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Операторы Хаусдорфа на пространствах типа Харди[1]

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Аннотация

Значительная часть теории операторов Хаусдорфа в последние 20 лет сосредоточена на оценках их ограниченности на пространстве Харди $H^1(\mathbb{R}^d)$. Естественными расширениями этого пространства во многих отношениях являются пространства, введённые Суизи. Они заполняют всю шкалу между $H^1(\mathbb{R}^d)$ и $L^1_0(\mathbb{R}^d)$. В отличие от $H^1(\mathbb{R}^d)$, для них известна только атомная характеризация. Для оценок операторов Хаусдорфа на $H^1(\mathbb{R}^d)$ всегда применялись и другие характеризации. Поскольку эта возможность исключена для пространств Суизи, в настоящей статье разработан подход к оценкам операторов Хаусдорфа, использующий только атомные разложения. Если на $H^1(\mathbb{R}^d)$ этот подход применим для однотипных атомов, то на пространствах Суизи он не менее эффективно работает на бесконечных суммах разнородных атомов. Для одного и того же оператора Хаусдорфа условие ограниченности не зависит от пространства, а только от параметров самого оператора. Пространство же, на котором оператор действует, характеризуется выбором атомов. Приведён пример (для простоты двумерный) с матрицей растяжения аргумента только по одной переменной.

Ключевые слова: Оператор Хаусдорфа; действительное пространство Харди; атомарное разложение.

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Hausdorff operators on Hardy type spaces[2]

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Abstract

During last 20 years, an essential part of the theory of Hausdorff operators is concentrated on their boundedness on the real Hardy space $H^1(\mathbb{R}^d)$. The spaces introduced by Sweezy are, in many respects, natural extensions of this space. They are nested in full between $H^1(\mathbb{R}^d)$ and $L_0^1(\mathbb{R}^d)$. Contrary to $H^1(\mathbb{R}^d)$, they are subject only to atomic characterization. For the estimates of Hausdorff operators on $H^1(\mathbb{R}^d)$, other characterizations have always been applied. Since this option is excluded in the case of Sweezy spaces, in this paper an approach to the estimates of Hausdorff operators is elaborated, where only atomic decompositions are used. While on $H^1(\mathbb{R}^d)$ this approach is applicable to the atoms of the same type, on the Sweezy spaces the same approach is not less effective for the sums of atoms of various types. For a single Hausdorff operator, the boundedness condition does not depend on the space but only on the parameters of the operator itself. The space on which this operator acts is characterized by the choice of atoms. An example is given (two-dimensional, for simplicity), where a matrix dilates the argument only in one variable.

Keywords: Hausdorff operator; real Hardy space; atomic decomposition.

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1. Introduction

In the last two decades, the study of various aspects of the theory of Hausdorff operators has constantly been developed. The paper [15] plays a special role in this topic not because such operators were introduced there or studied for the first time. In fact, in the one-dimensional case, Hausdorff operators on the real line were introduced in [9] (in a sense, they can be found in a dual form in [10]). The main feature of [15] is that such operators in a more or less full generality were studied on the real Hardy space. This showed the prospects of such a theory on more sophisticated spaces than the Lebesgue ones, on which Hausdorff summability had been started earlier. The progress of such studies can partially be seen in the survey papers [5] and [14]. However, the

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approach elaborated in [15] did not allow to solve a similar problem of the boundedness of the operator on $H^1(\mathbb{R}^d)$, $d \geqslant 2$. First attempts [16] and [20] had not solved the problem nor even led to the needed space $H^1(\mathbb{R}^d)$. A sort of a solution was given in [2] but for very exotic Hausdorff type operators, more precisely, for those with one-dimensional averaging. Finally, in [12] a "genuine" multidimensional Hausdorff type operator

$$(\mathcal{H}f)(x) = (\mathcal{H}_\Phi f)(x) = (\mathcal{H}_{\Phi,A} f)(x) = \int_{\mathbb{R}^d} \Phi(u) f\bigl(xA(u)\bigr)\, du, \tag{1}$$

