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Геометрия интегральных многообразий контактного распределения

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Аннотация

В данной работе рассматриваются различные классы почти контактных метрических структур в предположении вполне интегрируемости их контактного распределения. Получен аналитический критерий вполне интегрируемости контактного распределения почти контактного метрического многообразия. Выяснено, какие почти эрмитовы структуры индуцируются на интегральных многообразиях контактного распределения некоторых почти контактных метрических многообразий. В частности доказано, что почти эрмитова структура, индуцируемая на интегральных подмногообразиях максимальной размерности первого фундаментального распределения многообразия Кенмоцу, является келеровой структурой. А почти эрмитова структура, индуцируемая на интегральных подмногообразиях максимальной размерности первого фундаментального распределения норамльного многообразия, является эрмитовой структурой. Слабо косимплектическая структура с инволютивным первым фундаментальным распределением является точнейше косимплектической структурой и на его интегральных подмногообразиях максимальной размерности вполне интегрируемого контактного распределения индуцируется приближенно келерова структура. Также доказано, что контактное распределение квази-сасакиева многообразия интегрируемо тогда и только тогда, когда это многообразие является косимплектическим. На максимальных интегральных многообразиях контактного распределения косимплектического многообразия индуцируется келерова структура. А на интегральных многообразиях максимальной размерности контактного распределения локально конформно квази-сасакиевого многообразия, с инволютивным первым фундаментальным распределением, индуцируется структура класса W_4 почти эрмитовых структур в классификации Грея-Херверллы. Она будет келеровой тогда и только тогда, когда $grad\sigma \subset M$, где σ — определяющая функция соответствующего конформного преобразования.

Ключевые слова: вполне интегрируемое распределение, почти контактная метрическая структура, почти эрмитова структура, контактное распределение.

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Geometry of integral manifolds of contact distribution

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Abstract

In this paper, various classes of almost contact metric structures are considered under the assumption that their contact distribution is completely integrable. An analytical criterion for the completely integrability of the contact distribution of an almost contact metric manifold is obtained. It is found which almost Hermitian structures are induced on the integral manifolds of the contact distribution of some almost contact metric manifolds. In particular, it is proved that an almost Hermitian structure induced on integral submanifolds of maximum dimension of the first fundamental distribution of a Kenmotsu manifold is a Kähler structure. An almost Hermitian structure induced on integral manifolds of maximum dimension of a completely integrable first fundamental distribution of a normal manifold is a Hermitian structure. We show that a nearly cosymplectic structure with an involutive first fundamental distribution is the most closely cosymplectic one and approximately Kähler structure is induced on its integral submanifolds of the maximum dimension of a completely integrable contact distribution. It is also proved that the contact distribution of an inquasi-Sasakian manifold is integrable only in case of this manifold is cosymplectic. Kähler structure is induced on the maximal integral manifolds of the contact distribution of a cosymplectic manifold. If M is a $lcQS$-manifold with an involutive first fundamental distribution, then the structure of the class W_4 of almost Hermitian structures in the Gray-Hervella classification is induced on integral manifolds of the maximum dimension of its contact distribution. It is Kähler if and only if $grad\ \sigma \subset M$, where σ is an arbitrary smooth function on M of corresponding conformal transformation.

Keywords: completely integrable distribution, almost contact metric structure, almost Hermitian structure, contact distribution.

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1. Introduction

This work is devoted to the research of the contact distribution of almost contact metric structures under the assumption that it is completely integrable. Almost contact metric structures are one of the most meaningful examples of differential geometric structures. The theory of these structures development began only in the 1950s. Almost contact and almost contact metric manifolds were introduced A. Gray [1]. The theory of almost contact metric structures was actively developed in the works of S. Chern, W. Boothby, S. Sasaki [2, 3, 4].

Almost contact metric structures are naturally induced on hypersurfaces of almost Hermitian manifolds, as well as on spaces of principal toroidal bundles over almost Hermitian manifolds. These cases are most important examples of almost contact metric structures. We may say, that they determined the role of almost contact metric structures in differential geometry.

