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A DISCRETE UNIVERSALITY THEOREM FOR PERIODIC HURWITZ ZETA-FUNCTIONS

A. Laurinčikas, D. Mochov (Vilnius, Lithuania)

Dedicated to Gennadii Ivanovich Arkhipov and Sergei Mikhailovich Voronin

Abstract

In 1975, Sergei Mikhailovich Voronin discovered the universality of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, on the approximation of a wide class of analytic functions by shifts $\zeta(s + i\tau), \tau \in \mathbb{R}$. Later, it turned out that also some other zeta-functions are universal in the Voronin sense. If τ takes values from a certain descrete set, then the universality is called discrete.

In the present paper, the discrete universality of periodic Hurwitz zeta-functions is considered. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined by the series with terms $a_m(m + \alpha)^{-s}$, where $0 < \alpha \leq 1$ is a fixed number, and $\mathfrak{a} = \{a_m\}$ is a periodic sequence of complex numbers. It is proved that a wide class of analytic functions can be approximated by shifts
$\zeta(s+ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})$ with $k = 2, 3, ...$, where $h > 0$ and $0 < \beta_1 < 1$, $\beta_2 > 0$ are fixed numbers, and the set $\{\log(m+\alpha) : m = 0, 1, 2\}$ is linearly independent over the field of rational numbers. It is obtained that the set of such k has a positive lower density. For the proof, properties of uniformly distributed modulo 1 sequences of real numbers are applied.

Keywords: periodic Hurwitz zeta-function, space of analytic functions, limit theorem, universality.

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ДИСКРЕТНАЯ ТЕОРЕМА УНИВЕРСАЛЬНОСТИ ДЛЯ ПЕРИОДИЧЕСКИХ ДЗЕТА ФУНКЦИЙ ГУРВИЦА

А. Лауринчикас, Д. Мохов (г. Вильнюс)

Аннотация

В 1975 г. Сергей Михайлович Воронин открыл свойство универсальности дзета-функции Римана $\zeta(s)$, $s = \sigma + it$, о приближении широкого класса аналитических функций сдвигами $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. Позже оказалось, что и некоторые другие дзета-функции обладают свойством универсальности в смысле Воронина. Если сдвиг τ принимает значения из некоторого дискретного множества, то универсальность называется дискретной.

В работе изучается дискретная универсальность периодических дзета-функций Гурвица. Периодическая дзета-функция Гурвица $\zeta(s, \alpha; \mathfrak{a})$ определяется рядом с членами $a_m(m + \alpha)^{-s}$, $m = 0, 1, 2, \dots$, где $0 < \alpha \leq 1$ – фиксированное число, а $\mathfrak{a} = \{a_m\}$ – периодическая последовательность комплексных чисел. Доказано, что широкий класс аналитических функций с заданной точностью приближается сдвигами $\zeta(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})$ с $k = 2, 3, \dots$, где $h > 0$ и $0 < \beta_1 < 1$, $\beta_2 > 0$ – фиксированные числа, а множество $\{\log(m + \alpha) : \ m = 0, 1, 2, \dots\}$ линейно независимо над полем рациональных чисел. Получено, что множество таких сдвигов, приближающих данную аналитическую функцию, имеет положительную нижнюю плотность. При доказательстве используются свойства равномерно распределенных по модулю 1 последовательностей действительных чисел.

Ключевые слова: периодическая дзета-функция Гурвица, предельная теорема, пространство аналитических функций, универсальность.

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1. Introduction

Let $s = \sigma + it$ be a complex variable, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ with parameter $\alpha, 0 < \alpha \leq 1$ is defined,
for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and was introduced in [7]. In virtue of the equality

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{q^s} \sum_{m=0}^{q-1} a_m \zeta\Big(s, \frac{m+\alpha}{q}\Big), \sigma > 1,$$

where $\zeta(s, \alpha)$ is the classical Hurwitz zeta-function given, for $\sigma > 1$, by

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and meromorphically continued to the whole complex plane with unique simple pole at the point $s = 1$ with residue 1, the function $\zeta(s, \alpha; \mathfrak{a})$ also has meromorphic continuation to the whole complex plane with possible simple pole at the point $s = 1$ with residue

$$\frac{1}{q} \sum_{m=0}^{q-1} a_m.$$

If the latter quantity is equal to zero, the function $\zeta(s, \alpha; \mathfrak{a})$ is entire one.