where $A = A(u) = (a_{ij})_{i,j=1}^d = \bigl(a_{ij}(u)\bigr)_{i,j=1}^d$ is the $d \times d$ matrix with the entries $a_{ij}(u)$ being Borel measurable functions of u, was introduced (similar to that independently introduced in [3] for the study on the Lebesgue spaces). This matrix $A(u)$ may be singular at most on a set of measure zero; $xA(u)$ is the row d-vector obtained by multiplying the row d-vector x by the matrix A. Of course, xA can be written as $A^T x^T$, where both the matrix and the vector are transposed, the latter to the column vector. Applying the duality approach, the authors of [12] obtained a condition for the boundedness of the Hausdorff operator (1) on $H^1(\mathbb{R}^d)$ in terms of a matrix norm of A^{-1}. A very similar but slightly different condition appeared in [13] by means of the atomic characterization of $H^1(\mathbb{R}^d)$. In fact, all the conditions in question are given in the form $\Phi \in L^1_w$, where $w \geqslant 0$ is a weight (a non-negative and locally integrable function) and the norm of Φ in the weighted space L^1_w is defined as

$$\|\Phi\|_{L^1_w} = \int_{\mathbb{R}^d} |\Phi(u)| w(u)\, du.$$

The weight is always given in terms of the matrix $A(u)$. For example, the weight in [13] is assigned as follows. For a $d \times d$ matrix M, let $\|M\| = \max_{|x|=1} |Mx^T|$, where $|\cdot|$ denotes the Euclidean norm. It is known (see, e.g., [11, Ch.5, 5.6.35]) that this norm does not exceed any other matrix norm. Now, a bit of history of the H^1 results for $\mathcal{H}_{\Phi,A}$. We start not from the first one in [12] but with that in [13]. The H^1 boundedness is proved there provided $\Phi \in L^1_W$, where the weight

$$W(u) = \|A^{-1}(u)\|^d.$$

In fact, the result in [12] looks similarly, just the matrix norm (weight) differs a little. It is worth to compare these assumptions on Φ and A with the condition $\Phi \in L^1_V$, with

$$V(u) = |\det A^{-1}(u)|.$$

The former ensures the boundedness of the Hausdorff operator in $H^1(\mathbb{R}^d)$, while the latter provides the boundedness on $L^1(\mathbb{R}^d)$. Since we are going to deal with the real Hardy type spaces, subspaces of $L^1(\mathbb{R}^d)$, the integrability of the Hausdorff operator must be always guaranteed. For this, the condition $\Phi \in L^1_V$, with $V(u) = |\det A^{-1}(u)|$, will be a priori implied into our consideration. It was proved in [17] that the same condition $\Phi \in L^1_V$ ensures the boundedness of Hausdorff type operators on $H^1(\mathbb{R}^d)$ but for a very special diagonal matrices A with all diagonal entries equal to one another. This by no means can be true in the general case. One can see the difference between the two conditions from the well-known inequality

$$\|M\|^d \geqslant |\det M|. \tag{2}$$

By this, the two conditions are like the two poles. Any improvement means to take a step closer to the condition $\Phi \in L^1_V$. One step towards this was taken in [4], where the condition in [13] was relaxed to

$$\int_{\mathbb{R}^d} |\Phi(u)| \, \|A^{-1}(u)\|^{d(1-\frac{1}{q})} |\det A^{-1}(u)|^{\frac{1}{q}} \, du < \infty. \tag{3}$$

Shortly after, the next step had been taken. In [6], the following condition came into play:

$$\int_{\mathbb{R}^d} |\Phi(u) \det A^{-1}(u)| \, \ln \left(1 + \frac{\|A^{-1}(u)\|^d}{|\det A^{-1}(u)|}\right) du < \infty; \tag{4}$$

it is also proven in [6] that the result is sharp in the sense that for every better estimate, there is a corresponding Hausdorff operator not bounded in $H^1(\mathbb{R}^d)$.

