ЧЕБЫШЕВСКИЙ СБОРНИК
Том 22. Выпуск 3.

УДК 517.55 DOI 10.22405/2226-8383-2021-22-3-20-31

Оценки ядра Бергмана для классических областей Э. Картана

Ж. Ш. Абдуллаев

Абдуллаев Жонибек Шокирович — докторант, Национальный университет Узбекистана им. М. Улугбека (г. Ташкент, Узбекистан).
e-mail: jonibek-abdullayev@mail.ru

Аннотация

Целю работы является найти оптималные оценки ядер Бергмана для классических областей $\Re_I(m,k)$, $\Re_{II}(m)$, $\Re_{III}(m)$ и $\Re_{IV}(n)$, соответственно, через ядра Бергмана в шарах из пространств $\mathbb{C}^{mk}$, $\mathbb{C}^{\frac{m(m+1)}{2}}$, $\mathbb{C}^{\frac{m(m-1)}{2}}$ и $\mathbb{C}^n$. Для этого изпользуются утверждения теоремы Зоммера-Меринга о продолжении ядро Бергмана и некоторые свойства ядра Бергмана.

Ключевые слова: Классические областиб ядро Бергмана, однородная область, симметрическая область, ортонормальная система.

Библиография: 25 названий.

Для цитирования:
Ж. Ш. Абдуллаев. Оценки ядра Бергмана для классических областей Э. Картана // Чебышевский сборник, 2021, т. 22, вып. 3, с. 20–31.

CHEBYSHEVSKII SBORNIK
Vol. 22. No. 3.

UDC 517.55 DOI 10.22405/2226-8383-2021-22-3-20-31

Estimates the Bergman kernel for classical domains É. Cartan's

J. Sh. Abdullayev

Abdullayev Jonibek Shokirovich — PhD-Student, National University of Uzbekistan named after M. Ulugbek (Tashkent, Uzbekistan).
e-mail: jonibek-abdullayev@mail.ru

Abstract

The aim of this work is to find optimal estimates for the Bergman kernels for the classical domains $\Re_I(m,k), \Re_{II}(m), \Re_{III}(m)$ and $\Re_{IV}(n)$ through the Bergman kernels of balls in the spaces $\mathbb{C}^{mk}, \mathbb{C}^{\frac{m(m+1)}{2}}, \mathbb{C}^{\frac{m(m-1)}{2}}$ and $\mathbb{C}^n$, respectively. For this, we use the statements of the Summer-Mehring theorem on the extension of the Bergman kernel and some properties of the Bergman kernel.

Keywords: Classical domains, Bergman's kernel, , homogeneous domain, symmetric domain, orthonormal system.

Bibliography: 25 titles.

For citation:
J. Sh. Abdullayev, 2021, "Estimates the Bergman kernel for classical domains É. Cartan's", *Chebyshevskii sbornik*, vol. 22, no. 3, pp. 20–31.

1. Introduction, preliminaries and problem statement

In the representation of multidimensional data (in some sections of mathematics, technology, economics, etc.) vectors and matrices are especially useful when modeling and studying abstract and real systems, the description of which requires a lot of information. It is convenient to represent this information using matrices. With this approach, the analysis of systems is reduced to the analysis of the properties of matrices. The main mathematical apparatus of algebra is matrices; they are used in the study of systems of linear equations, linear and quadratic forms and linear mappings of vector spaces.

In the works of É Cartan([1]), C.L. Siegel ([2]), Hua Luogeng ([3]), I.I.Pjateckiǐ-Šapiro ([4]), as well as in [5] the matrix approach of presenting the theory of multivariable complex analysis is widely used. It mainly deals with the classical domains and related questions of function theory and geometry. The importance of studying classical domains is that they are not reducible, i.e. these domains are, in a sense, model domains of multidimensional space.

In recent times, scientists have achieved many significant results in the classical fields, and at the same time, a number of open problems have been formulated. For example, in [6] the regularity and algebraicity of mappings in classical domains are studied, and in [7] harmonic Bergman functions in classical domains are studied from a new point of view. In the paper, [22] holomorphic and pluriharmonic functions are defined for classical domains of the first type, the Laplace and Hua Luogeng operators are studied also. A connection was found between these operators.

In complex analysis, studies of specific classes play an important role domains. The well-known Riemann theorem states that any simply connected domain $D \subset \mathbb{C}$, the boundary of which contains at least two points, is conformally equivalent to the unit disc. This theorem ceases to be true in $\mathbb{C}^n$, when $n > 1$: there is no biholomorphic map of the unit ball

$$\mathbb{B}^n(1) = \{z \in \mathbb{C}^n : |z| < 1\}$$

in a polycircle

$$\mathbb{U}^n = \{z \in \mathbb{C}^n : |z_1| < 1, ..., |z_n| < 1\},$$

where $|z| = \sqrt{\langle z, \bar{z}\rangle} = \sqrt{|z_1|^2 + |z_2|^2 + ... + |z_n|^2}$.

