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О построении многомерных периодических фреймов всплесков[1]

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Аннотация

Изучаются многомерные периодические системы всплесков с матричным коэффициентом растяжения. В работе используется конструкция периодического кратномасштабного анализа, наиболее общее определение которого дано И. Максименко и М. Скопиной в [25]. Описан алгоритмический метод построения двойственных фреймов всплесков по набору коэффициентов Фурье одной подходящей функции. Данная функция является первой функцией в масштабирующей последовательности, формирующей двойственные периодические кратномасштабные анализы, которые используется для конечного построения систем всплесков. Условия, накладываемые на исходную функцию, представляют собой ограничения на скорость убывания её коэффициентов Фурье, а также на взаимное расположение нулевых и ненулевых коэффициентов.

Ключевые слова: периодический кратномасштабный анализ, фреймы всплесков, Бесселева система, двойственные фреймы.

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On construction of multidimensional periodic wavelet frames

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Abstract

Multidimensional periodic wavelet systems with matrix dilation in the framework of periodic multiresolution analyses are studied. In this work we use notion of a periodic multiresolution analysis, the most general definition of which was given by Maksimenko and M. Skopina in [25]. An algorithmic method of constructing multidimensional periodic dual wavelet frames from a suitable set of Fourier coefficients of one function is provided. This function is used as the first function in a scaling sequence that forms two periodic multiresolution analyses, which are used to construct wavelet systems. Conditions that the initial function has to satisfy are presented in terms of a certain rate of decay of its Fourier coefficients, and also mutual arrangement of zero and non-zero coefficients.

Keywords: wavelet function, periodic multiresolution analysis, wavelet frame, Bessel system, dual frames.

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1. Introduction

A natural way to define periodic wavelet system is to periodize standard wavelet systems from $L_2(\mathbb{R})$, which is possible if wavelet functions have sufficient decay rate. Such systems are widely studied ([6, §9.3], [13], [19], [20], [22], [12]). But many periodic objects that can reasonably be classified as wavelet systems cannot be obtained that way, and thus there exist other approaches to defining periodic wavelets in a more general sense. Just as in nonperiodic case, wavelets can be obtained on the basis of multiresolution analyses. Specifically, orthogonal bases and tight frames are built using one periodic multiresolution analysis (for brevity, PMRA in the sequel), and biorthogonal bases and dual frames are built using two PMRAs (see [4], [14], [8], [23], [21]). In this paper we use the definition of PMRA given by I. Maksimenko and M. Skopina in [25] (also see [24, Chapter 9]). In [2] N. Atreas has shown that in order to establish that dual wavelet systems are frames, one should check that, along with a few technical conditions, these systems are Bessel. It is worth noting that similar constructions of tight frames do not require this check. Algorithmic methods for the construction of PMRA-based tight wavelet frames were suggested in [7], and in [2] for multidimensional case. However, the condition of systems being Bessel is critical for the construction of dual wavelet frames. Sufficient conditions, under which multidimensional periodic wavelet system is Bessel, were established in [1]. Basing on this result, we provide an algorithmic method of constructing multidimensional periodic dual wavelet frames, starting with any suitable set of Fourier coefficients. In the provided scheme these coefficients define a function that induces two scaling sequences, which generate dual frames.

2. Notation and auxiliary results

As usual, $\mathbb{N}$ is a set of positive integers, $\mathbb{R}^d$ is a d-dimensional euclidean space, $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$ are its elements (vectors), $(x, y) = x_1y_1 + \dots x_dy_d$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$, $|x| = \sqrt{(x,x)}$, $\mathbb{Z}^d$ is integer lattice in $\mathbb{R}^d$, $\mathbb{Z} = \mathbb{Z}^1$, $\mathbb{Z}_+ = \{0, 1, \dots\}$, $\mathbb{T}^d = (-\frac{1}{2}; \frac{1}{2}]^d$ is a d-dimensional unit torus, $\delta_{n,k}$ is Kronecker delta, $\widehat{f}(k) = \int_{\mathbb{T}^d} f(t)e^{-2\pi i(k,t)}dt$ is k-th Fourier coefficient of $f \in L_2(\mathbb{T}^d)$, $\langle f, g\rangle$ is inner product in $L_2(\mathbb{T}^d)$.

If A is $d \times d$ matrix, then $\|A\|$ is its euclidean operator norm from $\mathbb{R}^d$ to $\mathbb{R}^d$, A^* is its Hermitian adjoint, $A^{*j} = (A^*)^j$, I_d is $d \times d$ identity matrix. If A is a $d \times d$ nonsingular integer matrix, we say that vectors k, $n \in \mathbb{Z}^d$ are congruent modulo A and write $k \equiv n \pmod{A}$ if $k - n = Al$, $l \in \mathbb{Z}^d$. We denote by $\mathbb{Z}^d_{\mathbf{0},A}$ set of all $l \in \mathbb{Z}^d$, such that $l \equiv \mathbf{0} \pmod{A}$. The integer lattice $\mathbb{Z}^d$ is partitioned into cosets with respect to this congruence. The number of these cosets equals to $|\det A|$ (see, for instance, [11, Proposition 2.1.1]). Any set containing only one representative of each coset is called a set of digits of the matrix A. When it does not matter which set of digits is chosen, we assume that it is chosen arbitrarily and denote it by $D(A)$. Let us also note that $H(A) := \mathbb{Z}^d \cap A\mathbb{T}^d$ is a set of digits (see [11, Proposition 2.1.1]). Also, there is a following lemma that establishes connection between sets of digits of matrices A, A^j and A^{j+1}.