The following natural continuation of the above work is in order. In [19], a scale of Hardy type spaces H_r, $1 \leqslant r < \infty$, was studied (for further study of this scale, see [1]). These spaces are nested between H^1 and L^1 (in fact, even L^1_0, where L^1_0 is a space of Lebesgue integrable functions with cancelation property) so that their duals are similarly nested between L^∞ and BMO (the former is known to be dual of L^1, while the latter is dual of H^1). Thus, our goal is to give boundedness conditions for Hausdorff operators on the H_r spaces. The main problem here is as follows. In each of the aforementioned results for H^1, the bound was achieved by making use of two different characterizations of the real Hardy space: atomic and Riesz transforms in [13] and [6]; atomic and maximal function in [4]. The obstacle is that the spaces H_r can be treated by means of atomic representations only, but even in the atomic expansion of one function a variety of atoms is used. None of the additional characterizations helpful in the earlier works exists in this case. In order to overcome this obstacle, we have found a way to use only atomic characterizations.

In the aforementioned H^1 results, the crucial role is played by estimates of the $H^1(\mathbb{R}^d)$ norms of all possible automorphisms $f_A(x) := f(xA(u))$ of the H^1 function f. The better is the constant C_A in the estimate

$$\|f(\cdot A(u))\|_{H^1(\mathbb{R}^d)} \leqslant C_A \|f\|_{H^1(\mathbb{R}^d)},$$

the stronger is the result. We continue this line; of course, for the H_r spaces the corresponding norms of $f_A(x) := f(xA(u))$ are evaluated.

The structure of the paper is as follows. In the next section, we give certain preliminaries of the theory of the real Hardy space and of H_r spaces. In Section 3, we describe the mentioned above atomic approach. In Section 4, we apply the obtained estimates for atoms to their sequences in order to establish the boundedness of Hausdorff operators on the H_r spaces. In the last section, we present an example.

In what follows $a \ll b$ means that $a \leqslant Cb$ for some absolute constant C but we are not interested in explicit indication of this constant.

2. Preliminaries on the Hardy type spaces

We start with some basics of the atomic characterization of $H^1(\mathbb{R}^d)$. Let $a(x)$ denote an atom (an $(1, q)$-atom), $1 < q \leqslant \infty$, a function that is of compact support:

$$\operatorname{supp} a \subset B(t_0, R), \tag{5}$$

where $B(t_0, R)$ is the ball in $\mathbb{R}^d$ with center t^0 and radius R; and satisfies the following size condition (L^q normalization)

$$\|a\|_q \leqslant \frac{1}{|B(t_0,R)|^{1-\frac{1}{q}}}, \tag{6}$$

where $|B|$ denotes the Lebesgue measure of B (we believe that no confusion will appear with the same notation $|\cdot|$ for the Euclidean norm, where $\cdot$ means a vector in $\mathbb{R}^d$), and the cancelation condition

$$\int_{\mathbb{R}^d} a(x)\,dx = 0. \tag{7}$$

These conditions make each atom integrable, with the L^1 norm bounded by 1. Indeed, by (5), Hölder's inequality and (6), we get, with $\frac{1}{p}+\frac{1}{q}=1$,

$$\int_{\mathbb{R}^d} |a(x)|\,dx = \int_{B(t_0,R)} |a(x)|\,dx \leqslant \left(\int_{B(t_0,R)} dx\right)^{\frac{1}{p}} \left(\int_{B(t_0,R)} |a(x)|^q\,dx\right)^{\frac{1}{q}} \leqslant 1. \tag{8}$$

It is well known that (see, e.g., [7] or [18])

$$\|f\|_{H^1} \sim \inf\left\{\sum_k |c_k| : f(x) = \sum_k c_k a_k(x)\right\}, \tag{9}$$

where a_k are the above described $(1,q)$-atoms. In fact, the atoms may be even of different q, but $q>1+\delta$ should hold for all q, for some fixed $\delta>0$. In addition, to compare the "genuine" norm of an H^1 function (say, by means of the maximal function or by means of the Riesz transforms) with that for the $(1,q)$-atomic decomposition, one should take into account that the factor $\frac{1}{q-1}$, in addition to an absolute constant, appears in the latter (for discussion on this, see, for example, [19]). This means that a more precise form of (3) should include this factor before the integral in question; this, for instance, shows that the case $q=1$ in (3) is excluded.

