) dt \right)^{\frac{1}{p}}, 1 \le p < \infty \tag{5}$$

and for $p = \infty$

$$\|f\|_{\mathbf{X}_h^p} = ess \sup_{0 \le t < \infty} |f(t) h^{'}(t)|.$$

In particular, when $h(x) = x \, (1 \le p < \infty)$ the space $\mathbf{X}_h^p(0,\infty)$ coincides with the $L_p([0,\infty))$

Definition 4. Let $f \in \mathbf{X}_h^p(0,\infty)$ and $h(x)$ be an increasing and positive function on $[0,\infty)$ and also derivative $h^{'}(x)$ is continuous on $[0,\infty)$ and $h(0) = 0$. The Riemann–Liouville fractional integral of a function $f(x)$ with respect to another function $h(x)$ is defined by

$$I_h^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (h(x) - h(t))^{\alpha-1} f(t) h^{'}(t) dt. \tag{6}$$

For the convenience of establishing the results, we give the semi-group property:

$$I_h^{\alpha} I_h^{\beta} f(x) = I_h^{\alpha+\beta} f(x), \quad \alpha \ge 0, \beta \ge 0, \tag{7}$$

which implies the commutative property

$$I_h^{\alpha} I_h^{\beta} f(x) = I_h^{\beta} I_h^{\alpha} f(x). \tag{8}$$

For details, one can consult [3].

Remark 1. If $f(x) = 1$, by (6), we get

$$I_h^{\alpha} f(x) = \frac{h^{\alpha}(x)}{\Gamma(\alpha+1)}, \quad \alpha \ge 0. \tag{9}$$

REMARK 2. If $f(x) = h^\gamma(x)$, then we have

$$I_h^\alpha[h^\gamma(x)] = \frac{h^{\gamma+\alpha}(x)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}, \qquad \alpha \geq 0, \gamma \geq 0. \tag{10}$$

For details to see([3]).

REMARK 3. Note that for $h(x) = x, I_h^\alpha f(x)$ is the usual Riemann–Liouville fractional operator I^α.

In [6, 7], the following definition was introduced.

DEFINITION 5. A real valued function $f : [0, \infty) \to \mathbb{R}$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such $f(x) = x^p f_1$, where $f_1 \in C([0, \infty))$.

DEFINITION 6. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ for a function $f \in C_\mu, (\mu \geq -1)$ is defined as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt; \; \alpha > 0,$$
$$I^0 f(x) = f(x),$$

where $\Gamma(\alpha) := \int_0^\infty e^{-t} t^{\alpha-1} dt$.

In [6, 7], the following theorems and corollary were proved.

THEOREM 1. Let f and g be two synchronous functions on $[0, \infty[$. Then for all $x > 0, \quad \alpha > 0$,

$$I^\alpha(fg)(x) \geq \frac{\Gamma(\alpha+1)}{x^\alpha} I^\alpha f(x) I^\alpha g(x). \tag{11}$$

THEOREM 2. Let f and g be two synchronous functions on $[0, \infty[$. Then for all $x > 0, \quad \alpha > 0$ *and* $\beta > 0$,

$$\frac{x^\alpha}{\Gamma(\alpha+1)} I^\beta(fg)(x) + \frac{x^\beta}{\Gamma(\beta+1)} I^\alpha(fg)(x) \geq I^\alpha f(x) I^\beta g(x) + I^\beta f(x) I^\alpha g(x). \tag{12}$$

THEOREM 3. Let $(f_i)_{i=1,2,\ldots,n}$ be n positive increasing functions on $[0, \infty[$. Then for any $x > 0, \alpha > 0$,

$$I^\alpha(\prod_{i=1}^n f_i)(x) \geq (I^\alpha(1))^{1-n} \prod_{i=1}^n I^\alpha f_i(x). \tag{13}$$

THEOREM 4. Let $p \geq 1$, such that $\frac{1}{p} + \frac{1}{p'} = 1$, if $|f|^p(x)$ and $|g|^{p'}(x)$ are two functions in $C_{\lambda,}(\lambda \geq 1)$, then for $\alpha > 0$, the fractional integral inequality

$$I^\alpha|fg|(x) \leq (I^\alpha|f|^p(x))^{\frac{1}{p}} (I^\alpha|g|^{p'}(x))^{\frac{1}{p'}}, \tag{14}$$

holds.

THEOREM 5. Let f and g be two functions defined on $[0, \infty)$, such that f is increasing and g is differentiable and there exists a real number $m := \inf_{x \geq 0} g^{'}(x)$. Then the inequality

$$I^\alpha(fg)(x) \geq \frac{\Gamma(\alpha+1)}{t^\alpha} I^\alpha f(x) I^\alpha g(x) - \frac{mx}{\alpha+1} I^\alpha f(x) + m I^\alpha(x f(x)) \tag{15}$$

is valid for all $x > 0, \; \alpha > 0$.