In this paper we investigate following questions:

1. Which case is the first fundamental distribution $\mathcal{L}$ completely integrable in?
2. What is relationship between the class of an almost contact metric structure on the manifold M and the corresponding class of the almost Hermitian structure on the integral manifold of maximal dimension of the first fundamental distribution of M?

2. The terms for the completely integrability of the contact distribution

The article uses the following designations:
M — a smooth manifold, dimension $2n+1$;
$C^\infty(M)$ — the algebra of smooth functions on the manifold M;
$\mathcal{X}(M)$ — $C^\infty(M)$-module of smooth vector fields on M;
$g = \langle\cdot,\cdot\rangle$ — the Riemannian metric on the manifold M;
$[\cdot,\cdot]$ — taking Lie brackets operation;
d — external differentiation operator;
∇ — the Riemannian connection of the metric g.
Let's define the basic concepts used in the presentation of the article.

Definition 1. A contact structure *on a manifold M is a differential 1-form η on M is such that at each point of the manifold* $\eta \wedge \underbrace{d\eta \wedge ... \wedge d\eta}_{n\ times} \neq 0$. *A manifold with a fixed contact structure on it is called a* contact manifold *[5].*

Definition 2. An almost contact metric structure *(AC-structure) on a manifold M is a quadruple (η, ξ, Φ, g) on this manifold, where η is a differential 1-form called a* contact form, *ξ is a vector field, called* characteristic, *Φ is an endomorphism of the module $\mathcal{X}(M)$, called* a structural endomorphism, *and $g = \langle\cdot,\cdot\rangle$ is a Riemannian metric on M so the following conditions are satisfied*

$$1)\ \eta(\xi) = 1; \quad 2)\ \eta \circ \Phi = 0; \quad 3)\ \Phi(\xi) = 0; \quad 4)\ \Phi^2 = -id + \eta \otimes \xi;$$

$$5)\ \langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y);\ X, Y \in \mathcal{X}(M).$$

If we put $Y = \xi$ in formula 5, then $\langle X, \xi\rangle = \eta(X)$.

It is well known that specifying a contact structure on a manifold generates an almost contact metric structure, which explains the term «almost contact metric structure». Generally speaking, the converse is not true.

A manifold with an almost contact metric structure fixed on it is called *an almost contact metric manifold*.

If we define an almost contact metric structure on the manifold M, then naturally a pair of mutually complementary projectors arises in the module $\mathcal{X}(M)$: $m = \eta \otimes \xi$; $l = id - \eta \otimes \xi$. Obviously, $m + l = id$. Moreover, it is an easy way to show $l = -\Phi^2$. These will be projections onto the distributions $\mathcal{L} = Im\Phi = ker\eta$ and $\mathcal{M} = ker\Phi$, respectively, we will call them *the first (or the contact)* and *the second fundamental distributions* of the AC-structure. Thus, for the module $\mathcal{X}(M)$ of smooth vector fields, $\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M}$, where $dim\mathcal{L} = 2n$ and $dim\mathcal{M} = 1$. Moreover, if we introduce in consideration $\mathcal{X}(M)^C = C \oplus \mathcal{X}(M)$ — the complexification of the module $\mathcal{X}(M)$, then $\mathcal{X}(M)^C = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{\sqrt{-1}} \oplus D_\Phi^0$, where $D_\Phi^{\sqrt{-1}}$, $D_\Phi^{\sqrt{-1}}$, D_Φ^0 are the proper distributions of the structural endomorphism Φ corresponding to the eigenvalues $\sqrt{-1}$, $\sqrt{-1}$ and 0. Moreover, the endomorphisms $\pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$, $\bar{\pi} = -\frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi)$, $m = \eta \otimes \xi$ [6].