Clearly, if $a_m \equiv 1$, the function $\zeta(s, \alpha; \mathfrak{a})$ becomes the Hurwitz zeta-function. If $a_m = e^{2\pi i \frac{m}{q}}, m \in \mathbb{N}_0$, then $\zeta(s, \alpha; \mathfrak{a})$ reduces to the Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}, \sigma > 1,$$

with $\lambda = \frac{1}{q}$. Thus, the periodic Hurwitz zeta-function is a generalization of classical zeta-functions.

The function $\zeta(s, \alpha; \mathfrak{a})$, as the majority of other zeta-functions, is universal in the Voronin sense, i.e., its shifts $\zeta(s + i\tau, \alpha; \mathfrak{a}), \tau \in \mathbb{R}$, approximate a wide class of analytic functions. We recall some results on the universality of $\zeta(s, \alpha; \mathfrak{a})$. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $\mathscr{K}$ the class of compact subsets of the strip D with connected complements, and by $H(K)$, $K \in \mathscr{K}$, the class of continuous functions on K which are analytic in the interior of K. Moreover, let

$$L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}.$$

Then in [11], the following theorem was obtained.

Theorem 1. *Suppose that the set $L(\alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathscr{K}$ and $f(s) \in H(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \sup_{s \in K} \Big|\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)\Big| < \epsilon\Big\} > 0.$$

Here meas A denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. It is not difficult to see that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ with transcendental α. This case was discussed in [2] and [3].

Theorem 1 is of continuous character because the shift τ in $\zeta(s+i\tau,\alpha;\mathfrak{a})$ can take arbitrary real values. Also, discrete versions of Theorem 1 are known when τ takes values from the set $\{kh : k \in \mathbb{N}_0\}$ with fixed $h>0$. The first result in this direction has been obtained in [10].

THEOREM 2. *Suppose that α is a transcendental number, and* $\exp\left\{\frac{2\pi}{h}\right\}$ *is a rational number. Let $K \in \mathscr{K}$ and $f(s) \in H(K)$. Then, for every $\epsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N : \sup_{s\in K}\Big|\zeta(s+ikh,\alpha;\mathfrak{a})-f(s)\Big|<\epsilon\Big\}>0.$$

Here $\#A$ denotes the cardinality of the set A. In [13], a more general result was obtained. Let

$$L(\alpha,h,\pi)=\Big\{\Big(\log(m+\alpha) : m\in\mathbb{N}_0\Big),\frac{\pi}{h}\Big\}.$$

THEOREM 3. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Let $K\in\mathscr{K}$ and $f(s)\in H(K)$. Then the assertion of Theorem 2 is true .*

The aim of this paper is to replace the set $\{kh : k\in\mathbb{N}_0\}$ in Theorems 2 and 3 by a more complicated one. Let $0<\beta_1<1$, $\beta_2>0$ and $h>0$ be fixed numbers.

THEOREM 4. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Let $K\in\mathscr{K}$ and $f(s)\in H(K)$. Then, for every $\epsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N-1}\#\Big\{2\leqslant k\leqslant N : \sup_{s\in K}\Big|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-f(s)\Big|<\epsilon\Big\}>0.$$

For the proof of Theorem 4, we will apply good distribution properties of the sequence $\{hk^{\beta_1}\log^{\beta_2}k : k=2,3,...\}$. In general, we will use the probabilistic method based on a limit theorem for probability measures in the space of analytic functions. Let $\mathcal{B}(X)$ denote the Borel $\sigma-$field of the space X, and let $H(D)$ be space of analytic functions on D endowed with the topology of uniform convergence on compacta.

We note that the universality of zeta and L-functions was discovered by Sergei Mikhailovich Voronin who in [15] obtained universality of the Riemann zeta-function and Dirichlet L -functions, see also [6].

