ЧЕБЫШЕВСКИЙ СБОРНИК
Том 22. Выпуск 3.

УДК 517.58 DOI 10.22405/2226-8383-2021-22-3-464-466

Лемма Болла в качестве упражнения

И. Р. Лифлянд

Лифлянд Илья Рафаилович — факультет математики, Университет Бар-Илан (г. Рамат-Ган, Израиль).
e-mail: liflyand@math.biu.ac.il

Аннотация

Предложено очень короткое доказательство леммы Болла средствами исключительно гармонического анализа.

Ключевые слова: Лемма Болла, преобразование Фурье, неравенство Хаусдорфа-Янга, постоянная Бабенко-Бекнера.

Библиография: 5 названий.

Для цитирования:
И. Р. Лифлянд. Лемма Болла в качестве упражнения // Чебышевский сборник, 2021, т. 22, вып. 3, с. 464–466.

CHEBYSHEVSKII SBORNIK
Vol. 22. No. 3.

UDC 517.58 DOI 10.22405/2226-8383-2021-22-3-464-466

Ball's lemma as an exercise

E. Liflyand

Liflyand Elijah Rafailovich — department of mathematics, Bar-Ilan University (Ramat-Gan, Israel).
e-mail: liflyand@math.biu.ac.il

Abstract

We suggest an extremely short proof of Ball's lemma by means of harmonic analysis only.

Keywords: Ball's lemma, Fourier transform, Hausdorff-Young inequality, Babenko-Beckner constant.

Bibliography: 5 titles.

For citation:
E. Liflyand, 2021, "Ball's lemma as an exercise" , *Chebyshevskii sbornik*, vol. 22, no. 3, pp. 464–466.

It was Erwin Lutwak who brought my attention to Ball's Lemma 3 in [2]. Then I saw it in [5, Ch.VI, 7.5]. Ball's paper is about convex geometry but the main ingredient is the mentioned purely analytic lemma. The proof given is quite technical. It turns out that by standard means of harmonic analysis it becomes a simple exercise, in a sense. The author thanks D. Gorbachev for extremely valuable discussions.

Our version of the result reads as follows.

Lemma 1. *For $p \geqslant 2$, there holds*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right|^{p}dt \leqslant \left(\frac{p}{p-1}\right)^{\frac{p-1}{2}}\frac{1}{\sqrt{p}}. \tag{1}$$

Proof. In fact, the assumption $p \geqslant 2$ ought to bear a superficial resemblance of the Hausdorff-Young inequality in the dual form:

$$\|\widehat{f}\|_{L^p(\mathbb{R})} \leqslant C_{p'}\|f\|_{L^{p'}(\mathbb{R})}, \tag{2}$$

where $C_{p'} = (2\pi)^{\frac{1}{p}}\frac{(p')^{\frac{1}{2p'}}}{p^{\frac{1}{2p}}}$. This sharp constant is due to Beckner [3] (an earlier partial result is due to K. Babenko [1]). What remains now is to figure out of what $\frac{\sin t}{t}$ is the Fourier transform. But this is a well-known fact: the formula (see (5) in [4, Ch.I, §4]; it is mentioned in Remark 12 in the cited literature of [4] that the formula goes back to Fourier)

$$\int_0^{\infty}\frac{\sin ax}{x}\cos tx\,dx = \begin{cases}\frac{\pi}{2}, & t<a;\\ \frac{\pi}{4}, & t=a;\\ 0, & t>a,\end{cases}$$

answers the question. In our case $a=1$, and hence, taking into account the factor $\frac{1}{2\pi}$ for the inverse Fourier transform, we obtain $\|f\|_{L^{p'}(\mathbb{R})} = 2^{-\frac{1}{p}}$. In addition, taking into account the factor $\frac{1}{\pi}$ on the left-hand side of (1) and $(2\pi)^{\frac{1}{p}}$ in $C_{p'}$, we arrive at the desired constant on the right-hand side of (1). □

To be precise, Lemma 3 in [2] states that if $p \geqslant 2$, then

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right|^{p}dt \leqslant \frac{\sqrt{2}}{\sqrt{p}},$$

and there is equality if and only if $p=2$. In fact, the constant $\left(\frac{p}{p-1}\right)^{\frac{p-1}{2}}$ monotonically increases from $\left(\frac{2}{2-1}\right)^{\frac{2-1}{2}} = \sqrt{2}$ at $p=2$ to $\sqrt{e}$ as $p\to\infty$. However, the dependance on $\frac{1}{\sqrt{p}}$ of the right-hand side of (1) is achieved by minimal means. Though the constant $C_{p'}$ in (2) is sharp, it does not provide a sharp constant on the right-hand side of (1), since the extremal function is different from that on the left-hand side of (1).

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Получено 10.09.20 г.
Принято в печать 20.09.2021 г.