For convenience, we present the following definitions (see, for example, [4], [8]):

Definition 1 (homogeneous domain). *The domain $D \subset \mathbb{C}^n$ is called homogeneous if the group $Aut(D)$ of automorphisms of this domain is transitive, that is, for any pair of points $z_1, z_2 \in D$ there exists an automorphism $\varphi \in Aut(D)$, such that $\varphi(z_1) = z_2$.*

Definition 2 (symmetric domain). *The homogeneous domain $D \subset \mathbb{C}^n$ is called symmetric if for any point $\zeta \in D$ such an automorphism $\varphi \in Aut(D)$, that:*

1) $\varphi(\zeta) = \zeta$ but $\varphi(z) \neq z$, if $z \in D$ is different from ζ;

2) $\varphi \circ \varphi = e$, where $e \in Aut(D)$ is the identity mapping.

Definition 3 *The domain $D \subset \mathbb{C}^n$ is called an irreducible domain if it is not a direct product of bounded symmetric domains of lower dimension.*

Definition 4 *The bounded domain $D \subset \mathbb{C}^n$ is called classical if the complete group of its holomorphic automorphisms is a classical Lie group and it is transitive on it.*

In homogeneous domains, the automorphism groups ([9], [10]) can be used to find integral formulas. Domains with rich automorphism groups are often implemented as matrix domains ([3], [11]). They turned out to be useful in solving various problems of function theory.

Complex homogeneous bounded domain are of great interest from different points of view. This is due to the fact that they are a relatively wide class of domains in $\mathbb{C}^n$, for which a number of meaningful, essentially multidimensional results have been obtained ([3], [12], [13] and etc.).

E. Cartan (see [1]) in 1935 initiated a systematic study of bounded homogeneous domains, found all homogeneous homogeneous domains in the spaces $\mathbb{C}^2$ and $\mathbb{C}^3$. He gave a classification of all bounded symmetric regions. These domains are divided into equivalence classes with respect to biholomorphic mappings. Each such class can be specified by specifying one area that belongs to it. Further, it is obvious that it is sufficient to consider only irreducible classes, that is, classes of domains that are not products of bounded symmetric domains of lower dimensions. In general, as E. Cartan established [1], there are six types of classes of irreducible bounded symmetric domains. Domains belonging to four of them are called classical because their automorphism groups are classical semisimple Lie groups. Two of these types are special in the sense that each of them occurs in the space $\mathbb{C}^n$ of only one dimension n, respectively for $n = 16$ and $n = 27$.

Consider the classical domains (see.[3], [1]):

$$\Re_I(m,k) = \left\{ Z \in \mathbb{C}[m \times k] : I^{(m)} - Z\overline{Z}' > 0 \right\},$$

$$\Re_{II}(m) = \left\{ Z \in \mathbb{C}[m \times m] : I^{(m)} - Z\overline{Z} > 0,\ \forall Z' = Z \right\},$$

$$\Re_{III}(m) = \left\{ Z \in \mathbb{C}[m \times m] : I^{(m)} + Z\overline{Z} > 0,\ \forall Z' = -Z \right\},$$

$$\Re_{IV}(n) = \left\{ \ z \in \mathbb{C}^n \ : \ |\langle z, z\rangle|^2 - 2|z|^2 + 1 > 0, \ \ |\langle z, z\rangle| < 1 \right\},$$

here $I^{(m)}$ is the identity matrix of order m, $\overline{Z}'$ is the complex conjugate matrix of the transposed matrix Z' ($H > 0$ for a Hermitian matrix H means, as usual, that H is positive definite). All these domains are homogeneous symmetric convex complete circular domains centered at O (O is zero matrix).

If we write the elements of the matrices $Z \in \mathbb{C}[m \times k]$ as a point in the space $\mathbb{C}^{mk}$:

$$z = \{z_{11}, ..., z_{1k}, z_{21}..., z_{2k}, ..., z_{m1}..., z_{mk}\} \in \mathbb{C}^{mk}, \tag{1}$$

then we can assume that Z is an element of the space in $\mathbb{C}^{mk}$, i.e., we arrive to the isomorphism

$$\mathbb{C}[m \times k] \cong \mathbb{C}^{mk}.$$

Therefore, the dimensions of the classical domains above four, are equal, respectively,

$$mk, \frac{m(m\ +\ 1)}{2}, \frac{m(m\ -\ 1)}{2}, n.$$

Writing out explicitly the transitive group of automorphisms of four types of classical domains and matrix balls (see, for example, [14], [15]) associated with classical domains, by direct computation one can find the Bergman and Cauchy-Szegő kernels for these domains. And then (using the properties of the Poisson kernel), we find the Carleman formula, which restores the value of a holomorphic function in the domain itself from its values on some boundary sets of uniqueness (see [16], [17], [18], [19]). In this case, the scheme for finding the Bergman and Cauchy-Szegő kernels from [3], [12], [20] is used. In [21] the volumes of a matrix ball of the third type and a generalized Lie ball are calculated. The full volumes of these domains are necessary to find the kernels of integral formulas for these domains (Bergman, Cauchy-Szegő, Poisson kernels, etc.) and is used for the integral representation of functions holomorphic in these domains, in the mean value theorem, and in other important concepts.

The Bergman space on bounded symmetric domains is a fundamental concept in the analysis. It is equipped with a natural projection, i.e. the Bergman projection, determined by the property of the reproducing nucleus. On the other hand, the weighted Bergman spaces are also important in harmonic analysis (see, for example [13]).