2. A limit theorem

We start with a limit theorem of discrete type on the torus

$$\Omega=\prod_{m\in\mathbb{N}_0}\gamma_m,$$

where $\gamma_m=\{s\in\mathbb{C} : |s|=1\}$ for all $m\in\mathbb{N}_0$. With the product topology and pointwise multiplication, the torus Ω is a compact topological group. Therefore, on $(\Omega,\mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and we have the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Denote by $\omega(m)$ the projection of an element $\omega\in\Omega$ to the coordinate space $\gamma_m, m\in\mathbb{N}_0$. For $A\in\mathcal{B}(\Omega)$, we set

$$Q_N(A)=\frac{1}{N-1}\#\Big\{2\leqslant k\leqslant N : \Big((m+\alpha)^{-ihk^{\beta_1}\log^{\beta_2}k} : m\in\mathbb{N}_0\Big)\in A\Big\}.$$

LEMMA 1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then the measure Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

PROOF. We remind that a sequence $\{x_m : m \in \mathbb{N}\}$ is uniformly distributed modulo 1 if, for every interval $I = [a, b) \subset [0, 1)$ of length $|I|$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I\big(\{x_k\}\big) = |I|,$$

where $\{u\}$ is the fractional part of $u \in \mathbb{R}$ and χ_I is the indicator function of I. By the Weyl criterion, see, for example, [5], the sequence $\{x_m\}$ is uniformly distributed modulo 1 if and only if, for every $k \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} e^{2\pi i k x_m} = 0.$$

It is well known [5] that the sequence $\{ak^{\beta_1} \log^{\beta_2} k : k = 2, 3, ...\}$ with $a \neq 0$ is uniformly distributed modulo 1.

For the proof of the lemma, we apply the method of the Fourier transforms. Let $\underline{k} = \{k_m : m \in \mathbb{N}_0\}$ with integers k_m. Then the Fourier transform $g_N(\underline{k})$ of the measure Q_N is of the form

$$g_N(\underline{k}) = \int\limits_{\Omega} \prod_{m\in\mathbb{N}_0} \omega^{k_m}(m) dQ_N,$$

where only a finite number of integers k_m are distinct from zero. Hence, we have that

$$\begin{aligned} g_N(\underline{k}) &= \frac{1}{N-1} \sum_{k=2}^{N} \prod_{m\in\mathbb{N}_0} (m+\alpha)^{-ihk_m k^{\beta_1} \log^{\beta_2} k} \\ &= \frac{1}{N-1} \sum_{k=2}^{N} \exp\Big\{ -ihk^{\beta_1} \log^{\beta_2} k \sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha) \Big\}. \end{aligned} \tag{1}$$

The linear independence over $\mathbb{Q}$ of the set $L(\alpha)$ implies that

$$\sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha) = 0$$

if and only if $\underline{k} = \underline{0}$. Therefore, if $\underline{k} \neq \underline{0}$, then

$$h \sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha) \neq 0.$$

By the above remark, the sequence

$$\Big\{ \frac{hk^{\beta_1} \log^{\beta_2} k}{2\pi} \sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha) : k = 2, 3, ... \Big\}$$

is uniformly distributed modulo 1. Thus, in view of (1) and the Weyl criterion,

$$\lim_{N\to\infty} g_N(\underline{k}) = 0 \tag{2}$$

for $\underline{k} \neq \underline{0}$. Obviously, by (1),

$$g_N(\underline{0}) = 1.$$

This and (2) show that

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Clearly,

$$g(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

is the Fourier transform of the Haar measure m_H. Therefore, the lemma follows by a general continuity theorem for probability measures on compact groups, see, for example, [4].

□

Furthermore, we will deal with a limit theorem for absolutely convergent Dirichlet series. For a fixed $\hat{\sigma} > \frac{1}{2}$ and $m \in \mathbb{N}_0, n \in \mathbb{N}$, let

$$v_n(m,\alpha) = \exp\Big\{ -\Big(\frac{m+\alpha}{n+\alpha}\Big)^{\hat{\sigma}}\Big\}.$$

Define two functions

$$\zeta_n(s,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m v_n(m,\alpha)}{(m+\alpha)^s}$$

and

$$\zeta_n(s,\alpha,\omega;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) v_n(m,\alpha)}{(m+\alpha)^s}.$$

Then the latter series are absolutely convergent for $\sigma > \frac{1}{2}$ [2]. From this, it follows that the function $u_n : \Omega \to H(D)$ given by the formula

$$u_n(\omega) = \zeta_n(s,\alpha,\omega;\mathfrak{a}), \omega \in \Omega,$$

is continuous one. For $A \in \mathcal{B}(H(D))$, let

$$P_{N,n}(A) = \frac{1}{N-1}\#\Big\{2 \leqslant k \leqslant N : \zeta(s+ihk^{\beta_1}\log^{\beta_2} k,\alpha;\mathfrak{a}) \in A\Big\}.$$