Consider the differential 1-form $\omega = \eta \circ \pi_*$, where π is the natural projection in the principal bundle of frames over the manifold M, π_* is the entrainment of Φ-connected vector fields on the manifold M. Obviously, this form is the Pfaffian form of the first fundamental distribution $\mathcal{L}$, that is, the basis of the codistribution associated with the first fundamental distribution $\mathcal{L}$ [7].

By the classical Frobenius theorem, the completely integrability of the first fundamental distribution is equivalent to the condition for the existence of a form θ so that $d\omega = \theta \wedge \omega$, that is, the exterior differential of the form ω must belong to the ideal of the Grassmann algebra of the manifold M [6].

Consider the first group of structure equations for an almost contact metric manifold on the space of the associated G-structure [6], [8]:

$$\begin{aligned} d\omega^a &= -\theta_b^a \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B^{ab}\omega \wedge \omega_b + B^a{}_b\omega \wedge \omega^b; \\ d\omega_a &= \theta_a^b \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c + B_{ab}\omega \wedge \omega^b + B_a{}^b\omega \wedge \omega_b; \\ d\omega &= C_{ab}\omega^a \wedge \omega^b + C^{ab}\omega_a \wedge \omega_b + C_a^b\omega^a \wedge \omega_b + C_a\omega \wedge \omega^a + C^a\omega \wedge \omega_a. \end{aligned} \tag{1}$$

We draw attention to the third equation of system (1). In it, the first three terms do not correspond to the form $d\omega = \theta \wedge \omega$, and the last two $\underbrace{C_a\omega^a}_{\theta}\wedge\omega$, $\underbrace{C^a\omega_a}_{\theta}\wedge\omega$ do. Thus, the first fundamental distribution $\mathcal{L}$ is completely integrable only in case of $C_{ab} = 0$, $C^{ab} = 0$, $C_a^b = 0$. Whence we get that

$$\begin{aligned} C_{ab} = -\sqrt{-1}\Phi^0_{[a,b]} = 0,\ C^{ab} = \sqrt{-1}\Phi^0_{[\hat{a},\hat{b}]} = 0, \\ C_a^b = -\sqrt{-1}(\Phi^0_{\hat{b},a} + \Phi^0_{a,\hat{b}}) = 0. \end{aligned} \tag{2}$$

Here $\Phi^i_{j,k}$ are the components of the covariant differential of the structural endomorphism Φ on the space of the frame bundle.

Consider equalities (2). Since $C_a^b = 0$, then $\Phi^0_{\hat{b},a} = -\Phi^0_{a,\hat{b}}$. Let's write the resulting expression in non-indexed form — the left and right sides taking into account that $\pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$, $\bar{\pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi)$, $m = \eta \otimes \xi$:

$$\begin{aligned} m(\nabla_{\pi Y}(\Phi)(\bar{\pi}X)) &= \eta\left(\nabla_{\left(-\frac{1}{2}\Phi^2 - \frac{\sqrt{-1}}{2}\Phi\right)(Y)}(\Phi)\left(-\tfrac{1}{2}\Phi^2 + \tfrac{\sqrt{-1}}{2}\Phi\right)(X)\right)\xi = \\ &= \eta(\nabla_{(-\frac{1}{2}\Phi^2)(Y)}(\Phi)\left(-\tfrac{1}{2}\Phi^2\right)(X) + \nabla_{(-\frac{1}{2}\Phi^2)(Y)}(\Phi)\left(\tfrac{\sqrt{-1}}{2}\Phi\right)(X) + \\ &+\nabla_{\left(-\frac{\sqrt{-1}}{2}\Phi\right)(Y)}(\Phi)\left(-\tfrac{1}{2}\Phi^2\right)(X) + \nabla_{\left(-\frac{\sqrt{-1}}{2}\Phi\right)(Y)}(\Phi)\left(\tfrac{\sqrt{-1}}{2}\Phi\right)(X))\xi = \\ &= \eta(\tfrac{1}{4}\nabla_{\Phi^2(Y)}(\Phi)\Phi^2(X) - \tfrac{\sqrt{-1}}{4}\nabla_{\Phi^2(Y)}(\Phi)\Phi(X) + \\ &+\tfrac{\sqrt{-1}}{4}\nabla_{\Phi(Y)}(\Phi)\Phi^2(X) - \tfrac{1}{4}\nabla_{\Phi(Y)}(\Phi)\Phi(X))\xi; \end{aligned}$$