Lemma 1 ([11], Lemma 2.1.3). *Let A be a nonsingular integer $d \times d$ matrix, $|\det A| > 1$. Then the set $\{r + A^jp\}$ for all possible $r \in D(A^j)$ and $p \in D(A)$ is a set of digits of the matrix A^{j+1}.*

In this paper M denotes a square integer matrix with eigenvalues greater than one in modulus. We will also denote $m := |\det M|$. Note that matrix M^{-1} has all eigenvalues less than one in modulus, and there is only finite number of them, and hence spectral radius of matrix M^{-1} is also less than one. This implies that

$$\lim_{n\to\infty} \|M^{-n}\| = 0. \tag{1}$$

For any $l \in \mathbb{Z}^d$, l_j is a vector such that $l_j \in H(M^{*j})$, $l_j \equiv l \operatorname{mod} M^{*j}$ (note that it is unique).

A matrix M is called isotropic if it is similar to a diagonal matrix such that numbers $\lambda_1, \dots, \lambda_d$ are placed on the main diagonal and $|\lambda_1| = \dots = |\lambda_d|$. Thus, $\lambda_1, \dots, \lambda_d$ are eigenvalues of M and the spectral radius of M is equal to $|\lambda|$, where λ is one of the eigenvalues of M. Note that if matrix M is isotropic then M^* is isotropic and M^j is isotropic for all $j \in \mathbb{Z}$. It is well known that for an isotropic matrices M and for any $j \in \mathbb{Z}$ we have

$$C_1^M|\lambda|^j \leqslant \|M^j\| \leqslant C_2^M|\lambda|^j, \tag{2}$$

where λ is one of the eigenvalues of M.

For any sequence of functions $\{f_j\}_{j\in\mathbb{Z}_+} \subset L_2(\mathbb{T}^d)$ we will denote its shifts by $f_{jk} := f_j(\cdot + M^{-j}k)$. By wavelet system we will mean a system of shifts $\{f_{jk}\}_{j\in\mathbb{Z}_+, k\in D(M^j)}$, associated with a sequence of functions $\{f_j\}_{j\in\mathbb{Z}_+} \subset L_2(\mathbb{T}^d)$, and denote it by $\{f_{jk}\}_{j,k}$. If we have several sequences $\{f_j^{(\nu)}\}_{j\in\mathbb{Z}_+}$, $\nu = 1, \dots, n$, $n \in \mathbb{N}$, the system that represents a union of wavelet systems of each sequence we will also call a wavelet system and denote it by $\{f_{jk}^{(\nu)}\}_{j,k,\nu}$. In the case if we will need to specify the sets of indices, we will write $\{f_{jk}^{(\nu)}\}_{j\in\mathbb{Z}_+, k\in D(M^j), \nu=1,\dots,n}$.

In this paper we rely on the following result that establishes sufficient conditions for wavelet systems to be Bessel.

Theorem 1 ([1]). *Let Fourier coefficients of functions $\psi_j \in L_2(\mathbb{T}^d)$, $j \in \mathbb{Z}_+$, satisfy the following conditions*

$$\forall j \in \mathbb{Z}_+, l \in \mathbb{Z}^d \quad |m^{j/2}\widehat{\psi_j}(l)| \leqslant C \min\left\{|M^{*-j}l|^{-(\frac{d}{2}+\varepsilon)}, |M^{*-j}l|^{\alpha}\right\} \tag{3}$$

for some $C > 0$, $\varepsilon > 0$, $\alpha > 0$. Then, the wavelet system $\{\psi_{jk}\}_{j,k}$ is Bessel.

Let us now proceed to defining periodic multiresolution analysis.

Definition 1 ([24], Definition 9.1.1). *A collections of sets* $\{V_j\}_{j=0}^{\infty}$, $V_j \subset L_2(\mathbb{T}^d)$, *is called PMRA, if the following properties hold:*

- ***MR1***. $V_j \subset V_{j+1}$;
- ***MR2***. $\overline{\bigcup_{j=0}^{\infty} V_j} = L_2(\mathbb{T}^d)$;
- ***MR3***. $\dim V_j = m^j$;
- ***MR4***. $\dim\{f \in V_j : f(\cdot + M^{-j}n) = \lambda_n f \ \ \forall n \in \mathbb{Z}^d\} \leqslant 1$, $\forall \{\lambda_n\}_{n\in\mathbb{Z}^d}, \lambda_n \in \mathbb{C}$;
- ***MR5***. $f \in V_j \Leftrightarrow f(\cdot + M^{-j}n) \in V_j \ \forall n \in \mathbb{Z}^d$;
- ***MR6***. *a)* $f \in V_j \Rightarrow f(M\cdot) \in V_{j+1}$; $f \in V_{j+1} \Rightarrow \sum_{s\in D(M)} f(M^{-1}\cdot + M^{-1}s) \in V_j$.