Moreover, we put $\hat{P}_n = m_H u_n^{-1}$, where the measure $m_H u_n^{-1}$ is defined by

$$m_H u_n^{-1}(A) = m_H(u_n^{-1}A), A \in \mathcal{B}(H(D)).$$

Lemma 2. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,n}$ converges weakly to $\hat{P}_n$ as $N \to \infty$.*

Proof. By the definition of the function u_n, we have

$$u_n\Big((m+\alpha)^{-ihk^{\beta_1}\log^{\beta_2} k} : m \in \mathbb{N}_0\Big) = \zeta(s+ihk^{\beta_1}\log^{\beta_2} k,\alpha;\mathfrak{a}).$$

Therefore, for $A \in \mathcal{B}(H(D))$,

$$\begin{aligned} P_{N,n}(A) &= \frac{1}{N-1}\#\Big\{2 \leqslant k \leqslant N : \Big((m+\alpha)^{-ihk^{\beta_1}\log^{\beta_2} k} : m \in \mathbb{N}_0\Big) \in A\Big\} \\ &= Q_N(u_n^{-1}A) = Q_N u_n^{-1}(A). \end{aligned}$$

This, the continuity of u_n, Lemma 1 and Theorem 5.1 of [1] show that $P_{N,n}$ converges weakly to $\hat{P}_n$ as $N \to \infty$.

□

Now we will approximate $\zeta(s+ihk^{\beta_1}\log^{\beta_2} k,\alpha;\mathfrak{a})$ by $\zeta_n(s+ihk^{\beta_1}\log^{\beta_2} k,\alpha;\mathfrak{a})$ in the mean. Let ρ be the metric on $H(D)$ which induces the topology of uniform convergence on compacta, see [10], or [8, 9].

Lemma 3. *The equality*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N-1}\sum_{k=2}^{N}\rho\Big(\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a}),\zeta_n(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})\Big)=0$$

holds.

Proof. It was obtained in [2] that, for $\sigma>\frac{1}{2}$,

$$\int_1^T\big|\zeta(\sigma+it,\alpha,\mathfrak{a})\big|^2dt=O(T). \tag{3}$$

To obtain a discrete version of the latter estimate, we will use the Gallagher lemma, see Lemma 1.4 in [14]. For $2\leqslant k\leqslant N$, with sufficiently large N we have

$$\begin{aligned}
&(k+1)^{\beta_1}\log^{\beta_2}(k+1)-k^{\beta_1}\log^{\beta_2}k\\
&=k^{\beta_1}\Big(1+\frac{1}{k}\Big)^{\beta_1}\Big(\log k+\log\Big(1+\frac{1}{k}\Big)\Big)^{\beta_2}-k^{\beta_1}\log^{\beta_2}k\\
&=k^{\beta_1}\Big(1+\frac{\beta_1}{k}+\frac{\beta_1(\beta_1-1)}{2k^2}+...\Big)\Big(\log k+\frac{1}{k}-\frac{1}{k^2}+...\Big)^{\beta_2}-k^{\beta_1}\log^{\beta_2}k\\
&=\Big(k^{\beta_1}+\frac{\beta_1}{k^{1-\beta_1}}+\frac{\beta_1(\beta_1-1)}{2k^{2-\beta_1}}+...\Big)\log^{\beta_2}k\Big(1+\frac{1}{k\log k}-\frac{1}{2k^2\log k}+...\Big)^{\beta_2}\\
&-k^{\beta_1}\log^{\beta_2}k\geq\frac{c\log^{\beta_2}N}{N^{1-\beta_1}}
\end{aligned}$$

with suitable constant $c>0$ not depending on N. Therefore, taking $\delta=\frac{ch\log^{\beta_2}N}{N^{1-\beta_1}}$ in Lemma 1.4 of [14], we find that