$$m(\nabla_{\tilde{\pi}X}(\Phi)(\pi Y)) = \eta\left(\nabla_{\left(-\frac{1}{2}\Phi^2+\frac{\sqrt{-1}}{2}\Phi\right)(X)}(\Phi)\left(-\frac{1}{2}\Phi^2-\frac{\sqrt{-1}}{2}\Phi\right)(Y)\right)\xi =$$
$$= \eta(\nabla_{(-\frac{1}{2}\Phi^2)(X)}(\Phi)\left(-\frac{1}{2}\Phi^2\right)(Y) + \nabla_{(-\frac{1}{2}\Phi^2)(X)}(\Phi)\left(-\frac{\sqrt{-1}}{2}\Phi\right)(Y) +$$
$$+\nabla_{\left(\frac{\sqrt{-1}}{2}\Phi\right)(X)}(\Phi)\left(-\frac{1}{2}\Phi^2\right)(Y) + \nabla_{\left(\frac{\sqrt{-1}}{2}\Phi\right)(X)}(\Phi)\left(-\frac{\sqrt{-1}}{2}\Phi\right)(Y))\xi =$$
$$= \eta(\tfrac{1}{4}\nabla_{\Phi^2(X)}(\Phi)\Phi^2(Y) + \tfrac{\sqrt{-1}}{4}\nabla_{\Phi^2(X)}(\Phi)\Phi(Y) -$$
$$-\tfrac{\sqrt{-1}}{4}\nabla_{\Phi(X)}(\Phi)\Phi^2(Y) - \tfrac{1}{4}\nabla_{\Phi(X)}(\Phi)\Phi(Y))\xi.$$

Let us equate the real and imaginary parts of these identities, respectively:

$$\nabla_{\Phi^2(Y)}(\Phi)\Phi^2(X) - \nabla_{\Phi(Y)}(\Phi)\Phi(X) =$$
$$= \nabla_{\Phi^2(X)}(\Phi)\Phi^2(Y) - \nabla_{\Phi(X)}(\Phi)\Phi(Y); \tag{3}$$

$$\nabla_{\Phi^2(Y)}(\Phi)\Phi(X) - \nabla_{\Phi(Y)}(\Phi)\Phi^2(X) =$$
$$= \nabla_{\Phi(X)}(\Phi)\Phi^2(Y) - \nabla_{\Phi^2(X)}(\Phi)\Phi(Y);\ \forall X, Y \in \mathcal{X}(M). \tag{4}$$

Replacing Y by $\Phi(Y)$ in equality (4), taking into account that:

$$\Phi(\Phi(Y)) = \Phi^2(Y), \quad \Phi^2(\Phi(Y)) = -\Phi(Y); \quad \forall X, Y \in \mathcal{X}(M) \tag{5}$$

we get for $\forall X, Y \in \mathcal{X}(M)$

$$\nabla_{\Phi^2(Y)}(\Phi)\Phi^2(X) + \nabla_{\Phi(Y)}(\Phi)\Phi^2(X) =$$
$$= \nabla_{\Phi^2(X)}(\Phi)\Phi^2(Y) + \nabla_{\Phi(X)}(\Phi)\Phi(Y); \tag{6}$$

Adding equalities (3) and (6) term-by-term, we obtain:

$$\nabla_{\Phi^2(Y)}(\Phi)\Phi^2(X) = \nabla_{\Phi^2(X)}(\Phi)\Phi^2(Y); \quad \forall X, Y \in \mathcal{X}(M). \tag{7}$$

Subtracting equality (5) term by term from equality (3), we obtain:

$$\nabla_{\Phi(Y)}(\Phi)\Phi(X) = \nabla_{\Phi(X)}(\Phi)\Phi(Y); \quad \forall X, Y \in \mathcal{X}(M). \tag{8}$$

Taking into account (5), we conclude that identities (7) and (8) are equivalent.
The described procedure is called the identity restoration procedure [6, 8].
So, we can formulate the result.