COROLLARY 1. Let f and g be two functions defined on $[0, \infty)$. Suppose that f is decreasing and g is differentiable and there exists a real number $M := \sup_{x \geq 0} g^{'}(x)$, then for all $x > 0, \; \alpha > 0$, we have

$$I^\alpha(fg)(x) \geq \frac{\Gamma(\alpha+1)}{t^\alpha} I^\alpha f(x) I^\alpha g(x) - \frac{Mx}{\alpha+1} I^\alpha f(x) + M I^\alpha(x f(x)). \tag{16}$$

2. Main Results

The aim of this work is to generalize the Theorems 1, 2, 3, 4, 5 and Corollary 1 (the results of [6, 7]) for the operator I_h^α.

THEOREM 6. Let f and g be two synchronous functions on $[0,\infty[$ and $h(x)$ be an increasing and positive function on $[0,\infty[$, having a continuous derivative $h'(x)$ on $[0,\infty[$ and also $h(0)=0$, $x>0, \alpha>0$, then

$$I_h^\alpha(fg)(x) \geq \frac{\Gamma(\alpha+1)}{h^\alpha(x)} I_h^\alpha f(x) I_h^\alpha g(x). \tag{17}$$

The inequality (17) is reversed if the functions are asynchronous on $[0,\infty)$ (i.e.$((f(x)-f(y))$ $(g(x)-g(y))\leq 0)$ for $x,y\in[0,\infty)$).

PROOF. Since f and g are synchronous on $[0,\infty[$, then for $\mu\geq 0, \nu\geq 0$, we have

$$(f(\mu)-f(\nu))(g(\mu)-g(\nu))\geq 0, \tag{18}$$

equivalently

$$f(\mu)g(\mu)+f(\nu)g(\nu)\geq f(\mu)g(\nu)+f(\nu)g(\mu). \tag{19}$$

Multiplying both sides of (19) by $\frac{(h(x)-h(\mu))^{\alpha-1}h'(\mu)}{\Gamma(\alpha)}$, $\mu\in(0,x), x>0$ and integrating the resulting inequality with respect to μ from 0 *to* x, we have

$$I_h^\alpha(fg)(x)+f(\nu)g(\nu)\frac{h^\alpha(x)}{\Gamma(\alpha+1)}\geq g(\nu)I_h^\alpha(f)(x)+f(\nu)I_h^\alpha(g)(x). \tag{20}$$

Now, multiplying (20) by $\frac{(h(x)-h(\nu))^{\alpha-1}h'(\nu)}{\Gamma(\alpha)}$, $\nu\in(0,x)$ and integrating the resulting inequality with respect to ν from 0 *to* x, we get

$$I_h^\alpha(fg)(x)\geq\frac{\Gamma(\alpha+1)}{h^\alpha(x)}I_h^\alpha f(x)I_h^\alpha g(x). \tag{21}$$

This proves the theorem.

REMARK 4. If $h(x)=x$ in Theorem 6,we obtain the inequality (11) (Theorem 1).

COROLLARY 2. Let $f\in X_h^\alpha(0,\infty)$, then we have

$$I_h^\alpha(f^n)(x) \quad\geq\quad (I_h^\alpha(1))^{1-n}\left(I_h^\alpha f(x)\right)^n; \quad n\geq 2. \tag{22}$$

PROOF. We prove (22) by induction. For $n=2$ inequality(22) is proved in Theorem 6. Assume that the inequality is true for some $n\geq 2$ such that $g=f^n$, $g\in X_h^\alpha(0,\infty)$. Then by applying the inequality(17) to the functions f and g, we have

$$\begin{aligned}
I_h^\alpha(fg)(x) &\geq (I_h^\alpha(1))^{-1}\left(I_h^\alpha f\right)(x)\left(I_h^\alpha g\right)(x)\\
&= (I_h^\alpha(1))^{-1}\left(I_h^\alpha f\right)(x)\left(I_h^\alpha f^n\right)(x)\\
&\geq (I_h^\alpha(1))^{-n}\left(I_h^\alpha f(x)\right)^{n+1},
\end{aligned}$$

where the induction hypothesis for $n\geq 2$ is used for the second inequality.