$$\begin{aligned}
&\sum_{k=2}^{N}\big|\zeta(\sigma+ihk^{\beta_1}\log^{\beta_2}k+it,\alpha;\mathfrak{a})\big|^2\\
&\ll N^{1-\beta_1}\log^{-\beta_2}N\int_1^{hN^{\beta_1}\log^{\beta_2}N}\big|\zeta(\sigma+i\tau+it,\alpha;\mathfrak{a})\big|^2d\tau+\\
&+\left(\int_1^{hN^{\beta_1}\log^{\beta_2}N}\big|\zeta(\sigma+i\tau+it,\alpha;\mathfrak{a})\big|^2d\tau\int_1^{hN^{\beta_1}\log^{\beta_2}N}\big|\zeta'(\sigma+i\tau+it,\alpha;\mathfrak{a})\big|^2d\tau\right)^{\frac{1}{2}}\\
&\ll N+|t|\ll N(1+|t|)
\end{aligned}$$

for $\sigma>\frac{1}{2}$ because of (3) and the estimate

$$\int_1^T\big|\zeta'(\sigma+it,\alpha,\mathfrak{a})\big|dt=O(T)$$

implied by (3). Let K be a compact subset of the strip D. Then, repeating the proof of Theorem 4.1 from [6], we obtain that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N-1}\sum_{k=2}^{N}\sup_{s\in K}\Big|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-\zeta_n(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})\Big|=0.$$

This and the definition of the metric ρ prove the lemma.
□

Now we state the main limit theorem. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \alpha, \omega; \mathfrak{a})$ by the formula

$$\zeta(s, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}.$$

The latter series, for almost all $\omega \in \Omega$, converges uniformly on compact subsets of the strip D, and therefore, defines a $H(D)$-valued random element. Denote by P_ζ the distribution of the random element $\zeta(s, \alpha, \omega; \mathfrak{a})$, i.e.,

$$P_\zeta(A) = m_H\Big\{\omega \in \Omega : \zeta(s, \alpha, \omega; \mathfrak{a}) \in A\Big\}, A \in \mathcal{B}(H(D)).$$

For $A \in \mathcal{B}(H(D))$, let

$$P_N(A) = \frac{1}{N-1}\#\Big\{2 \leqslant k \leqslant N : \zeta(s + ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}) \in A\Big\}.$$

Theorem 5. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then P_N converges weakly to P_ζ as $N \to \infty$. Moreover, the support of P_ζ is the whole of $H(D)$.*

Proof. Let θ_N be a random variable defined on a certain probability space $\big(\hat{\Omega}, \mathcal{F}, \mathbb{P}\big)$ and having the distribution

$$\mathbb{P}\big(\theta_N = hk^{\beta_1}\log^{\beta_2} k\big) = \frac{1}{N-1}, k = 2, ..., N.$$

Define the $H(D)$-valued random element $X_{N,n}$ by the formula

$$X_{N,n} = X_{N,n}(s) = \zeta_n(s + i\theta_N, \alpha; \mathfrak{a}).$$

Moreover, let $\hat{X}_n$ be the $H(D)$-valued random element having the distribution $\hat{P}_n$, where $\hat{P}_n$ is the limit measure in Lemma 2. Then the assertion of Lemma 2 can be written in the form

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} \hat{X}_n, \tag{4}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution. We will prove that the family of probability measures $\big\{\hat{P}_n : n \in \mathbb{N}\big\}$ is tight, i.e., for every $\epsilon > 0$, there exists a compact set $K = K(\epsilon) \subset H(D)$ such that

$$\hat{P}_n(K) > 1 - \epsilon$$

for all $n \in \mathbb{N}$. Since the series for $\zeta_n(s, \alpha; \mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$, we have that

$$\begin{aligned}\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \big|\zeta(\sigma + it, \alpha; \mathfrak{a})\big|^2 dt &= \sup_{n\in\mathbb{N}}\sum_{m=0}^{\infty}\frac{|a_m|^2 v_n^2(m,\alpha)}{(m+\alpha)^{2\sigma}} \\ &\le \sum_{m=0}^{\infty}\frac{|a_m|^2}{(m+\alpha)^{2\sigma}} \le C < \infty.\end{aligned}$$

This together with the Gallagher lemma [14] implies the bound

$$\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{N-1}\sum_{k=2}^{N}\Big|\zeta_n(s + ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\Big|^2 \le C_1 < \infty.$$

Hence,

$$\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{N-1}\sum_{k=2}^{N}\left|\zeta_n(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})\right|\le C_2<\infty. \tag{5}$$