The scale of H_r spaces is constructed by an additional parameter r coming into play. More precisely, a sequence of the above defined atoms is used so that the corresponding q-s approach to 1. The parameter r appears in a different size condition used in place of (6):

$$\|a\|_q \leqslant \frac{1}{p^{\frac{1}{r}}|B(t_0,R)|^{1-\frac{1}{q}}}. \tag{10}$$

The atoms defined by (5), (10) and (7) will be called $(1,q,r)$-atoms. The H_r spaces are defined as

$$\|f\|_{H_r} \sim \inf\left\{\sum_k |c_k| : f(x) = \sum_k c_k a_k(x)\right\}, \tag{11}$$

where a_k are the $(1,q_k,r)$-atoms, with $1<q_k\leqslant 2$ and $q_k\to 1$ as $k\to\infty$. In each of the cases, the convergence can be understood in the distributional sense or in the L^1 norm. What mainly makes the H_r spaces different is the influence of different r on p_k which tends to infinity as $q_k\to 1$; recall that $\frac{1}{p_k}+\frac{1}{q_k}=1$. It is proven in the afore-mentioned papers that

$$H^1 \subset H_{r_1} \subset H_{r_2} \subset L^1_0,$$

for any $1 < r_1 < r_2 < \infty$.

We add, just for completeness, that the parameter 1 in the notation of atoms means that some exponential p smaller than 1 may appear instead, for the study of H^p spaces, with $0 < p < 1$. However, we are not going to use such an opportunity in this work.

3. Hausdorff operators on the real Hardy space

In order to show how one can estimate Hausdorff operators using only atomic decompositions, we first restrict ourselves to the case of $H^1(\mathbb{R}^d)$. We shall deal with $(1,q)$-atoms rather than with $(1,\infty)$-atoms, exactly as in [4]. Thus, we begin with decomposing f in (1).

Let $f = \sum\limits_k c_k a_k$, where a_k are $(1,q)$-atoms and $\sum\limits_k |c_k| < \infty$. First of all, we note that

$$\int_{\mathbb{R}^d} a\bigl(xA(u)\bigr)\,dx = |\det A^{-1}| \int_{\mathbb{R}^d} a\bigl(x\bigr)\,dx = 0. \tag{12}$$

By changing the order of integration and summation, we arrive at

$$(\mathcal{H}f)(x) = \int_{\mathbb{R}^d} \Phi(u) \sum_k c_k a_k\bigl(xA(u)\bigr)\,du = \sum_k c_k \int_{\mathbb{R}^d} \Phi(u) a_k\bigl(xA(u)\bigr)\,du, \tag{13}$$

To justify the change of the order, we observe that both the right-hand side and the intermediate integral are Lebesgue integrable in x. This follows from (12) because the function $\Phi\,|\det A^{-1}(\cdot)|$ is integrable on $\mathbb{R}^d$, the series $\sum_k c_k$ is absolutely convergent, and, by (8), the atoms a_k are uniformly integrable. Thus, due to the du Bois-Reymond lemma (see, e.g., [8]), it suffices to verify that the last equality in (13) holds true in the distributional sense. But this follows immediately if one uses again the above arguments.