Definition 2 ([24], Definition 9.1.3). *Let* $\{V_j\}_{j=0}^{\infty}$ *be a PMRA in* $L_2(\mathbb{T}^d)$. *Sequence of functions* $\{\varphi_j\}_{j\in\mathbb{Z}_+}$, $\varphi_j \in V_j$, *is called a* scaling sequence, *if functions* φ_{jk}, $k \in D(M^j)$, *form a basis for* V_j.

Theorem 2 ([24], Theorem 9.1.4). *Functions* $\{\varphi_j\}_{j=0}^{\infty} \subset L_2(\mathbb{T}^d)$ *form a scaling sequence for some PMRA if and only if:*

- ***S1***. $\widehat{\varphi_0}(k) = 0$, *for all* $k \neq 0$;
- ***S2***. *for all* $j \in \mathbb{Z}_+$, *and for all* $n \in \mathbb{Z}^d$ *exists* $m \equiv n \pmod{M^{*j}}$, *such that* $\widehat{\varphi_j}(k) \neq 0$;
- ***S3***. *for all* $k \in \mathbb{Z}^d$ *exists* $j \in \mathbb{Z}_+$, *such that* $\widehat{\varphi_j}(k) \neq 0$;
- ***S4***. *For all* $j \in \mathbb{Z}_+$, $n \in \mathbb{Z}^d$, *exists* $\gamma_n^j \neq 0$, *such that* $\gamma_n^j \widehat{\varphi_j}(k) = \widehat{\varphi_{j+1}}(M^*k)$ *for all* $k \equiv n \pmod{M^{*j}}$;
- ***S5***. *For all* $j \in \mathbb{N}$, $n \in \mathbb{Z}^d$, *exists* μ_n^j, *such that* $\widehat{\varphi_{j-1}}(k) = \mu_n^j \widehat{\varphi_j}(k)$ *for all* $k \equiv n \pmod{M^{*j}}$.

Let us note that in Theorem 2 the sequences of numbers $\{\gamma_k^j\}_{k\in\mathbb{Z}^d}$, $\{\mu_k^j\}_{k\in\mathbb{Z}^d}$ are M^{*j}-periodic with respect to k for every $j \in \mathbb{Z}_+$.

Now we define how scaling sequences generate wavelet systems. Let $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ be two scaling sequences, s_k – arbitrarily enumerated digits of the matrix M^*, and matrices $A^{(r)} = \{a_{nk}^{(r)}\}_{n,k=0}^{m-1}$, $\widetilde{A}^{(r)} = \{\widetilde{a}_{nk}^{(r)}\}_{n,k=0}^{m-1}$ are such that

$$a_{0k}^{(r)} = \mu_{r+M^{*j}s_k}^{j+1}, \quad \widetilde{a}_{0k}^{(r)} = \widetilde{\mu}_{r+M^{*j}s_k}^{j+1}, \tag{4}$$

and for any $r \in D(M^{*j})$ it is true that $A^{(r)}\widetilde{A}^{(r)*} = mI_m$. For $\nu = 1, \ldots, m-1$, let

$$\alpha_{r+M^{*j}s_k}^{\nu,j} = a_{\nu k}^{(r)}, \quad \widetilde{\alpha}_{r+M^{*j}s_k}^{\nu,j} = \widetilde{a}_{\nu k}^{(r)}. \tag{5}$$

By lemma 1, vectors $r + M^{*j}s_k$ form a set of digits $D(M^{*j+1})$, i. e. we can M^{*j+1}-periodically extend these sequences to $\mathbb{Z}^d$. Let us define functions $\psi_j^{(\nu)}$, $\widetilde{\psi}_j^{(\nu)}$ by defining its Fourier coefficients

$$\widehat{\psi_j^{(\nu)}}(l) = \alpha_l^{\nu,j}\widehat{\varphi_{j+1}}(l), \quad \widehat{\widetilde{\psi}_j^{(\nu)}}(l) = \widetilde{\alpha}_l^{\nu,j}\widehat{\widetilde{\varphi}_{j+1}}(l). \tag{6}$$

Systems $\{\varphi_0\}\cup\{\psi_{jk}^{(\nu)}\}_{j\in\mathbb{Z}_+,k\in D(M^j),\nu=1,\ldots,m-1}$ and $\{\widetilde{\varphi}_0\}\cup\{\widetilde{\psi}_{jk}^{(\nu)}\}_{j,k,\nu}$ we will call *dual wavelet systems that are generated by scaling sequences* $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$. Now let us cite a theorem that establishes frame conditions for such systems.