Let K_l, $l\in\mathbb{N}$, be compact sets from the definition of the metric ρ [10]. Then (6) together with the Cauchy integral formula shows that

$$\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{N-1}\sum_{k=2}^{N}\sup_{s\in K_l}\left|\zeta_n(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})\right|\le R_l<\infty. \tag{6}$$

Let $\epsilon>0$ be an arbitrary number, and $M_l=M_l(\epsilon)=2^lR_l\epsilon^{-1}$. Then, taking into account (7), we find that, for $l\in\mathbb{N}$,

$$\begin{aligned}&\limsup_{N\to\infty}\mathbb{P}\Big(\sup_{s\in K_l}\big|X_{N,n}(s)\big|>M_l\Big)\\ &=\limsup_{N\to\infty}\frac{1}{N-1}\#\Big\{2\leqslant k\leqslant N:\sup_{s\in K_l}\big|\zeta_n(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})\big|>M_l\Big\}\\ &\le\limsup_{N\to\infty}\frac{1}{(N-1)M_l}\sum_{k=2}^{N}\sup_{s\in K_l}\big|\zeta_n(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})\big|\le\frac{\epsilon}{2^l}.\end{aligned}$$

Hence, by the relation (5), we obtain that, for $l\in\mathbb{N}$,

$$\mathbb{P}\Big(\sup_{s\in K_l}\big|\hat{X}_n(s)\big|>M_l\Big)\le\frac{\epsilon}{2^l}. \tag{7}$$

Putting

$$K(\epsilon)=\Big\{g\in H(D):\sup_{s\in K_l}\big|g(s)\big|\le M_l,l\in\mathbb{N}\Big\},$$

we have that $K(\epsilon)$ is a compact subset of $H(D)$ because it is uniformly bounded on compact subsets of the strip D. Moreover, (8) shows that, for all $m\in\mathbb{N}$,

$$\mathbb{P}\Big(\hat{X}_n(s)\in K(\epsilon)\Big)\ge 1-\epsilon,$$

or, for all $n\in\mathbb{N}$,

$$\hat{P}\Big(K(\epsilon)\Big)\ge 1-\epsilon.$$

Thus, the sequence $\{\hat{P}_n:n\in\mathbb{N}\}$ is tight.

Since the sequence $\{\hat{P}_n:n\in\mathbb{N}\}$ is tight, by the Prokhorov theorem, see [1, Theorem 6.1], it is relatively compact. Therefore, there exists a subsequence $\{\hat{P_{n_r}}\}\subset\{\hat{P}_n\}$ such that $\hat{P_{n_r}}$ converges weakly to a certain probability measure P on $\big(H(D),\mathcal{B}(H(D))\big)$ as $n\to\infty$. From this,

$$\hat{X}_{n_r}(s)\xrightarrow[r\to\infty]{\mathcal{D}}P. \tag{8}$$

Let the $H(D)$-valued random element X_N be defined by the formula

$$X_N=X_N(s)=\zeta(s+i\theta_N,\alpha;\mathfrak{a}).$$

Then, by Lemma 3, for every $\epsilon>0$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\mathbb{P}\Big(\rho\big(X_N(s),X_{N,n}(s)\big)\geqslant\epsilon\Big)$$

$$= \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N-1} \#\Big\{2 \leqslant k \leqslant N : \rho\Big(\zeta(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}),$$
$$\zeta_n(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\Big) \geq \epsilon\Big\}$$

$$\leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N-1)\epsilon} \sum_{k=2}^{N} \rho\Big(\zeta(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}),$$
$$\zeta_n(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\Big) = 0.$$

This equality, (5), (9) and Theorem 4.2 from [1] imply the relation

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P, \tag{9}$$

thus, P_N converges weakly to P as $N \to \infty$. Moreover, (10) shows that the measure P is independent of the choice of the subsequence $\{\hat{P_{n_r}}\}$. Therefore, the relation

$$\hat{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P,$$

is true, and we have that the measure $\hat{P}_n$ converges weakly to P as $n \to \infty$.

It remains to identify the measure P. In [11], under the hypothesis that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, it was obtained that the measure

$$\frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \zeta(s+i\tau, \alpha; \mathfrak{a}) \in A\Big\}, A \in \mathcal{B}(H(D)),$$

as $T \to \infty$, also converges weakly to the measure P which is the limit measure of $\hat{P}_n$ as $n \to \infty$, and that P coincides with P_ζ. Since P_N, as $n \to \infty$, converges weakly to P, hence we have that P_N also converges weakly to P_ζ as $N \to \infty$. Moreover, in [11], it was obtained, that the support of P_ζ is the whole of $H(D)$. The theorem is proved.