Let us analyze the properties of the distorted atoms $a(xA(u))$, where a satisfies (5). First, it has mean zero because of (12). Obviously, its support is contained in a ball of radius $R\|A^{-1}(u)\|$. To make atoms from $a(xA(u))$, we scale them as follows:

$$\widetilde{a}(x,u) := \frac{1}{|\det A^{-1}(u)|^{\frac{1}{q}} \|A^{-1}(u)\|^{\frac{d}{p}}} a(xA(u)).$$

Since

$$\left(\int_{\mathbb{R}^d} |a(xA(u))|^q\,dx\right)^{\frac{1}{q}} \leqslant |\det A^{-1}(u)|^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |a(x)|^q\,dx\right)^{\frac{1}{q}},$$

using (6), we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^d} |\widetilde{a}(x,u)|^q\,dx\right)^{\frac{1}{q}} &\leqslant \frac{1}{|B(A^{-1}(u)t_0,\, R\|A^{-1}(u)\|)|^{\frac{1}{p}}} \\ &= \frac{1}{|B(t_0,1)|^{\frac{1}{p}}(\,R\|A^{-1}(u)\|)^{\frac{d}{p}}},\end{aligned} \tag{14}$$

which yields that $\widetilde{a}(x,u)$ is an $(1,q)$-atom.

With this in hand, let us consider

$$\int_{\mathbb{R}^d} \Phi(u) a\bigl(xA(u)\bigr)\,du,$$

with a continuing to be an $(1,q)$-atom. Taking into account (3), let us denote

$$A_m(x) := \frac{1}{\alpha_m} \int_{2^m \leqslant R\|A^{-1}(u)\| < 2^{m+1}} \Phi(u) |\det A^{-1}(u)|^{\frac{1}{q}} \|A^{-1}(u)\|^{\frac{d}{p}} \widetilde{a}(x,u)\, du.$$

and

$$\alpha_m = \int_{2^m \leqslant R\|A^{-1}(u)\| < 2^{m+1}} |\Phi(u)|\, |\det A^{-1}(u)|^{\frac{1}{q}} \|A^{-1}(u)\|^{\frac{d}{p}}\, du,$$

for $m \in \mathbb{Z}$. If a_k is taken rather than a, we shall denote the corresponding values by $A_{m,k}$ and $\alpha_{m,k}$. We thus have

$$\int_{\mathbb{R}^d} A_m(x)\, dx = 0.$$

As above, the support of $A_m(x)$ is related to the support of $\widetilde{a}(x,u)$. Since the latter is within a ball of radius $R\|A^{-1}(u)\|$, we have that the support of $A_m(x)$ is within a ball of radius 2^{m+1}. Now, by Jensen's inequality, we obtain

$$\left(\int_{\mathbb{R}^d} |A_m(x)|^q\, dx\right)^{\frac{1}{q}} \leqslant \frac{1}{\alpha_m} \int_{2^m \leqslant R\|A^{-1}(u)\| < 2^{m+1}} |\Phi(u)|\, |\det A^{-1}(u)|^{\frac{1}{q}} \|A^{-1}(u)\|^{\frac{d}{p}}$$
$$\times \left(\int_{\mathbb{R}^d} |\widetilde{a}(x,u)|^q\, dx\right)^{\frac{1}{q}} du \ll \frac{1}{2^{(m+1)\frac{d}{p}}}.$$

By these, $A_m(x)$ (and each of the $A_{m,k}(x)$) is an atom. Subsequently, we get

$$(\mathcal{H}f)(x) = \sum_k c_k \int_{\mathbb{R}^d} \Phi(u) a_k\big(xA(u)\big)\, du = \sum_k c_k \sum_{m \in \mathbb{Z}} a_{m,k} A_{m,k}(x),$$

which is an atomic decomposition, up to an absolute constant multiple. Indeed, since $\sum\limits_m \alpha_{m,k}$ is dominated by the left-hand side of (3) and $A_{m,k}(x)$ are atoms, we have a series of atoms with the sequence of coefficients summable.