Definition 4 ([23]). *Let $\{\varphi_\nu(z), \nu = 0, 1, 2, ...\}$ be a complete orthonormal system of functions in $L^2(D)$. The Bergman kernel (or kernel function[1]) $K_D(z, \bar\zeta)$ is the sum of the series*

$$\sum_{\mu=1}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(\zeta)} = K_D(z, \bar\zeta), \tag{2}$$

which is holomorphic by z and antiholomorphic by ζ

For example (see [8], [23]), the Bergman kernel for a ball with radius R, $\mathbb{B}^n(R) = \{z \in \mathbb{C}^n : |z| < R\}$, has the form

$$K_{\mathbb{B}^n(R)}(z, \bar\zeta) = \frac{n!R^2}{\pi^n\left(R^2 - \sum\limits_{k=1}^{n} z_k\bar\zeta_k\right)^{n+1}}.$$

The aim of this work is to find optimal estimates for the Bergman kernels for the classical domains $\Re_I(m,k)$, $\Re_{II}(m)$, $\Re_{III}(m)$ and $\Re_{IV}(n)$, respectively, through the Bergman kernels in balls from the spaces $\mathbb{C}^{mk}$, $\mathbb{C}^{\frac{m(m+1)}{2}}$, $\mathbb{C}^{\frac{m(m-1)}{2}}$ and $\mathbb{C}^n$. For this, we use the statements of the Sommer-Mehring theorem (see [24]) on the extension of the Bergman kernel and some properties of the Bergman kernel.

Theorem 1 (Sommer-Mehring). *Let $D \subset G$ are domains in $\mathbb{C}^n$, where the domain D is bounded and univalent. If the Bergman kernel (kernel function) $K_D(z, \bar z)$ can be continued into the domain G as a real analytic function, it follows that: :*

a) Every function $f \in L^2(D)$ can be holomorphically continued to the domain G;

b) If $\{\varphi_\nu(z), \nu = 1, 2, ...\}$ – is a closed orthonormal system of functions in the domain D equality

$$K_D(z, \bar\zeta) = \sum_{\nu=1}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(\zeta)}$$

takes place in the domain $G_z \times G_\zeta$;

c) for every function $f \in L^2(D)$ the expansion $f(z) = \sum\limits_{\nu=1}^{\infty} a_\nu\varphi_\nu(z)$ remains valid in the whole domain G;

d) The function $K_D(z, \bar\zeta)$ holomorphically to the variable z and antiholomorphic by to the variable ζ in the domain $G_z \times G_\zeta$;

e) (reproducing property Bergman kernel) The integral representation

$$f(z) = \int\limits_{D_\zeta} f(\zeta) K_D(z, \bar\zeta)\, d\mu_\zeta$$

is valid at all points $z \in G$;

f) (extreme property of the Bergman kernel) for functions $f(z) \in L^2(D)$, satisfying the condition $\|f\|_{L^2(D)} \leqslant 1$, at all points $z \in G$ the equality holds

$$K_D(z, \bar z) = \max|f(z)|^2.$$

In other words, the Bergman kernel for any transitive circular region is equal to the ratio of the volume density to the Euclidean volume of the domain (we recall that if the domain $D \subset \mathbb{C}^n$ admits

[1] In classical Russian literature, the Bergman kernel is usually called the domain kernel function (see, for example, [23], [8]). At the present time, such a term would not be very apt, since we have three kernels (Bergman, Cauchy-Szegő and Poisson) that could bear this name.

the transformation group $z = e^{i\theta}w$, then we call D a circular domain, if, in addition, with a point, z and the point rz $(0 \leqslant r \leqslant 1)$ lies in D, then we call D a the complete circular domain). Hua Luogeng in [3] constructed Bergman kernels for four types of classical domains, being guided only by this consideration and without resorting to complete orthonormal systems, and in this book one can also find explicit expressions for the Bergman kernel, groups of automorphisms of the domain $\Re_I(m,k), \Re_{II}(m), \Re_{III}(m)$ и $\Re_{IV}(n)$.

2. Estimate for the Bergman Kernel for the Lie Ball

Let $\Re_{IV}(n)$ be a classical domain of the fourth type (this domain is called a Lie ball (see [3])). It is known [3], that the Bergman kernel for the Lie ball $\Re_{IV}(n)$ has the form

$$K_{\Re_{IV}(n)}(z,\bar\zeta) = \frac{1}{V(\Re_{IV}(n))\left(1 - 2\left\langle z,\bar\zeta\right\rangle + \langle z,z\rangle\left\langle\bar\zeta,\bar\zeta\right\rangle\right)^n},$$

where $V(\Re_{IV}(n)) = \frac{\pi^n}{2^{n-1}n!}$ – full volume of the Lie ball $\Re_{IV}(n)$.

Let us first prove the following lemma.

Lemma 1. *The Lie ball $\Re_{IV}(n)$ is contained in the unit ball $\mathbb{B}^n(1)$ from the space $\mathbb{C}^n$.*

Proof. Let any $z \in \Re_{IV}(n)$. Then we have the following relations

$$\left(1 - |z|^2\right)^2 = (1 - \langle z,\bar z\rangle)^2 = 1 - 2\langle z,\bar z\rangle + \langle z,\bar z\rangle^2 >$$

$$> \langle z,\bar z\rangle^2 - |\langle z,z\rangle|^2 > \langle z,\bar z\rangle^2 - 1 = |z|^4 - 1.$$

Hence $|z|^2 - 1 < 0$, i.e. $z \in \mathbb{B}^n(1)$. Hence,

$$\Re_{IV}(n) \subset \mathbb{B}^n(1). \tag{3}$$

Lemma 1 is proved. □

The following is true.