□

3. Proof of universality

First we state two lemmas.

LEMMA 4. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\epsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} \Big| f(s) - p(s) \Big| < \epsilon.$$

The lemma is the Mergelyan theorem, see [12].

LEMMA 5. *Let $P_n, n \in \mathbb{N}$, and P be a probability measures on $(X, \mathcal{B}(X))$. Then P_n, as $n \to \infty$, converges weakly to P if and only if, for every open set $G \subset X$,*

$$\liminf_{n\to\infty} P_n(G) \geq P(G).$$

The lemma is a part of Theorem 2.1 from [1].

Proof of Theorem 4. By Lemma 4, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}\Big|f(s)-p(s)\Big|<\frac{\epsilon}{2}. \tag{10}$$

Define the set

$$G=\Big\{g\in H(D):\sup_{s\in K}\Big|g(s)-p(s)\Big|<\frac{\epsilon}{2}\Big\}.$$

Then G is an open set in $H(D)$, therefore, in view of Theorem 5 and Lemma 5,

$$\liminf_{N\to\infty}P_N(G)\geq P_\zeta(G). \tag{11}$$

Moreover, G is an open neighbourhood of the polynomial $p(s)$ which, again by Theorem 4, is an element of the support of the measure P_ζ. Thus, $P_\zeta(G)>0$. This, (12) and the definition of G imply the inequality

$$\liminf_{N\to\infty}\frac{1}{N-1}\#\Big\{2\leqslant k\leqslant N:\sup_{s\in K}\Big|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-p(s)\Big|<\frac{\epsilon}{2}\Big\}>0.$$

This and (11) prove the theorem.

□

4. Conclusions

Let $\mathfrak{a}=\{a_m\}$ be a periodic sequence of complex numbers, $0<\alpha\leq 1$ and $s=\sigma+it$. The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$ is defined, for $\sigma>1$, by the series

$$\zeta(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. Moreover, let

$$L(\alpha)=\{\log(m+\alpha):m=0,1,2,...\}.$$

In the paper, the following discrete universality theorem for the function $\zeta(s,\alpha;\mathfrak{a})$ is obtained. Suppose that $\mathscr{K}$ be the class of compact subsets of the strip D with connected complement, and $H(K)$, $K\in\mathscr{K}$, be the class of continuous functions on K which are analytic in the interior of K. Moreover, we assume that the set $L(\alpha)$ is linearly independent over the field of rational numbers, and that $0<\beta_1<1$, $\beta_2>0$ and $h>0$ are fixed numbers. Then the function $\zeta(s,\alpha;\mathfrak{a})$ is universal in the Voronin sense, i.e., if $K\in\mathscr{K}$, $f(s)\in H(K)$, then, for every $\epsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N-1}\#\Big\{2\leqslant k\leqslant N:\sup_{s\in K}\Big|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-f(s)\Big|<\epsilon\Big\}>0.$$

REFERENCES

1. Billingsley P., 1968, "Convergence of Probability Measures" , *New York: Wiley.*

2. Javtokas A., Laurinčikas A., 2006, "On the periodic Hurwitz zeta-fucntion" , *Hardy-Ramanujan J.* Vol. 29. P. 18-36.

3. Javtokas A., Laurinčikas A., 2006, "Universality of the periodic Hurwitz zeta-function" , *Integral Transforms Spec. Funct.* Vol. 17, No. 10. P. 711-722.

4. Heyer H.,1977, "Probability Measures on Locally Compact Groups" , *Berlin, Heidelberg, New York: Springer-Verlag.*

5. Kuipers L., Niederreiter H., 1974, "Uniform Distribution of Sequences" , *Pure and Applied Mathematics.* New York, London, Sydney: Wiley-Interscience.

6. Karatsuba A.A., Voronin S.M., 1992, "The Riemann zeta-function" , *Berlin: Walter de Gruyter.*

7. Laurinčikas A., 2006, "The joint universality for periodic Hurwitz zeta-functions" , *Analysis* (Munich) Vol. 26, No. 3. P. 419–428.