Theorem 3 ([2]). *Let $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ be scaling sequences that satisfy the condition*

$$\lim_{j\to+\infty} m^j \widehat{\varphi_j}(k)\overline{\widehat{\widetilde{\varphi}_j}(k)} = 1 \quad \forall k \in \mathbb{Z}^d, \tag{7}$$

and let $\{\varphi_0\} \cup \{\psi_{jk}^{(\nu)}\}_{j,k,\nu}$ and $\{\widetilde{\varphi}_0\} \cup \{\widetilde{\psi}_{jk}^{(\nu)}\}_{j,k,\nu}$ be Bessel dual wavelet systems generated by them. Then these systems are dual frames.

3. Main result

Theorem 4. *Let M be an isotropic matrix such that $\mathbb{T}^d \subset M^*\mathbb{T}^d$, and $\varphi_1 \in L_2(\mathbb{T}^d)$ with Fourier coefficients given by*

$$\widehat{\varphi_1}(l) = \begin{cases} a_0, & \text{if} \quad l = \mathbf{0}, \\ a_l(\frac{1}{|l|})^{\alpha}, & \text{if} \quad l \notin \mathbb{Z}^d_{\mathbf{0},M^*},\ l \in Q, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha > d/2$, $0 < C_1 \leqslant |a_l| \leqslant C_2$ for $l = \mathbf{0}$ and all $l \in Q$, where $Q \subset \mathbb{Z}^d$ is such that $Q \cap \mathbb{Z}^d_{\mathbf{0},M^*} = \emptyset$, $H(M^*) \subset Q$ and satisfies the condition:

(Z) *If $l \notin Q$ and $l \in H(M^{*j})$ for some $j \in \mathbb{N}$, then $l + M^j k \notin Q$ for every $k \in \mathbb{Z}^d$. Then there exist scaling sequences $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ that generate wavelet systems $\{\varphi_0\} \cup \{\psi_{jk}\}_{j,k}$ and $\{\widetilde{\varphi}_0\} \cup \{\widetilde{\psi}_{jk}\}_{j,k}$, which are dual frames.*

For any vector $l \notin \mathbb{Z}^d_{\mathbf{0},M^*}$, $l \notin Q$ we set $a_l = C_1$, and define $\{a_l^*\}$, $l \in \mathbb{Z}^d$, by

$$a_l^* = \begin{cases} a_l, & \text{if } l = \mathbf{0} \text{ or } l \notin \mathbb{Z}^d_{\mathbf{0},M^*}, \\ a_k, & \text{if } l = M^{*n}k,\ n \in \mathbb{N},\ k \notin \mathbb{Z}^d_{\mathbf{0},M^*}. \end{cases}$$

Next, we construct scaling sequences $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ by defining their Fourier coefficients. We start with setting

$$\widehat{\widetilde{\varphi}_j}(l) := \begin{cases} m^{-\frac{j+1}{2}} {a_l^*}^{-1}, & \text{if } l \in H(M^{*j}), \\ 0, & \text{if } l \notin H(M^{*j}), \end{cases}$$

and, since $\mathbb{T}^d \subset M^*\mathbb{T}^d$,

$$\widetilde{\mu}_l^j = \begin{cases} \sqrt{m}, & \text{if } l \in H(M^{*j-1}), \\ 0, & \text{if } l \notin H(M^{*j-1}). \end{cases} \tag{8}$$

Thus, the functions $\widetilde{\varphi}_j$ are defined, and they are trigonometric polynomials.

Construction of $\{\varphi_j\}_j$ is slightly more sophisticated. First of all we define the function on 0-th level,

$$\widehat{\varphi_0}(\mathbf{0}) := \sqrt{m} \cdot \widehat{\varphi_1}(\mathbf{0}), \quad \widehat{\varphi_0}(l) := 0, \quad l \neq \mathbf{0}.$$

Note that the already have Fourier coefficients of φ_1. Next we define coefficients $\widehat{\varphi_j}(l)$ for the rest of the scaling sequence, recursively by j.

I. ($l \notin \mathbb{Z}^d_{\mathbf{0},M^*}$) Define $\widehat{\varphi_j}(l)$ and find μ_l^j for $l \notin \mathbb{Z}^d_{\mathbf{0},M^*}$, $k \in \mathbb{Z}^d$, $j > 1$.

1) Let $l \in H(M^{*j})$. Two cases may occur:

$$a) \quad \widehat{\varphi_{j-1}}(l) \neq 0 \quad \Rightarrow \quad \widehat{\varphi_j}(l) := \frac{|l_j|^{\alpha}\widehat{\varphi_{j-1}(l)}}{\sqrt{m}|l_{j-1}|^{\alpha}}, \quad \mu_l^j = \sqrt{m}\Big(\frac{|l_{j-1}|}{|l_j|}\Big)^{\alpha}; \tag{9}$$

$$b) \quad \widehat{\varphi_{j-1}}(l) = 0 \quad \Rightarrow \quad \widehat{\varphi_j}(l) := m^{-\frac{j-1}{2}}\Big(\frac{|l_j|}{|l|}\Big)^{\alpha} a_l^*, \quad \mu_l^j = 0. \tag{10}$$

Note that the case $l=\mathbf{0}$ is not described here, and hence $|l_{j-1}|\neq 0$.