Theorem 1. *The contact distribution of an almost contact metric manifold is completely integrable if and only if the following relation holds:* $\nabla_{\Phi(Y)}(\Phi)\Phi(X) = \nabla_{\Phi(X)}(\Phi)\Phi(Y)$, $\forall X, Y \in \mathcal{X}(M)$.

Let M be an almost contact metric manifold with a completely integrable first fundamental distribution $\mathcal{L}$. Taking into account (1), from the above mentioned, we obtain that the first group of structure equations of such a manifold has the following form:

$$d\omega^a = -\theta^a_b \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B^{ab}\omega \wedge \omega_b + B^a{}_b\omega \wedge \omega^b;$$
$$d\omega_a = \theta^b_a \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c + B_{ab}\omega \wedge \omega^b + B_a{}^b\omega \wedge \omega_b;$$
$$d\omega = C_a\omega \wedge \omega^a + C^a\omega \wedge \omega_a.$$

Let $N \subset M$ be an integral manifold of maximal dimension of the first fundamental distribution of M. Then an almost Hermitian structure $\langle J, \tilde{g}\rangle$ is canonically induced on it, where $J = \Phi|_{\mathcal{L}}$,

$\tilde{g} = g|_{\mathcal{L}}$. Since the form ω is the Pfaffian form of the first fundamental distribution, the first group of structural equations of an almost Hermitian structure on N has the form:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + B^{ab}{}_c \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c; \\ d\omega_a &= \theta^b_a \wedge \omega_b + B_{ab}{}^c \omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c; \\ d\omega &= 0. \end{aligned} \tag{9}$$

Bearing in mind the Gray-Herwella classification of almost Hermitian structures, written in the form of a table [6, p. 450], we can reveal the connection between the class of an almost contact metric structure on the manifold M and the corresponding class of the almost Hermitian structure on the manifold N. Consider some examples.

3. Structures of Kenmotsu

Definition 3. *[6]. A* Kenmotsu structure *is an almost contact metric structure for which the identity $\nabla_X(\Phi)Y = \langle \Phi X, Y\rangle - \eta(Y)\Phi X$; $\forall X, Y \in \mathcal{X}(M)$ holds. A manifold with a fixed Kenmotsu structure is called* Kenmotsu manifold.

This class of manifolds was introduced by the Japanese geometer Kenmotsu in 1971. An example of such manifolds is the odd-dimensional Lobachevsky space of curvature (-1) [9]. The first group of structural equations of Kenmotsu structures is as follows [6]:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + \omega \wedge \omega^b; \\ d\omega_a &= \theta^b_a \wedge \omega_b + \omega \wedge \omega_b; \\ d\omega &= 0. \end{aligned} \tag{10}$$

The third equation of system (10) indicates that the first fundamental distribution of Kenmotsu structures is completely integrable. The first group of structural equations of an almost Hermitian structure induced on its integral submanifolds looks as follows:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b; \\ d\omega_a &= \theta^b_a \wedge \omega_b; \\ d\omega &= 0. \end{aligned} \tag{11}$$

Equations (9) turn into equations (11) if $B^{abc} = 0$ and $B^{ab}{}_c = 0$. Taking into account the above table ([6], p. 450), we see that the almost Hermitian structure induced on integral submanifolds of the manifold M is Kähler. Thus, the theorem is proved.