Theorem 2. *If the Bergman kernel $K_{\Re_{IV}(n)}(z,\bar z)$ extends to the domain $\mathbb{B}^n(1)$ as a real analytic function, then at the points $z \in \mathbb{B}^n(1)$ the inequality*

$$K_{\mathbb{B}^n(1)}(z,\bar z) \leqslant K_{\Re_{IV}(n)}(z,\bar z).$$

Proof. Using (3) we compare the Bergman kernels defined in these balls and by Theorem 1 (property *e)*), due to the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$K_{\mathbb{B}^n(1)}(z,\bar z) = \int\limits_{\Re^n_{IV}} K_{\mathbb{B}^n(1)}(\zeta,\bar z)\, K_{\Re_{IV}(n)}\left(z,\bar\zeta\right) d\nu(\zeta) \leqslant$$

$$\leqslant \left(\int\limits_{\Re_{IV}(n)} \left|K_{\mathbb{B}^n(1)}(\zeta,\bar z)\right|^2 d\nu(\zeta)\right)^{\frac12} \cdot \left(\int\limits_{\Re_{IV}(n)} \left|K_{\Re_{IV}(n)}\left(z,\bar\zeta\right)\right|^2 d\nu(\zeta)\right)^{\frac12} \leqslant$$

$$\leqslant \left(\int\limits_{\mathbb{B}^n(1)} \left|K_{\mathbb{B}^n(1)}(\zeta,\bar z)\right|^2 d\nu(\zeta)\right)^{\frac12} \cdot \left(\int\limits_{\Re_{IV}(n)} \left|K_{\Re_{IV}(n)}\left(z,\bar\zeta\right)\right|^2 d\nu(\zeta)\right)^{\frac12}. \tag{4}$$

Now, using holomorphy in the variable z and antiholomorphism in the variable ζ (i.e., $K\left(z,\bar{\zeta}\right)=\overline{K\left(\zeta,\,\bar{z}\right)}$- is the property of conjugate symmetry of the Bergman kernel) from (4) we obtain the equality:

$$\left(\int\limits_{\mathbb{B}^n(1)}\left|K_{\mathbb{B}^n(1)}\left(\zeta,\bar{z}\right)\right|^2 d\nu\left(\zeta\right)\right)^{\frac{1}{2}}\cdot\left(\int\limits_{\Re_{IV}(n)}\left|K_{\Re_{IV}(n)}\left(z,\bar{\zeta}\right)\right|^2 d\nu\left(\zeta\right)\right)^{\frac{1}{2}}=$$

$$=\left(\int\limits_{\mathbb{B}^n(1)}K_{\mathbb{B}^n(1)}\left(\zeta,z\right)\overline{K_{\mathbb{B}^n(1)}\left(\zeta,\bar{z}\right)}d\nu\left(\zeta\right)\right)^{\frac{1}{2}}\left(\int\limits_{\Re_{IV}(n)}K_{\Re_{IV}(n)}\left(z,\zeta\right)\overline{K_{\Re_{IV}(n)}\left(z,\bar{\zeta}\right)}d\nu\left(\zeta\right)\right)^{\frac{1}{2}}=$$

$$=\left(\int\limits_{\mathbb{B}^n(1)}K_{\mathbb{B}^n(1)}\left(\zeta,\bar{z}\right)K_{\mathbb{B}^n(1)}\left(z,\bar{\zeta}\right)d\nu\left(\zeta\right)\right)^{\frac{1}{2}}\left(\int\limits_{\Re_{IV}(n)}K_{\Re_{IV}(n)}\left(z,\bar{\zeta}\right)K_{\Re_{IV}(n)}\left(\zeta,\bar{z}\right)d\nu\left(\zeta\right)\right)^{\frac{1}{2}}=$$

$$=\left(K_{\mathbb{B}^n(1)}\left(z,\bar{z}\right)\right)^{\frac{1}{2}}\left(K_{\Re_{IV}(n)}\left(z,\bar{z}\right)\right)^{\frac{1}{2}}. \tag{5}$$

Thus, from the relations(4) and (5) the following inequality follows:

$$K_{\mathbb{B}^n(1)}\left(z,\bar{z}\right)\leqslant K_{\Re_{IV}(n)}\left(z,\bar{z}\right).$$

Theorem 2 is proved. □

Note that for $n=1$

$$K_{\mathbb{B}^1(1)}\left(z,\bar{z}\right)=K_{\Re_{IV}(1)}\left(z,\bar{z}\right)=\frac{1}{\pi\left(1-|z|^2\right)}.$$

On the other hand, based on the above, we can estimate from above the Bergman kernel $K_{\Re_{IV}(n)}(z,\bar{z})$. For points $z\in B^n\left(\frac{1}{\sqrt{2}}\right)$, takes place

$$|z|^2=\sum_{k=1}^{n}|z_k|^2<\frac{1}{2}.$$

In this way

$$1+|\langle z,z\rangle|^2-2|z|^2\geqslant 1-2|z|^2>0. \tag{6}$$

Considering that $|\langle z,z\rangle|\leqslant|z|^2$, we have

$$1-|\langle z,z\rangle|^2\geqslant 1-|z|^4>\frac{3}{4}>0. \tag{7}$$

From the inequality(6) and (7) we get $z\in\Re_{IV}\left(n\right)$. Hence,

$$\mathbb{B}^n\left(\frac{1}{\sqrt{2}}\right)\subset\Re_{IV}\left(n\right).$$

In the same way, we obtain the following statement.