2) Let $l\notin H(M^{*j})$. Since numbers μ_l^j should be M^{*j}-periodic with respect to l, we will periodically extend them from $l\in H(M^{*j})$, where we defined these numbers at previous step. Again, two cases may occur:

$$a)\quad \mu_l^j\neq 0 \quad\Rightarrow\quad \widehat{\varphi_j}(l):=\frac{\widehat{\varphi_{j-1}}(l)}{\mu_l^j}; \tag{11}$$

$$b)\quad \mu_l^j=0 \quad\Rightarrow\quad \widehat{\varphi_j}(l):=0. \tag{12}$$

II. $(l\in\mathbb{Z}^d_{\mathbf{0},M^*})$ Now we define $\widehat{\varphi_j}(l)$ and find μ_l^j for $l\in\mathbb{Z}^d_{\mathbf{0},M^*}$, $j>1$.

$$\widehat{\varphi_j}(l):=\frac{1}{\sqrt{m}}\widehat{\varphi_{j-1}}(M^{*-1}l),\quad \mu_l^j=\mu^{j-1}_{M^{*-1}l}. \tag{13}$$

Note that $\gamma_l^j=\frac{1}{\sqrt{m}}$ for all $l\in\mathbb{Z}^d$ due to this formula.

Thus, we have defined all $\widehat{\varphi_j}(l)$. Obviously, the corresponding functions φ_j are in L_2. For $l=\mathbf{0}$ we, by definition (13), have a simple formula $\widehat{\varphi_j}(0)=\frac{1}{\sqrt{m}}\widehat{\varphi_{j-1}}(0)$. Next, let us show that for the following inequality holds for all $l\neq\mathbf{0}$,

$$|\widehat{\varphi_j}(l)|\leqslant C^* m^{-\frac{j-1}{2}}\Big(\frac{|l_j|}{|l|}\Big)^{\alpha}|a_l^*|, \tag{14}$$

where for $l\notin\mathbb{Z}^d_{\mathbf{0},M^*}$ inequality turns into equality with $C^*=1$, and $C^*=(C_2^{M^*})^{2\alpha}$ for $l\in\mathbb{Z}^d_{\mathbf{0},M^*}$. For $l\notin\mathbb{Z}^d_{\mathbf{0},M^*}$, it follows directly from the formulas (9)-(12). Now let $l=M^{*n}k$, $k\notin\mathbb{Z}^d_{\mathbf{0},M^*}$, $k\in\mathbb{Z}^d$, and let $\widehat{\varphi_j}(l)\neq 0$. Using definition (13) n times, we have

$$\widehat{\varphi_j}(l)=\Big(\frac{1}{\sqrt{m}}\Big)^n\widehat{\varphi_{j-n}}(M^{*-n}l)=m^{-\frac{n}{2}}m^{-\frac{j-n-1}{2}}\Big(\frac{|(M^{*-n}l)_{j-n}|}{|M^{*-n}l|}\Big)^{\alpha}a^*_{M^{*-n}l}.$$

According to definition of a_l^*, $a^*_{M^{*-n}l}=a_l^*$. Also, due to properties of matrix M and definition of l_j, we know that $(M^{*-n}l)_{j-n}=M^{*-n}l+M^{*j-n}r$, where $r\in\mathbb{Z}^d$ is such that $M^{*-n}l+M^{*j-n}r\in M^{*j-n}\mathbb{T}^d$. This means that $M^{*n}(M^{*-n}l)_{j-n}=l+M^{*j}r\in M^{*j}\mathbb{T}^d$, and hence $l+M^{*j}r=l_j$. Thus, $(M^{*-n}l)_{j-n}=M^{*-n}l_j$. Using these facts, we obtain

$$\begin{aligned}|\widehat{\varphi_j}(l)|&=m^{-\frac{j-1}{2}}\Big(\frac{|M^{*-n}l_j|}{|M^{*-n}l|}\Big)^{\alpha}|a_l^*|\\&\leqslant m^{-\frac{j-1}{2}}\Big(\frac{\|M^{*-n}\||l_j||l|}{|M^{*-n}l||l|}\Big)^{\alpha}|a_l^*|\\&\leqslant m^{-\frac{j-1}{2}}\Big(\frac{\|M^{*-n}\||M^{*-n}M^{*n}l||}{|M^{*-n}l|}\frac{|l_j|}{|l|}\Big)^{\alpha}|a_l^*|\\&\leqslant m^{-\frac{j-1}{2}}\Big(\|M^{*-n}\|\|M^{*n}\|\frac{|l_j|}{|l|}\Big)^{\alpha}|a_l^*|.\end{aligned} \tag{15}$$

It remains to recall that M is an isotropic matrix, which implies that $\|M^{*-n}\|\|M^{*n}\|\leqslant(C_2^{M^*})^2$.