Theorem 2. *An almost Hermitian structure induced on integral manifolds of maximum dimension of the first fundamental distribution of a Kenmotsu manifold is a Kähler structure.*

4. Normal structures

Definition 4. *[6]. An AC-structure is called* normal *if the Nijenhuis tensor N_Φ of its structural endomorphism satisfies the condition $N_\Phi + 2d\eta \otimes \xi = 0$.*

The concept of normality was introduced by Sasaki and Hatakeyama in 1961 [4]. It is one of the most fundamental concepts of contact geometry. Examples of normal structures are cosymplectic and Sasakian structures, which are widely studied in modern research.

The structural equations of normal manifolds have the form [8]:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b + B^a{}_b\omega \wedge \omega^b; \\ d\omega_a &= \theta^b_a \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_a{}^b\omega \wedge \omega_b; \\ d\omega &= C^b_a\omega^a \wedge \omega_b. \end{aligned} \tag{12}$$

Under the assumption that the first fundamental distribution is completely integrable, equations (12) take the form:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b; \\ d\omega_a &= \theta^b_a \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b; \\ d\omega &= 0. \end{aligned} \tag{13}$$

Equations (13) are obtained from structural equations (9) under the condition $B^{abc} = 0$. From the table ([6], p. 450), we find that in this case a Hermitian structure is induced on integral manifolds of maximum dimension. Thus, we can formulate a theorem.

THEOREM 3. *An almost Hermitian structure induced on integral manifolds of maximum dimension of a completely integrable first fundamental distribution of a normal manifold is a Hermitian structure.*

5. Nearly cosymplectic manifolds

DEFINITION 5. *[11]. An almost contact metric structure is called* nearly cosymplectic *if* $\nabla_X(\Phi)X = 0$; $\forall X \in \mathcal{X}(M)$; *if* $\nabla_X(\Phi)Y = 0$; $\forall X, Y \in \mathcal{X}(M)$, *the structure is called* cosymplectic.

An example of a nearly cosymplectic manifold is the five-dimensional sphere S^5 embedded in S^6 as a totally geodesic submanifold equipped with a canonical nearly Kähler structure [12].

The structural equations of nearly cosymplectic manifolds have the form [8]:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c + \tfrac{3}{2}C^{ab}\omega_b \wedge \omega; \\ d\omega_a &= \theta^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c + \tfrac{3}{2}C_{ab}\omega^b \wedge \omega; \\ d\omega &= C_{ab}\omega^a \wedge \omega^b + C^{ab}\omega_a \wedge \omega_b. \end{aligned} \tag{14}$$

Considering the third equation of system (14), we see that in order the first fundamental distribution to be completely integrable, it is necessary and sufficient that the terms $C_{ab}\omega^a \wedge \omega^b$ and $C^{ab}\omega_a \wedge \omega_b$ should be zero. This means that the coefficients C_{ab} and C^{ab} must be zeroed.

Thus, under the assumption that the first fundamental distribution is completely integrable, the equations of a nearly cosymplectic structure take the form:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c; \\ d\omega_a &= \theta^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c; \\ d\omega &= 0. \end{aligned} \tag{15}$$

It is known that this equation is the most closely cosiplectic structure [13].

Thus, taking into account the Frobenius theorem, we can formulate the result.

THEOREM 4. *Let M be a nearly cosymplectic manifold, then its first fundamental distribution is involutive if and only if M is the most closely cosymplectic manifold.*

Further, the structure equations of an almost Hermitian structure induced on integral manifolds of an almost contact metric manifold have the form (9). As we can see, equations (9) turn into equations (15) if $B^{ab}_c = 0$. Taking into account that for nearly cosymplectic manifolds the condition $B^{abc} = B^{[abc]}$ [6] is satisfied, according to the table ([6], p. 450), we find that in this case a nearly Kähler structure is induced on integral manifolds of maximum dimension.

Thus, the theorem is proved.