Theorem 3. *If the Bergman kernel is $K_{\mathbb{B}^n\left(\frac{1}{\sqrt{2}}\right)}\left(z,\bar{z}\right)$ continues to the domain $\Re_{IV}\left(n\right)$ as a real analytic function, then at the points $z\in\Re_{IV}\left(n\right)$ the inequality holds*

$$K_{\Re_{IV}(n)}\left(z,\bar{z}\right)\leqslant K_{\mathbb{B}^n\left(\frac{1}{\sqrt{2}}\right)}\left(z,\bar{z}\right).$$

3. Estimates for the Bergman kernel for the classical domains $\Re_I(m,k)$, $\Re_{II}(m)$ and $\Re_{III}(m)$

Now we give estimates for the Bergman kernel for other types of classical domains. For this, we first prove the following lemma, which establishes relations between the classical domains $\Re_I(m,k)$, $\Re_{II}(m)$, $\Re_{III}(m)$ and balls from the space $\mathbb{C}^{mk}$, $\mathbb{C}^{\frac{m(m+1)}{2}}$, $\mathbb{C}^{\frac{m(m-1)}{2}}$, respectively.

Lemmma 2. *The following relations hold:*

a)

$$B^{mk}(1) \subset \Re_I(m,k);$$

b)

$$\mathbb{B}^{\frac{m(m+1)}{2}}\left(\frac{1}{\sqrt{2}}\right) \subset \Re_{II}(m);$$

c)

$$\mathbb{B}^{\frac{m(m-1)}{2}}(1) \subset \Re_{III}(m).$$

Proof. a) It is known [25], that for any $Z \in \mathbb{C}[m\times k]$ when $m \leqslant k$, there exist unitary matrices U of order m and V of order k such that

$$Z = U\begin{pmatrix} \lambda_1 & 0 & ... & 0 & 0 & ... & 0\\ 0 & \lambda_2 & ... & 0 & 0 & ... & 0\\ ... & ... & ... & ... & ... & ... & ...\\ 0 & 0 & ... & \lambda_m & 0 & 0 & 0 \end{pmatrix}V,$$

for some $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_m \geqslant 0$. Hence it follows that

$$\det\left(I^{(m)} - Z\bar{Z}'\right) = \left(1-\lambda_1^2\right)...\left(1-\lambda_m^2\right) = \det\left(I^{(m)} - \bar{Z}'Z\right).$$

Therefore, for a given $Z \in \mathbb{C}[m\times k]$ the relation $I^{(m)} - Z\bar{Z}' > 0$ is executed if and only if $1-\lambda_s^2 > 0$ or $\lambda_s < 1$, $s = 1,...,m$.

On the other hand, for $Z = \{z_{11},...,z_{1k}, z_{21}...,z_{2k},...,z_{m1}...,z_{mk}\} \in \mathbb{B}^{mk}(1)$, we have

$$\|Z\|^2 = \sum_{s=1}^{m}\sum_{j=1}^{k}|z_{sj}|^2 = \sum_{s=1}^{m}\sum_{j=1}^{k} z_{sj}\bar{z}_{sj} = Sp\left(Z\bar{Z}'\right) = \sum_{s=1}^{m}\lambda_s^2 < 1.$$

Whence $\lambda_s < 1\,(s = 1,...,m)$ and $Z \in \Re_I(m,k)$. Hence,

$$B^{mk} \subset \Re_I(m,k).$$

b) Let $Z \in \Re_{II}(m)$. It is known [25], that for any symmetric matrix $Z \in \mathbb{C}[m\times m]$, there exists a unitary matrix $U \in U(m)$ and real numbers $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_m \geqslant 0$, such that

$$Z = U'diag(\lambda_1,...,\lambda_m)\,U = U'\Lambda U.$$

In this way,

$$I^{(m)} - Z\bar{Z} = I^{(m)} - U'\Lambda^2\bar{U} = I^{(m)} - U'\left(\lambda_1^2,...,\lambda_m^2\right)^2\bar{U} =$$
$$= U'diag\left(1-\lambda_1^2,...,1-\lambda_m^2\right)\bar{U}.$$

This equality implies that $Z \in \Re_{II}(m)$ if and only if $1-\lambda_s^2>0$ and $\lambda_s<1\,(s=1,...,m)$. On the other hand, for $Z \in \mathbb{B}^{\frac{m(m+1)}{2}}\left(\frac{1}{\sqrt{2}}\right)$ we have

$$\|Z\|^2=\sum_{1\leqslant s\leqslant j\leqslant m}|z_{sj}|^2=\sum_{1\leqslant s\leqslant j\leqslant m}z_{sj}\bar{z}_{sj}=\frac{1}{2}Sp\left(Z\bar{Z}\right)+\frac{1}{2}\sum_{s=1}^{m}|z_{ss}|^2=$$

$$=\frac{1}{2}\sum_{s=1}^{m}\lambda_s^2+\frac{1}{2}\sum_{s=1}^{p}|z_{ss}|^2<\frac{1}{2},$$

and thus $\sum\limits_{s=1}^{m}\lambda_s^2<1$. Therefore, $\lambda_s<1, s=1,...,m$ and $\mathbb{B}^{\frac{m(m+1)}{2}}\left(\frac{1}{\sqrt{2}}\right)\subset\Re_{II}(m)$.