Let us show that $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ are scaling sequences. Condition **S1** is obviously fulfilled. Since

$$\widehat{\varphi_j}(l)\neq 0,\ \widehat{\widetilde{\varphi}_j}(l)\neq 0$$

whenever $l\in H(M^{*j})$, conditions **S2** and **S3** are also granted. Condition **S4** (periodicity of γ_k^j) is also fulfilled, because all γ_k^j are equal to each others. The last, condition **S5** (periodicity of μ_k^j) is granted by the fact that for every $j\in\mathbb{Z}_+$ we defined μ_k^j on $H(M^{*j})$, and then extended it to $\mathbb{Z}^d$. It is also worth noting that the fulfillment of condition *(Z)* grants us absence of collisions during the process of defining μ_k^j.

Noting that $l_j = l$ for sufficiently large j, we can see that the equality

$$\lim_{j\to\infty} m^j \widehat{\varphi_j}(l)\overline{\widehat{\widetilde{\varphi}_j}(l)} = 1 \quad \forall l \in \mathbb{Z}^d$$

follows from inequality (14) for $l \notin \mathbb{Z}^d_{\mathbf{0},M^*}$, which, as it was mentioned above, turns into equality with $C^* = 1$; and from equality (15) for $l \in \mathbb{Z}^d_{\mathbf{0},M^*}$.

Now, we introduce and analyze wavelet systems generated by the scaling sequences $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$.

Let us define Fourier coefficients of ψ_j, $\widetilde{\psi}_j$. It will be suitable for us to represent a set of digits of the matrix M^{*j} as given in Lemma 1, i. e.

$$D(M^{*j}) = \bigcup_{\substack{r\in D(M^{*j-1})\\ p\in D(M^*)}} \{r + M^{*j-1}p\}. \tag{16}$$

But we should note that this set is not necessarily the same as $H(M^{*j})$. However, when speaking about μ_n^j, due to its M^{*j}-periodicity we can safely regard it as defined on any set of digits (particularly on $H(M^{*j})$), whenever they are defined on at least one set of digits.

It follows from (8) that

$$\begin{aligned} &\widetilde{\mu}_k^{j+1} \neq 0 \quad \text{for } k \in H(M^{*j}),\\ &\widetilde{\mu}_k^{j+1} = 0 \quad \text{for } k \in H(M^{*j+1}) \setminus H(M^{*j}). \end{aligned} \tag{17}$$

Using lemma 1, with $D(M^{*j}) = H(M^{*j})$, $D(M^*) = H(M^*)$, we can rewrite it as

$$\forall r \in H(M^{*j}) \qquad \widetilde{\mu}_{r+M^{*j}p}^{j+1} \begin{cases} \neq 0, & \text{for } p = \mathbf{0},\\ = 0, & \text{for } p \neq \mathbf{0},\ p \in H(M^*). \end{cases} \tag{18}$$

Let us now build matrices $A^{(r)}$ and $\widetilde{A}^{(r)}$ for every $r \in H(M^{*j})$. First, enumerate digits $p \in H(M^*)$ such that $p_0 = \mathbf{0}$. Then we define the first row as

$$a_{0k}^{(r)} = \mu_{r+M^{*j}p_k}^{j+1}, \quad \widetilde{a}_{0k}^{(r)} = \widetilde{\mu}_{r+M^{*j}p_k}^{j+1}, \quad k = 0, 1, \ldots, m-1. \tag{19}$$

It is easy to see that, due to (18), $\widetilde{a}_{0k}^{(r)} = 0$ for $k = 1, \ldots, m-1$. Extend these matrices to square matrices in the following fashion

$$A^{(r)} = \begin{bmatrix} \mu_r^{j+1} & \mu_{r+M^{*j}p_1}^{j+1} & \ldots & \mu_{r+M^{*j}p_{m-1}}^{j+1}\\ 0 & -\widetilde{\mu}_r^{j+1} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & -\widetilde{\mu}_r^{j+1} \end{bmatrix},$$

$$\widetilde{A}^{(r)} = \begin{bmatrix} \widetilde{\mu}_r^{j+1} & 0 & \ldots & 0\\ \mu_{r+M^{*j}p_1}^{j+1} & -\mu_r^{j+1} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{r+M^{*j}p_{m-1}}^{j+1} & 0 & \ldots & -\mu_r^{j+1} \end{bmatrix}.$$

Due to (9), $\mu_r^{j+1} = \sqrt{m}\Big(\frac{|r_j|}{|r_{j+1}|}\Big) = \sqrt{m}$, since $r \in H(M^{*j})$. Using this equality and (8), it is easy to check that $A^{(r)}\widetilde{A}^{(r)*} = mI_m$. Now we let

$$\alpha^{\nu,j}_{r+M^{*j}p_k} = a^{(r)}_{\nu k}, \quad \widetilde{\alpha}^{\nu,j}_{r+M^{*j}p_k} = \widetilde{a}^{(r)}_{\nu k}.$$

Vectors $r + M^{*j}p_k$, $k = 1, \dots, m-1$ are a set of digits $D(M^{*j+1})$, since $r \in H(M^{*j})$, $p_k \in H(M^{*j})$. Thus, we can M^{*j+1}-periodically extend the coefficients $\alpha^{\nu,j}_l$, $\widetilde{\alpha}^{\nu,j}_l$ to $\mathbb{Z}^d$.