THEOREM 5. *An almost Hermitian structure induced on integral manifolds of maximum dimension of a completely integrable first fundamental distribution of a nearly cosymplectic manifold is a nearly Kähler structure.*

6. Quasi-Sasakian structures

DEFINITION 6. A quasi-Sasakian structure *is a normal almost contact metric structure with a closed fundamental form.*

This class of structures was introduced by Blair in his thesis, as well as in [14]. This class of structures is interesting, because it includes the classes of cosymplectic and Sasakian structures.

The structural equations of the quasi-Sasakian structure are as follows [8]:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + B^a{}_b \omega \wedge \omega^b; \\ d\omega_a &= \theta^b_a \wedge \omega_b + B_a{}^b \omega \wedge \omega_b; \\ d\omega &= 2B^a{}_b \omega^a \wedge \omega_b. \end{aligned} \tag{16}$$

Considering the third equation of system (16), we note that for the contact distribution to be completely integrable it is necessary and sufficient that $B^a{}_b = 0$. It is known that this condition is equivalent to the condition $\nabla_{\Phi X}\xi = 0$ [15]. Then equations (16) take the form:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b; \\ d\omega_a &= \theta^b_a \wedge \omega_b; \\ d\omega &= 0. \end{aligned} \tag{17}$$

It is known that these are equations of cosymplectic structure [8]. Taking into account the table ([6], p. 450), we get the following results.

THEOREM 6. *Let M be a quasi-Sasakian manifold. Then the following conditions are equivalent:*
1) The contact distribution of the manifold M is involutive;
2) $\nabla_{\Phi X}\xi = 0,\ \forall X \in \mathcal{X}(M)$;
3) M is a cosymplectic manifold.

THEOREM 7. *The contact distribution of a quasi-Sasakian manifold is completely integrable if and only if this manifold is cosymplectic. In this case, a Kähler structure is induced on the maximal integral manifolds of the contact distribution.*

Since cosymplectic manifolds are a special case of quasi-Sasakian manifolds, we can formulate a corollary.

COROLLARY 1. *A Kähler structure is induced on the maximal integral manifolds of the contact distribution of a cosymplectic manifold.*

7. Locally conformally quasi-Sasakian structures

Let $S = (\eta, \xi, \Phi, g)$ be an AC-structure on a manifold M^{2n+1} of greater than three demension.

Definition 7. *[14]*. A conformal transformation *of an AC-structure $S = (\eta, \xi, \Phi, g)$ on manifold M is a transition from S to an AC-structure $\tilde{S} = (\tilde{\eta}, \tilde{\xi}, \tilde{\Phi}, \tilde{g})$, where $\tilde{\eta} = e^{-\sigma}\eta$, $\tilde{\xi} = e^{\sigma}\xi$, $\tilde{\Phi} = \Phi$, $\tilde{g} = e^{-2\sigma}g$, σ is an arbitrary smooth function on M, called the defining transformation function.*

Definition 8. *An AC-structure S on M is called a* locally conformally quasi-Sasakian structure*, in short, an $lcQS$-structure, if the restriction of this structure to some neighborhood U of an arbitrary point $p \in M$ admits a conformal transformation into a quasi-Sasakian structure.*

We will call this transformation *locally conformal*. A manifold on which a $lcQS$-structure is fixed is called a *$lcQS$-manifold*.

Kenmotsu structure [9] is an example of $lcQS$-structure.

The first group of structural equations of $lcQS$-structures have the form [14]:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + 2\sigma^{[a}\delta^{b]}_c \omega^c \wedge \omega_b + (\sigma_0 \delta^a_b + C^a_b)\omega \wedge \omega^b; \\ d\omega_a &= \theta^b_a \wedge \omega_b + 2\sigma_{[a}\delta^c_{b]}\omega_c \wedge \omega^b + (\sigma_0 \delta^b_a - C^b_a)\omega \wedge \omega_b; \\ d\omega &= 2C^b_a \omega^a \wedge \omega_b - \sigma_a \omega \wedge \omega^a - \sigma^a \omega \wedge \omega_a. \end{aligned} \tag{18}$$