c) Let $Z\in\Re_{III}(m)$. It is known [25], that for any skew-symmetric matrix $Z\in\mathbb{C}[m\times m]$, there exists a unitary matrix $U\in U(m)$ and $\lambda_1\geqslant\lambda_2\geqslant...\geqslant\lambda_\gamma\geqslant 0$, $\gamma=\left[\frac{m}{2}\right]$, such that

$$Z=U\left[\begin{pmatrix}0&\lambda_1\\-\lambda_1&0\end{pmatrix}\dot{+}...\dot{+}\begin{pmatrix}0&\lambda_\gamma\\-\lambda_\gamma&0\end{pmatrix}\right]\bar{U}'$$

when m-is even,

$$Z=U\left[\begin{pmatrix}0&\lambda_1\\-\lambda_1&0\end{pmatrix}\dot{+}...\dot{+}\begin{pmatrix}0&\lambda_\gamma\\-\lambda_\gamma&0\end{pmatrix}\dot{+}0\right]\bar{U}'$$

when m is odd. Here the direct sum of the matrices A and B, i.e., $\ll A\dot{+}B\gg$ is defined as $\begin{pmatrix}A&0\\0&B\end{pmatrix}$.

Further, when m is even, we have

$$I^{(m)}+Z\bar{Z}=U\left[\begin{pmatrix}1-\lambda_1^2&0\\0&1-\lambda_1^2\end{pmatrix}\dot{+}...\dot{+}\begin{pmatrix}1-\lambda_\gamma^2&0\\0&1-\lambda_\gamma^2\end{pmatrix}\right]\bar{U}'.$$

Similarly, when m is odd, we have

$$I^{(m)}+Z\bar{Z}=U\left[\begin{pmatrix}1-\lambda_1^2&0\\0&1-\lambda_1^2\end{pmatrix}\dot{+}...\dot{+}\begin{pmatrix}1-\lambda_\gamma^2&0\\0&1-\lambda_\gamma^2\end{pmatrix}\dot{+}1\right]\bar{U}'.$$

It follows that $Z\in\Re_{III}(m)$ if and only if $\lambda_s<1\,(s=1,...,\gamma)$.

For $Z\in\mathbb{B}^{\frac{m(m-1)}{2}}(1)$, we have

$$\|Z\|^2=\sum_{1\leqslant s\leqslant j\leqslant m}|z_{sj}|^2=\sum_{1\leqslant s\leqslant j\leqslant m}z_{sj}\bar{z}_{sj}=\frac{1}{2}Sp\left(Z\bar{Z}'\right)=\sum_{s=1}^{\gamma}\lambda_s^2<1.$$

This means that $\lambda_s<1\,(s=1,...,\gamma)$, и $Z\in\Re_{III}(m)$. Hence, $B^{\frac{m(m-1)}{2}}(1)\subset\Re_{III}(m)$.

Lemma 2 is proved. □

It is known [3], that the Bergman kernels for the domains $\Re_I(m,k)$, $\Re_{II}(m)$ and $\Re_{III}(m)$ have the form:

$$K_{\Re_I(m,k)}(z,\bar{z})=\frac{1}{V(\Re_I(m,k))}\frac{1}{\det^{m+k}\left(I^{(m)}-Z\bar{Z}'\right)},$$

$$K_{\Re_{II}(m)}(z,\bar{z})=\frac{1}{V(\Re_{II}(m))}\frac{1}{\det^{m+1}\left(I^{(m)}-Z\bar{Z}\right)},$$

$$K_{\Re_{III}(m)}(z,\bar{z})=\frac{1}{V(\Re_{III}(m))}\frac{1}{\det^{m-1}\left(I^{(m)}+Z\bar{Z}\right)},$$

where

$$V\left(\Re_{I}\left(m,k\right)\right)=\frac{1!2!...\left(m-1\right)!1!2!...\left(k-1\right)!}{1!2!...\left(m+k-1\right)!}\pi^{mk},$$

$$V\left(\Re_{II}\left(m\right)\right)=\pi^{\frac{m(m+1)}{2}}\frac{2!4!...\left(2m-2\right)!}{m!\left(m+1\right)!...\left(2m-1\right)!},$$

$$V\left(\Re_{III}\left(m\right)\right)=\pi^{\frac{m(m-1)}{2}}\frac{2!4!...\left(2m-4\right)!}{\left(m-1\right)!m!...\left(2m-3\right)!},$$

volumes of the domain $\Re_I(m,k)$, $\Re_{II}(m)$ and $\Re_{III}(m)$, respectively.

The following theorem is proved in the same way as Theorem 2.