8. Laurinčikas A., 1996, "Limit theorems for Riemann zeta-function" , *Dordrecht, Boston, London: Kluwer.*

9. Laurinčikas A., Garunkštis R., 2002, "The Lerch zeta-function" , *Dordrecht, Boston, London: Kluwer.*

10. Laurinčikas A., Macaitienė R., 2009, "The discrete universality of the periodic Hurwitz zeta-function" , *Integral Transforms Spec. Funct.* Vol. 20. P. 673-686.

11. Laurinčikas A., Macaitienė R., Mochov D., Šiaučiūnas D., 2013, "On universality of certain zeta-functions" , *Izv. Saratov. univ., ser. Matem., Mekhan. Inform.* P. 67-72.

12. Mergelyan S. N., 1952, "Uniform approximations to functions of a complex variable" , *Usp. Matem. Nauk.* Vol. 7, No. 2. P. 31–122 (Russian) ≡ *Amer. Math. Trans.* 1954. Vol. 101.

13. Mincevič A., Mochov D.,2015, "On the discrete universality of the periodic Hurwitz zeta-function" , *Šiauliai Math. Semin.* Vol. 10. P. 81-89.

14. Montgomery H. L., 1971, "Topics in Multiplicative Number Theory." , *Lecture Notes in Math.* Vol. 227. Berlin: Springer.

15. Voronin S. M., 1975, "Theorem on the "universality" of the Riemann zeta-function." , *Izv. Akad. Nauk SSSR.* Vol. 39. P. 475–486 (in Russian) ≡ *Math. USSR Izv.* 1975. Vol. 9. P. 443–453.

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Billingsley P. Convergence of Probability Measures. New York: Wiley, 1968.

2. Javtokas A., Laurinčikas A. On the periodic Hurwitz zeta-fucntion // Hardy-Ramanujan J. 2006. Vol. 29. P. 18-36.

3. Javtokas A., Laurinčikas A. Universality of the periodic Hurwitz zeta-function // Integral Transforms Spec. Funct. 2006. Vol. 17, No. 10. P. 711-722.

4. Heyer H. Probability Measures on Locally Compact Groups. Berlin, Heidelberg, New York: Springer-Verlag, 1977.

5. Кейперс Л., Нидеррейтер Г. Равномерное распределение последовательностей. Москва: Наука, 1985.

6. Воронин С.М., Карацуба А.А. Дзета-функция Римана. Москва: Физматлит, 1994.

7. Laurinčikas A. The joint universality for periodic Hurwitz zeta-functions // Analysis (Munich). 2006. Vol. 26, No. 3. P. 419–428.

8. Laurinčikas A. Limit theorems for Riemann zeta-function. Dordrecht, Boston, London: Kluwer, 1996.

9. Laurinčikas A., Garunkštis R. The Lerch zeta-function. Dordrecht, Boston, London: Kluwer.

10. Laurinčikas A., Macaitienė R. The discrete universality of the periodic Hurwitz zeta-function // Integral Transforms Spec. Funct. 2009. Vol. 20. P. 673-686.

11. Laurinčikas A., Macaitienė R., Mochov D., Šiaučiūnas D. On universality of certain zeta-functions // Изв. Саратовского унив., сер. Матем., Механ., Информ. 2013. P. 67-72.

12. Мергелян С.Н. Равномерные приближения функций комплексного переменного // УМН 1952. Т. 7, №. 2. С. 31–122 ≡ Amer. Math. Trans. 1954. Vol. 101.

13. Mincevič A., Mochov D. On the discrete universality of the periodic Hurwitz zeta-function // Šiauliai Math. Semin. 2015. Vol. 10. P. 81-89.

14. Montgomery H.L. Topics in Multiplicative Number Theory. Lecture Notes in Math. Vol. 227. Berlin: Springer, 1971.

15. Воронин С. М. Теорема об “универсальности” дзета-функции Римана // Изв. АН СССР. Сер. матем. 1975. Т. 39. С. 475–486. ≡ Math. USSR Izv. 1975. Vol. 9. P. 443–453.

Faculty of Mathematics and Informatics, Vilnius University, Naugarduko str. 24, LT-03225 Vilnius, Lithuania.
E-mail: antanas.laurincikas@mif.vu.lt, dmitrij.mochov@mif.vu.lt
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