As noted earlier, the contact distribution $\mathcal{L}$ is completely integrable if and only if there is a form θ such that $d\omega = \theta \wedge \omega$. Hence, the third equation of system (18) should have the form: $d\omega = mega_b - \sigma_a \omega^a \wedge \omega + \sigma^a \omega \wedge \omega_a$ (Since the first term of the third equation of system (18), generally speaking, does not satisfy the condition $d\omega = \theta \wedge \omega$). Hence, it is necessary that $C^b_a = 0$. Then the first group of structural equations of a $lcQS$-manifold takes the form:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + 2\sigma^{[a}\delta^{b]}_c \omega^c \wedge \omega_b + (\sigma_0 \delta^a_b + C^a_b)\omega \wedge \omega^b; \\ d\omega_a &= \theta^b_a \wedge \omega_b + 2\sigma_{[a}\delta^c_{b]}\omega_c \wedge \omega^b + (\sigma_0 \delta^b_a - C^b_a)\omega \wedge \omega_b; \\ d\omega &= \sigma_a \omega^a \wedge \omega + \sigma^a \omega_a \wedge \omega. \end{aligned}$$

This means that the structure equations of an almost Hermitian structure induced on integral manifolds of the maximum dimension of the contact distribution $\mathcal{L}$ can be written as follows:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b + 2\sigma^{[a}\delta^{b]}_c \omega^c \wedge \omega_b; \\ d\omega_a &= \theta^b_a \wedge \omega_b + 2\sigma_{[a}\delta^c_{b]}\omega_c \wedge \omega^b; \\ d\omega &= 0. \end{aligned} \tag{19}$$

Equations (19) are equations of the class W_4 of almost Hermitian structures in the Gray-Hervella classification [6]. Note that if we put $\sigma_a = \sigma^a = 0$ in these equations, then they will take the form:

$$\begin{aligned} d\omega^a &= -\theta^a_b \wedge \omega^b; \\ d\omega_a &= \theta^b_a \wedge \omega_b; \\ d\omega &= 0. \end{aligned}$$

These are the equations of the K?hler structure. Let us find out when the condition $\sigma_a = \sigma^a = 0$ is satisfied. Obviously, this condition can be written in the form $d\sigma || \eta$. This condition is equivalent to $grad\sigma \subset M$, which in turn is equivalent to the normality of the $lcQS$-structure [14]. Thus, we can formulate a conclusion.

Theorem 8. *Let M be an $lcQS$-manifold with an involutive first fundamental distribution, then the structure of the class W_4 of almost Hermitian structures in the Gray-Hervella classification is induced on integral manifolds of the maximum dimension of its contact distribution. It is K?hler if and only if $grad\sigma \subset M$.*

8. Conclusion

It is proved that an almost Hermitian structure induced on integral submanifolds of maximum dimension of the first fundamental distribution of a Kenmotsu manifold is a Kähler structure. An almost Hermitian structure induced on integral manifolds of maximum dimension of a completely integrable first fundamental distribution of a normal manifold is a Hermitian structure. We show that a nearly cosymplectic structure with an involutive first fundamental distribution is the most closely cosymplectic one and approximately Kähler structure is induced on its integral submanifolds of the maximum dimension of a completely integrable contact distribution. It is also proved that the contact distribution of an inquasi-Sasakian manifold is integrable only in case of this manifold is cosymplectic. Kähler structure is induced on the maximal integral manifolds of the contact distribution of a cosymplectic manifold. If M is a $lcQS$-manifold with an involutive first fundamental distribution, then the structure of the class W_4 of almost Hermitian structures in the Gray-Hervella classification is induced on integral manifolds of the maximum dimension of its contact distribution. It is Kähler if and only if $grad\, \sigma \subset M$, where σ is an arbitrary smooth function on M of corresponding conformal transformation.

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