Now, for $\nu = 1, \dots, m-1$, we let

$$\widehat{\psi^{(\nu)}_j}(l) = \alpha^{\nu,j}_l \widehat{\varphi_{j+1}}(l), \quad \widehat{\widetilde{\psi}^{(\nu)}_j}(l) = \widetilde{\alpha}^{\nu,j}_l \widehat{\widetilde{\varphi}_{j+1}}(l).$$

We can see that

$$\widehat{\psi^{(\nu)}_j}(l) = \begin{cases} -\sqrt{m}\widehat{\varphi_{j+1}}(l), & \text{for } l \equiv r + p_\nu \pmod{M^{*j+1}}, \\ & r \in H(M^{*j}); \\ 0, & \text{otherwise}; \end{cases} \tag{20}$$

$$\widehat{\widetilde{\psi}^{(\nu)}_j}(l) = \begin{cases} -\sqrt{m}\widehat{\widetilde{\varphi}_{j+1}}(l), & \text{for } l \in H(M^{*j+1}) \setminus H(M^{*j}); \\ \mu^{j+1}_{l+M^{*j}p_\nu}\widehat{\widetilde{\varphi}_{j+1}}(l), & \text{for } l \in H(M^{*j}), \\ 0, & \text{otherwise}; \end{cases} \tag{21}$$

To estimate them we consider two cases:

1) Let $l \in H(M^{*j})$. In this case, $|M^{*-j}l| \leqslant \frac{\sqrt{d}}{2}$, and hence,

$$|M^{*-j}l|^\alpha \leqslant C_{d,\alpha}|M^{*-j}l|^{-\alpha}, \tag{22}$$

where $C_{d,\alpha} = \Big(\frac{2}{\sqrt{d}}\Big)^{-2\alpha}$. From (20), $|\widehat{\psi^{(\nu)}_j}(l)| = 0$. Next,

$$|\widehat{\widetilde{\psi}^{(\nu)}_j}(l)| = |\mu^{j+1}_{l+M^{*j}p_\nu}|m^{-\frac{j+2}{2}}|a^{*-1}_l|,$$

$$|\mu^{j+1}_{l+M^{*j}p_\nu}| = \sqrt{m}\Big(\frac{|(l+M^{*j}p_\nu)_j|}{|(l+M^{*j}p_\nu)_{j+1}|}\Big)^\alpha.$$

It is not hard to see that $(l+M^{*j}p_\nu)_j = l$, and since $p_\nu \neq \mathbf{0}$, $(l+M^{*j}p_\nu)_{j+1} \in H(M^{*j+1}) \setminus H(M^{*j})$, which means that $|(l+M^{*j}p_\nu)_{j+1}| \geqslant \frac{1}{2\|M^{*-j}\|}$. Using this and the fact that M^* is isotropic, we have

$$\begin{aligned} |\mu^{j+1}_{l+M^{*j}p_\nu}| = \sqrt{m}\Big(\frac{|l|}{|(l+M^{*j}p_\nu)_{j+1}|}\Big)^\alpha &\leqslant \sqrt{m}\Big(2\|M^{*j}\|\|M^{*-j}\||M^{*-j}l|\Big)^\alpha \\ &\leqslant \sqrt{m}2^\alpha(C_2^{M^*})^{2\alpha}\Big(|M^{*-j}l|\Big)^\alpha, \end{aligned}$$

and thus, according to (21), we have

$$m^{j/2}|\widehat{\widetilde{\psi}_j}(l)| \leqslant m^{\frac{3}{2}}2^\alpha(C_2^{M^*})^{2\alpha}\Big(|M^{*-j}l|\Big)^\alpha|a^{*-1}_l| \leqslant C_{d,\alpha}m^{\frac{3}{2}}2^\alpha(C_2^{M^*})^{2\alpha}\Big(|M^{*-j}l|\Big)^{-\alpha}|a^{*-1}_l|.$$

2) Let $l \notin H(M^{*j})$. In this case $|M^{*-j}l| \geqslant \frac{1}{2}$, and hence

$$|M^{*-j}l|^{-\alpha} \leqslant \Big(4|M^{*-j}l|\Big)^\alpha.$$

By (21), $|\widehat{\widetilde{\psi}_j}(l)| = -\sqrt{m}\widehat{\widetilde{\varphi}_{j+1}}(l)$ for $l \in H(M^{*j+1}) \setminus H(M^{*j})$, i. e. where $|M^{*-j}l| \leqslant \|M^*\|\sqrt{d}$, and 0 otherwise. Thus, we have the following estimate

$$|m^{j/2}\widehat{\widetilde{\psi}_j}(l)| = |-m^{\frac{3}{2}}{a_l^*}^{-1}| \leqslant |-m^{\frac{3}{2}}{a_l^*}^{-1}|\|M^*\|^\alpha d^{\frac{\alpha}{2}}|M^{*-j}l|^{-\alpha} \leqslant |-m^{\frac{3}{2}}{a_l^*}^{-1}|\|M^*\|^\alpha d^{\frac{\alpha}{2}}\Big(4|M^{*-j}l|\Big)^\alpha$$