We thus have obtained an atomic decomposition for the Hausdorff operator, with the H^1 norm dominated by that of f times

$$\|A^{-1}\|^{d(1-\frac{1}{q})} |\det A^{-1}|^{\frac{1}{q}} = W^{1-\frac{1}{q}} V^{\frac{1}{q}}.$$

It is worth noting that minimizing the decomposition, if needed, we can prove the result in [4] (condition (3).

4. Hausdorff operators on the Hardy type spaces

We will now demonstrate how to adjust the estimates of the previous section for obtaining

Theorem 1. *For any $q > 1$ and any $1 \leqslant r < \infty$, the Hausdorff operator $\mathcal{H}f$ is bounded on the Hardy type space $H_r(\mathbb{R}^d)$ provided condition (3) holds.*

Proof. The proof goes along the same lines as that in the previous section. Indeed, in that proof the treatment of an individual atom was the main technicality. After that, since the atoms were of the same type, the condition concerned just that type of atom. Here, various $(1, q_k, r)$-atoms are involved in the decomposition. We take into account that in any atomic decomposition all the sums, where only $(1, q_k, r)$-atoms are involved with $q_k \geqslant q$, give a function in H^1. In other words, every function f in H_r can be represented as $f = f_1 + f_2$, where $f_1 \in H^1$ and $f_2 \in H_r$ with atomic expansions by means $(1, q_k, r)$-atoms, $q_k < q$. Obviously, we may consider $f_1 \in H^1$ be characterized by $(1, q)$-atoms. Thus, we apply the obtained estimates separately to each of them. Therefore, the norms to be estimated by $\sum_k |c_k|$ appear first as

$$(\mathcal{H}f)(x) = \sum_k c_k \sum_{m \in \mathcal{Z}} \alpha_{m,k} A_{m,q_k,r}(x).^3$$

Here $A_{m,q_k,r}(x)$ is an $(1, q_k, r)$-atom, $q_k < q$, and

$$\alpha_{m,k} = \int_{2^m \leqslant R_k \|A^{-1}(u)\| < 2^{m+1}} |\Phi(u)|\, |\det A^{-1}(u)|^{\frac{1}{q_k}} \|A^{-1}(u)\|^{d(1-\frac{1}{q_k})}\, du.$$

Subsequently, we get

$$\sum_k |c_k| \sum_m |\alpha_{k,m}| = \sum_k |c_k| p^{-\frac{1}{q_k}} \int_{\mathbb{R}^d} |\Phi(u)| \|A^{-1}(u)\|^{d(1-\frac{1}{q_k})} |\det A^{-1}(u)|^{\frac{1}{q_k}}\, du.$$

Since $|\det A^{-1}(u)| \leqslant \|A^{-1}(u)\|^d$, we have

$$\begin{aligned} &\|A^{-1}(u)\|^{d(1-\frac{1}{q_k})} |\det A^{-1}(u)|^{\frac{1}{q_k}} \\ \leqslant &\|A^{-1}(u)\|^{d(1-\frac{1}{q_k})} |\|A^{-1}(u)\|^{d(\frac{1}{q_k}-\frac{1}{q})} |\det A^{-1}(u)|^{\frac{1}{q}} \\ = &\|A^{-1}(u)\|^{d(1-\frac{1}{q})} |\det A^{-1}(u)|^{\frac{1}{q}}, \end{aligned}$$

and taking the upper bound (3), we obtain

$$\sum_k |c_k| \sum_m |\alpha_{k,m}| < \infty,$$

which completes the proof of Theorem 1. □

Remark 1. *One can see that condition (3) is the same for any r and the same as for H^1 in [4]. The parameter r appears only in the atoms, that is, the family of considered functions may be different. It is an open problem whether this result for the H_r spaces can be improved or not.*

5. Example

We give an example which illustrates the main result. It is two-dimensional, $d = 2$, for the sake of convenience and transparency.