Theorem 4. *The following statements are true:*

a) if the Bergman kernel is $K_{\mathbb{B}^{mk}(1)}(z,\bar z)$ continues to the domain $\Re_I(m,k)$ as a real analytic function, then at the points $z\in\Re_I(m,k)$ the following inequality holds

$$K_{\Re_{I}(m,k)}\left(z,\bar z\right)\leqslant K_{\mathbb{B}^{mk}(1)}\left(z,\bar z\right);$$

b) if the Bergman kernel is $K_{\mathbb{B}^{\frac{m(m+1)}{2}}\left(\frac{1}{\sqrt{2}}\right)}(z,\bar z)$ continues to domain $\Re_{II}(m)$ as a real analytic function, then at the points $z\in\Re_{II}(m)$ the inequality holds

$$K_{\Re_{II}(m)}\left(z,\bar z\right)\leqslant K_{\mathbb{B}^{\frac{m(m+1)}{2}}\left(\frac{1}{\sqrt{2}}\right)}\left(z,\bar z\right);$$

c) if the Bergman kerne $K_{\mathbb{B}^{\frac{m(m-1)}{2}}(1)}(z,\bar z)$ extends into the domain $\Re_{III}(m)$ as a real analytic function, then at the points $z\in\Re_{III}(m)$ the inequality holds

$$K_{\Re_{III}(m)}\left(z,\bar z\right)\leqslant K_{\mathbb{B}^{\frac{m(m-1)}{2}}(1)}\left(z,\bar z\right).$$

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Cartan É. Sur les domaines bornes homogenes de l 'espace de n variables complexes // *Abh. Math. Sem. Univ. Hamburg,* **11** (1935), pp. 116-162.

2. Зигель К. Автоморфные функции нескольких комплексных переменных // *М.:ИЛ*, 1954. 168 с.

3. Хуа Л. К. Гармонический анализ функций многих комплексных переменных в классических областях // *М.: ИЛ*, 1959. 163 с.

4. Пятецкий-Шапиро И. И. Геометрия классических областей и теория автоморфных функций // *М.: Наука*, 1961. 192 с.

5. Henkin G.M., The method of integral representations in complex analysis, Complex analysis-several variables – 1 // *Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., 7, VINITI, Moscow*, 1985, 23–124

6. Xiao Ming. Regularity of mappings into classical domains // *Mathematische Annalen*, 2020, 378(3-4), pp. 1271-1309

7. Xiao M. Bergman-Harmonic Functions on Classical Domains // *International Mathematics Research Notices*, Vol. 00, No. 0, pp. 1–36 (2019).

8. Фукс Б. А. Специальные главы теории аналитических функций многих комплексных переменных. *Физматгиз*, 1963. 428 с.

9. Айзенберг Л. А. Формулы Карлемана в комплексном анализе // *Новосибирск: Наука*, 1990. 248 с.

10. Айзенберг Л. А., Южаков А. П. Интегральные представления и вычеты в многомерном комплексном анализе // *Новосибирск: Наука*, 1979. 366 с.

11. Худайберганов Г., Хидиров Б. Б., Рахмонов У. С. Автоморфизмы матричных шаров // *Вестник НУУз.* 2010. № 3. с.205-210.

12. Rudin W. Function Theory in the Unit Ball of $\mathbb{C}^n$ // *New York, Berlin, Heidelberg: Springer-Verlag*, (1980) 436 p.

13. Krantz S. G. Harmonic and complex analysis in several variables, *Springer Monographs in Mathematics, Gewerbestrasse* // 11, 6330 Cham, Switzerland (2017), 429 p.

14. Sergeev A. G . On matrix and Reinhardt domains // *Preprint, Inst. Mittag-Leffler, Stockholm*, 7 pp. (1988).

15. Худайберганов Г., Кытманов А. М. , Шаимкулов Б. А. Анализ в матричных облястя // *Красноярск: Сибирский федеральный ун-т*, 296 с. (2017).

16. Khudayberganov G., Rakhmonov U. S. Carleman Formula for Matrix Ball of the Third Type // *Algebra, Complex Analysis, and Pluripotential Theory.* USUZCAMP 2017. Springer Proceedings in Mathematics & Statistics, vol. **264** (2017), pp. 101-108, Springer, Cham.

17. Khudayberganov G., Rakhmonov U. The Bergman and Cauchy-Szegö kernels for matrix ball of the second type // *J. Sib. Fed. Univ. Math. Phys.*, 7:3 (2014), 305-310.

18. Myslivets S. G. Construction of Szegö and Poisson kernels in convex domains // *J. Sib. Fed. Univ. Math. Phys.*, 2018, Volume 11, Issue 6, 792-795.

19. Khudayberganov G., Abdullayev J. Sh. Relationship between the Bergman and Cauchy-Szegö kernels in the domains $\tau^+(n-1)$ and $\Re^n_{IV}$ // *J. Sib. Fed. Univ. Math. Phys.*, 13:5 (2020), 559-567.

20. Мысливец С. Г. О ядрах Сеге и Пуассона в выпуклых областях в $\mathbb{C}^n$ // *Известия вузов.Математика.* 2019. № 1. С. 42-48.

21. Rakhmonov U. S., Abdullayev J. Sh. On volumes of matrix ball of third type and generalized Lie balls // *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Kompyuternye Nauki*, 2019, vol. 29, issue 4, pp. 548-557.

22. Khudayberganov G., Khalknazarov A.M., Abdullayev J.Sh., Laplace and Hua Luogeng operators // *Russian Mathematics (Izv. Vyssh. Uchebn. Zaved. Mat)* 2020, Vol **64** , no. 3, pp. 66-71. ©Allerton Press, Inc., 2020

23. Шабат Б. В. Введение в комплексный анализ. *М.: Наука.* Ч.2. 3-е изд., 1985 г., 464 с.