THEOREM 7. Let f and g be two synchronous functions on $[0,\infty[$ and $h(x)$ be an increasing and positive function on $[0,\infty[$, derivative $h^{'}(x)$ is continuous on $[0,\infty[$ and also $h(0)=0, x>0$, $\alpha>0,\beta>0$, then

$$\frac{h^\beta(x)}{\Gamma(\beta+1)}I_h^\alpha(fg)(x)+\frac{h^\alpha(x)}{\Gamma(\alpha+1)}I_h^\beta(fg)(x)\geq I_h^\alpha f(x)I_h^\beta g(x)+I_h^\beta f(x)I_h^\alpha g(x). \tag{23}$$

PROOF. Multiplying both sides of (20) by $\frac{(h(x)-h(\nu))^{\beta-1}h'(\nu)}{\Gamma(\beta)}$, $\nu \in (0,x), x>0$ and integrating the resulting inequality with respect to ν from 0 to x, we have

$$\frac{h^{\beta}(x)}{\Gamma(\beta+1)}I_h^{\alpha}(fg)(x)+\frac{h^{\alpha}(x)}{\Gamma(\alpha+1)}I_h^{\beta}(fg)(x)\geq I_h^{\alpha}f(x)I_h^{\beta}g(x)+I_h^{\beta}f(x)I_h^{\alpha}g(x).$$

This completes the proof of Theorem.

REMARK 5. If $h(x)=x$ in Theorem 7, we get the inequality (12) (Theorem 2).

THEOREM 8. Let $(f_i)_{i=1,2,\dots,n}$ be n positive increasing functions on $[0,\infty[$. Then for any $x>0, \alpha>0$, we get

$$I_h^{\alpha}\left\{\prod_{i=1}^{n}f_i(x)\right\}\geq (I_h^{\alpha}(1))^{1-n}\prod_{i=1}^{n}I_h^{\alpha}\{f_i(x)\}. \tag{24}$$

PROOF. We prove this theorem by induction. If $n=2$, in (24), we get

$$I_h^{\alpha}\{f_1(x)f_2(x)\}\geq (I_h^{\alpha}(1))^{-1}I_h^{\alpha}f_1(x)I_h^{\alpha}f_2(x).$$

Which holds in view of Theorem 6. We suppose that the inequality

$$I_h^{\alpha}\left\{\prod_{i=1}^{n-1}f_i(x)\right\}\geq (I_h^{\alpha}(1))^{2-n}\prod_{i=1}^{n-1}I_h^{\alpha}\{f_i(x)\}, \tag{25}$$

holds for some positive integer $n\geq 2$. Now $(f_i)_{i=1,2,3,\dots\dots,n}$ are increasing functions implies that the function $\prod_{i=1}^{n-1}f_i(x)$, is also an increasing function. Therefore, we can apply inequality (25) by putting $\prod_{i=1}^{n-1}f_i(x)=g(x)$ and $f_n(x)=f(x)$. Then

$$\prod_{i=1}^{n-1}f_i(x)f_n=\prod_{i=1}^{n}f_i(x)\geq (I_h^{\alpha}(1))^{-1}I_h^{\alpha}\prod_{i=1}^{n-1}f_i(x)I_h^{\alpha}f_n(x)\geq (I_h^{\alpha}(1))^{1-n}I_h^{\alpha}\prod_{i=1}^{n}f_i(x).$$

Where the induction hypothesis for n is used in the second inequality. The proof of Theorem 8, is complete.

REMARK 6. If we put $h(x)=x$ in (24), we obtain inquality (13) (Theorem 3).

To prove the next Theorem we shall use the following Lemma(for details see [2]).

LEMMA 1. Let $0<p<\infty$ and p,p' are conjugate, $f\in L_p([a,b]), g\in L_{p'}([a,b])$ and ω be a weight function (non negative measurable function). 1) $1\geq p$, then

$$\int_a^b fg\omega dx \quad\leq\quad \left(\int_a^b f^p\omega dx\right)^{\frac{1}{p}}\left(\int_a^b g^{p'}\omega dx\right)^{\frac{1}{p'}}, \tag{26}$$

2) $0<p<1$, then

$$\int_a^b fg\omega dx \quad\geq\quad \left(\int_a^b f^p\omega dx\right)^{\frac{1}{p}}\left(\int_a^b g^{p'}\omega dx\right)^{\frac{1}{p'}}. \tag{27}$$

THEOREM 9. Let $p\geq 1$, be such that $\frac{1}{p}+\frac{1}{p'}=1$, if $f\in \mathbf{X}_h^p(0,\infty), g\in \mathbf{X}_h^{p'}(0,\infty)$. Then for $\alpha>0$

$$I_h^{\alpha}|fg|(x)\leq \{I_h^{\alpha}|f|^p(x)\}^{\frac{1}{p}}\{I_h^{\alpha}|g|^{p'}(x)\}^{\frac{1}{p'}}. \tag{28}$$