Let

$$f(x) = \begin{cases} p_k^{-\frac{1}{r}}, & 1 + ... + \frac{1}{(k-1)^2} \leqslant x_1 < 1 + ... + \frac{1}{k^2}, 0 < x_2 < 1, \\ & k = 3, 4, ..., \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

[3]Next we set

$$A_{k,r,m}(x) := \frac{1}{\alpha_{k,r,m}} \sum_{2^m \leqslant \sqrt{1+\frac{N^2}{k^4}} < 2^{m+1}} \frac{1}{k^{\frac{2}{q_k}}} N^{\frac{1}{q_k}} (1 + \frac{N^2}{k^4})^{1-\frac{1}{q_k}} \widetilde{a}_{k,r}(x)\, du.$$

where

$$\alpha_{k,r,m} = \sum_{2^m \leqslant \sqrt{1+\frac{N^2}{k^4}} < 2^{m+1}} p^{-\frac{1}{q_k}} \frac{1}{k^{\frac{2}{q_k}}} N^{\frac{1}{q_k}} \left(1 + \frac{N^2}{k^4}\right)^{1-\frac{1}{q_k}},$$

The matrix A is

$$A(u) \equiv A = \begin{pmatrix} \frac{1}{N} & 0 \\ 0 & 1 \end{pmatrix}.$$

Set

$$a_{k,r}(x) = \begin{cases} \frac{k^{\frac{2}{q_k}}}{p_k^{\frac{1}{r}}}, & 1 + ... + \frac{1}{(k-1)^2} \leqslant x_1 < 1 + ... + \frac{1}{k^2}, 0 < x_2 < 1, \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

and check that $a_{k,r}$ is an $(1, q_k, r)$-atom, up to a constant multiple. Indeed, its support is in the ball of radius $\sqrt{1 + \frac{1}{k^4}}$, and its L^{q_k} norm is dominated by $p_k^{-\frac{1}{r}}$.

By this,

$$f(x) = \sum_k \frac{1}{k^{\frac{2}{q_k}}} a_{k,r}(x),$$

and we have

$$f(xA) = \sum_k \frac{1}{k^{\frac{2}{q_k}}} a_{k,r}(xA) = \sum_k \frac{1}{k^{\frac{2}{q_k}}} N^{\frac{1}{q_k}} \left(1 + \frac{N^2}{k^4}\right)^{1-\frac{1}{q_k}} \widetilde{a}_{k,r}(x),$$

where $\widetilde{a}_{k,r}$ is an $(1, q_k, r)$-atom (up to a constant multiple), with the support in the ball of radius $\sqrt{1 + \frac{N^2}{k^4}}$.

Hence, the corresponding Hausdorff operator is subject to the decomposition

$$\begin{aligned} (\mathcal{H}f)(x) &= \int_{\mathbb{R}^2} \Phi(u)\, du \sum_k k^{-\frac{2}{q_k}} a_{k,r}(xA) \\ &= \int_{\mathbb{R}^2} \Phi(u) \sum_k k^{-\frac{2}{q_k}} N^{\frac{1}{q_k}} \left(1 + \frac{N^2}{k^4}\right)^{1-\frac{1}{q_k}} \widetilde{a}_{k,r}(x) \\ &:= \sum_k c_k \widetilde{a}_{k,r}(x). \end{aligned}$$

Since the function $g(t) = \lambda^t (1 + \lambda^2)^{1-t}$ decreases on $(0, +\infty)$ for every $\lambda > 0$, we have

$$\|f\|_{H_r} \leqslant \sum_k |c_k| \leqslant \|\Phi\|_{L^1} \sum_{k=1}^{\infty} \left(\frac{N}{k^2}\right)^{\frac{1}{q}} \left(1 + \frac{N^2}{k^4}\right)^{1-\frac{1}{q}}.$$

The final condition follows from the bound for the latter series. Majorazing it by taking all $q_k = q$ and splitting the sum into two, over $1 \leqslant k \leqslant \sqrt{N}$ and over $\sqrt{N} < k < \infty$, we see that for the first sum the estimate is exactly $N^{2-\frac{1}{q}}$, while the bound for the second one is better.

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