24. Bremermann H.-J. Die charakterisierung von regularitätsgebieten durch pseudokonvexe runktionen // *Schriftenreihe Math. Inst. Munster*, № 5 (1951).

25. Hua Luogeng. On the theory of automorphic functions of a matrix variable I-geometrical basis // *American Journal of Mathematics*, Vol. **66**, No. 3 , pp. 470-488 (1944).

REFERENCES

1. Cartan É. Sur les domaines bornes homogenes de l 'espace de n variables complexes // *Abh. Math. Sem. Univ. Hamburg,* **11** (1935), pp. 116-162.

2. C.L.Siegel, Automorphic functions of several complex variables // *M.: Publishing house of foreign literature*, 1954. p. 168. (In Russian).

3. Hua Luogeng. Harmonic analysis of functions of several complex variables in classical domains // *Inostr. Lit., M.*, 1959 (In Russian)

4. Pjateckiĭ-Šapiro I.I. Geometry of classical domains and the theory of automorphic functions // *Moscow: State publishing house of physical and mathematical literature*, 1961. p. 191. (In Russian).

5. Henkin G.M. The method of integral representations in complex analysis, Complex analysis-several variables – 1 // *Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., 7, VINITI, Moscow*, 1985, 23–124

6. Xiao Ming. Regularity of mappings into classical domains // *Mathematische Annalen*, 2020, 378(3-4), pp. 1271-1309

7. Xiao M. Bergman-Harmonic Functions on Classical Domains // *International Mathematics Research Notices*, Vol. 00, No. 0, pp. 1–36 (2019).

8. Fuks B. A. Special Chapters in the Theory of Analytic Functions of Several Complex Variables // *Физматгиз*, 1963.

9. Aizenberg L.A. Carleman Formulas in complex analysis // *Novosibirsk, Science*. 1990. 248 p. (In Russian)

10. Aizenberg, L.A., Yuzhakov, A.P. Integral Representations and Residues in Multidimensional Complex Analysis. *Novosibirsk: Nauka*, 366 pp. (1979):

11. Khudayberganov G., Hidirov B.B., Rakhmonov U.S., Automorphisms of matrix balls Acta NUUz, 2010, no. 3, 205-210 (In Russian)

12. Rudin W. Function Theory in the Unit Ball of $\mathbb{C}^n$ // *New York, Berlin, Heidelberg: Springer-Verlag*, (1980) 436 p.

13. Krantz S. G. Harmonic and complex analysis in several variables, *Springer Monographs in Mathematics, Gewerbestrasse* // 11, 6330 Cham, Switzerland (2017), 429 p.

14. Sergeev A. G . On matrix and Reinhardt domains // *Preprint, Inst. Mittag-Leffler, Stockholm*, 7 pp. (1988).

15. Khudayberganov G., Kytmanov A.M., Shaimkulov B.A., Analysis in matrix domains // *Monograph. Krasnoyarsk: Siberian Federal University*, 2017. p. 296 (In Russian)

16. Khudayberganov G., Rakhmonov U. S. Carleman Formula for Matrix Ball of the Third Type // *Algebra, Complex Analysis, and Pluripotential Theory.* USUZCAMP 2017. Springer Proceedings in Mathematics & Statistics, vol. **264** (2017), pp. 101-108, Springer, Cham.

17. Khudayberganov G., Rakhmonov U. The Bergman and Cauchy-Szegö kernels for matrix ball of the second type // *J. Sib. Fed. Univ. Math. Phys.*, 7:3 (2014), 305-310.

18. Myslivets S. G. Construction of Szegö and Poisson kernels in convex domains // *J. Sib. Fed. Univ. Math. Phys.*, 2018, Volume 11, Issue 6, 792-795.

19. Khudayberganov G., Abdullayev J. Sh. Relationship between the Bergman and Cauchy-Szegö kernels in the domains $\tau^+(n-1)$ and $\Re_{IV}^n$ // *J. Sib. Fed. Univ. Math. Phys.*, 13:5 (2020), 559-567.

20. Myslivets S.G. On the Szegő and Poisson kernels in the convex domains in $\mathbb{C}^n$ // *Russian Mathematics (Izv. Vyssh. Uchebn. Zaved. Mat)* 2019, no. 1, pp. 42-48.

21. Rakhmonov U. S., Abdullayev J. Sh. On volumes of matrix ball of third type and generalized Lie balls // *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Kompyuternye Nauki*, 2019, vol. 29, issue 4, pp. 548-557.

22. Khudayberganov G., Khalknazarov A.M., Abdullayev J.Sh., Laplace and Hua Luogeng operators // *Russian Mathematics (Izv. Vyssh. Uchebn. Zaved. Mat)* 2020, Vol **64** , no. 3, pp. 66-71. Allerton Press, Inc., 2020

23. Shabat B.V. , Introduction to Complex Analysis Part II Functions of Several Variables, *Nauka, Fiz. Mat. Lit., M.*, pp. 464. 1985 (in Russian)

24. Bremermann H.-J. Die charakterisierung von regularitätsgebieten durch pseudokonvexe runktionen // *Schriftenreihe Math. Inst. Munster*, № 5 (1951).

25. Hua Luogeng. On the theory of automorphic functions of a matrix variable I-geometrical basis // *American Journal of Mathematics*, Vol. **66**, No. 3 , pp. 470-488 (1944).

Получено 05.04.21 г.
Принято в печать 20.09.2021 г.