Next, from (14), for non-zero coefficients we have $|\widehat{\varphi_{j+1}}(l)| \leqslant C^* m^{-\frac{j}{2}}\Big(\frac{|l_{j+1}|}{|l|}\Big)^\alpha |a_l^*|$, where $|l_{j+1}| \leqslant \frac{\sqrt{d}\|M^{*j+1}\|}{2}$, since $l_{j+1} \in H(M^{*j+1})$. Using this and the fact that M^* is isotropic,

$$\begin{aligned}|\widehat{\varphi_{j+1}}(l)| \leqslant C^* m^{-\frac{j}{2}}\Big(\frac{|M^{*-(j+1)}l||l_{j+1}|}{|M^{*-(j+1)}l||l|}\Big)^\alpha |a_l^*| &\leqslant C^* m^{-\frac{j}{2}}\Big(\frac{\sqrt{d}\|M^{*j+1}\|\|M^{*-(j+1)}\||l|}{2|M^{*-(j+1)}l||l|}\Big)^\alpha \\ &\leqslant C^* m^{-\frac{j}{2}}(C_2^{M^*})^{2\alpha}\Big(\frac{\sqrt{d}}{2}\Big)^\alpha \|M^{*-1}\|^{-\alpha}|M^{*-j}l|^{-\alpha} \\ &\leqslant C^* m^{-\frac{j}{2}}(C_2^{M^*})^{2\alpha}\Big(\frac{\sqrt{d}}{2}\Big)^\alpha 4^\alpha \|M^{*-1}\|^{-\alpha}|M^{*-j}l|^{\alpha}.\end{aligned}$$

By definition,

$$\begin{aligned}m^{j/2}|\widehat{\psi_j}(l)| = m^{j/2}|-\sqrt{m}\widehat{\varphi_{j+1}}(l)| &\leqslant C^*\sqrt{m}(C_2^{M^*})^{2\alpha}\Big(\frac{\sqrt{d}}{2}\Big)^\alpha \|M^{*-1}\|^{-\alpha}|M^{*-j}l|^{-\alpha} \\ &\leqslant C^*\sqrt{m}(C_2^{M^*})^{2\alpha}\Big(\frac{\sqrt{d}}{2}\Big)^\alpha 4^\alpha\|M^{*-1}\|^{-\alpha}|M^{*-j}l|^{\alpha}.\end{aligned}$$

As for coefficients that are equal to zero, the same estimates are obviously held.

Thus, we have shown that all conditions of theorems 1 and 3 are satisfied, and hence, wavelet systems $\{\varphi_0\} \cup \{\psi_{jk}\}_{j,k}$ and $\{\widetilde{\varphi}_0\} \cup \{\widetilde{\psi}_{jk}\}_{j,k}$ are dual frames.

COROLLARY 1. *Let M be an isotropic matrix such that $\mathbb{T}^d \subset M^*\mathbb{T}^d$, and $\varphi_1 \in L_2(\mathbb{T}^d)$ with Fourier coefficients given by*

$$\widehat{\varphi_1}(l) = \begin{cases} a_{\mathbf{0}}, & \textit{if} \quad l = \mathbf{0}, \\ a_l(\frac{1}{|l|})^\alpha, & \textit{if} \quad l \notin \mathbb{Z}^d_{\mathbf{0},M^*}, \\ 0, & \textit{if} \quad l \in \mathbb{Z}^d_{\mathbf{0},M^*},\ l \neq \mathbf{0}, \end{cases}$$

where $\alpha > d/2$, $0 < C_1 \leqslant |a_l| \leqslant C_2$ for $l = \mathbf{0}$ and all $l \notin \mathbb{Z}^d_{\mathbf{0},M^*}$. Then there exist scaling sequences $\{\varphi_j\}_{j=0}^\infty$, $\{\widetilde{\varphi}_j\}_{j=0}^\infty$ that generate wavelet systems $\{\varphi_0\} \cup \{\psi_{jk}\}_{j,k}$ and $\{\widetilde{\varphi}_0\} \cup \{\widetilde{\psi}_{jk}\}_{j,k}$, which are dual frames.

It suffices to check that, in this case, $Q = \{l : l \notin \mathbb{Z}^d_{\mathbf{0},M^*}\}$. This set obviously satisfies condition *(Z)* from Theorem 4.

4. Conclusion

We have presented a method of constructing periodic dual wavelet frames with an isotropic matrix dilation, starting with only one suitable function. Its Fourier coefficients have to have a sufficient rate of decay, and also satisfy the condition (Z) on mutual arrangement of zero and non-zero coefficients. The resulting wavelet systems can be built layer by layer, with the provided recurrent formulas for its Fourier coefficients.

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