PROOF. Let $h(x)-h(t)\leq M$, $0<M<\infty$. If $f\in \mathbf{X}_h^p(0,\infty)$, we have

$$\left|\int_0^x((h(x)-h(t))^{\alpha-1}f(t)^p h'(t)dt\right| \leq \int_0^x((h(x)-h(t))^{\alpha-1}|f(t)|^p h'(t)dt$$
$$\leq M^{\alpha-1}\int_0^x|f(t)|^p h'(t)dt<\infty,$$

therefore

$$\left|\int_0^x((h(x)-h(t))^{\alpha-1}f(t)^p h'(t)dt\right| < \infty \tag{29}$$

and

$$\left|\int_0^x((h(x)-h(t))^{\alpha-1}g(t)^{p'} h'(t)dt\right| < \infty. \tag{30}$$

The proof of (30) is similar to that of (29). Since $f \in \mathbf{X}_h^p(0,\infty), g \in \mathbf{X}_h^{p'}(0,\infty)$, then $f(t).h'(t)^{\frac{1}{p}} \in L_p(0,\infty)$ and $g(t).h'(t)^{\frac{1}{p'}} \in L_{p'}(0,\infty)$. Applying the inequality (26), it yields that,

$$\int_0^x|(h(x)-h(t))^{\alpha-1}f(t)g(t)h'(t)|dt$$
$$\leq\left\{\int_0^x(h(x)-h(t))^{\alpha-1}f(t)^p h'(t)dt\right\}^{\frac{1}{p}}\left\{\int_0^x(h(x)-h(t))^{\alpha-1}g(t)^{p'}h'(t)dt\right\}^{\frac{1}{p'}}<\infty.$$

Now multiplying the last inequality by $(\Gamma(\alpha))^{-1}=(\Gamma(\alpha))^{\frac{-1}{p}+\frac{-1}{p'}}$, which gives(28). This proves the Theorem.

Remark 7. If we put $h(x)=x$ in inquality (28), we get inquality (14) (Theorem 4).

Theorem 10. Let f and g be two functions defined on $X_h^p(0,\infty)$, such that f is increasing and g is differentiable and there exists a real number $m_\alpha := \inf_{x\geq 0} g'(x)$. Then the inequality

$$\begin{aligned}(I_h^\alpha fg)(x) &\geq \frac{\Gamma(\alpha+1)}{h^\alpha(x)}(I_h^\alpha f)(x)(I_h^\alpha g)(x)\\ &- m_\alpha\frac{\Gamma(\alpha+1)}{h^\alpha(x)}(I_h^\alpha f)(x)(I_h^\alpha x)(x)+m_\alpha(I_h^\alpha xf)(x)\end{aligned} \tag{31}$$

holds.

Proof. Let $k(x)=g(x)-mx$, $k(x)$ is differentiable and increasing on $X_h^p(0,\infty)$. Then using the Theorem 6, we obtain

$$\begin{aligned}(I_h^\alpha f(g-m))(x) &\geq \frac{\Gamma(\alpha+1)}{h^\alpha(x)}(I_h^\alpha f)(x)(I_h^\alpha(g-mx))(x)\\ &= \frac{\Gamma(\alpha+1)}{h^\alpha(x)}(I_h^\alpha f)(x)(I_h^\alpha g)(x)m\\ &- \frac{\Gamma(\alpha+1)}{h^\alpha(x)}(I_h^\alpha f)(x)(I_h^\alpha x)(x).\end{aligned} \tag{32}$$

We have

$$I_h^\alpha f(g-mx)(x) = (I_h^\alpha fg)(x)-m(I_h^\alpha xf)(x), \tag{33}$$

hence

$$(I_h^\alpha fg)(x) \geq \frac{\Gamma(\alpha+1)}{h^\alpha(x)}(I_h^\alpha f)(x)(I_h^\alpha g)(x) - m_\alpha \frac{\Gamma(\alpha+1)}{h^\alpha(x)}(I_h^\alpha f)(x)(I_h^\alpha x)(x) + m_\alpha (I_h^\alpha xf)(x).$$

Then the proof is complete.

REMARK 8. If we put $h(x) = x$ in (31), we get the inequality (15) (Theorem 5).

THEOREM 11. Let f and g be two functions defined on $X_h^p(0,\infty)$, such that f is decreasing and g is differentiable and there exists a real number $M := \sup_{x\geq 0} g^{'}(x)$. Then the inequality

$$(I_h^\alpha fg)(i) \geq \frac{\Gamma(\alpha+1)}{h^\alpha(i)}(I_h^\alpha f)(i)(I_h^\alpha g)(i) - M \frac{\Gamma(\alpha+1)}{h^\alpha(i)}(I_h^\alpha f)(i)(I_h^\alpha x)(i) + M (I_h^\alpha if)(i). \quad (31)$$

holds.

PROOF. We apply the Theorem 6 to the decreasing functions f and G such that:

$$G(x) := g(x) - M_\alpha x.$$

The rest is similar to the proof of Theorem 10.

REMARK 9. If we put in Theorem 11, $h(x) = x$,we have the inequality(16) (Corollary 1).

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