ЧЕБЫШЕВСКИЙ СБОРНИК
Том 22. Выпуск 2.

УДК 511 DOI 10.22405/2226-8383-2021-22-2-484-489

Замечание к теореме Давенпорта[1]

Ке Гонг

Гонг Ке — доктор наук, доцент, Хэнаньский университет (Кайфэн, КНР).
e-mail: kg@henu.edu.cn

Аннотация

Пусть Λ-n-мерная решетка, а $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}$ - любые $n-1$ векторов в n-мерном вещественном евклидовом пространстве. В работе доказано существование базиса $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$ решётки Λ такого, что неравенство

$$|\boldsymbol{\alpha}_i - N\boldsymbol{c}_i| = O(\log^2 N), \qquad (1 \leqslant i \leqslant n-1)$$

имеет место для любого вещественного $N \geqslant 2$, где константа в знаке O зависит лишь от Λ и $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}$.

Ключевые слова: Решетка, базис, аппроксимация, комбинаторное решето.

Библиография: 17 названий.

Для цитирования:
Ке Гонг. Замечание к теореме Давенпорта // Чебышевский сборник, 2021, т. 22, вып. 2, с. 484–489.

CHEBYSHEVSKII SBORNIK
Vol. 22. No. 2.

UDC 511 DOI 10.22405/2226-8383-2021-22-2-484-489

Note on a theorem of Davenport

Ke Gong

Gong Ke — doctor of science, Associate Professor, Henan University (Kaifeng, P. R. China).
e-mail: kg@henu.edu.cn

Abstract

Let Λ be a n-dimensional lattice, and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}$ be any $n-1$ vectors in n-dimensional real Euclidean space. We show that there exists a basis $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$ of Λ such that

$$|\boldsymbol{\alpha}_i - N\boldsymbol{c}_i| = O(\log^2 N), \qquad (1 \leqslant i \leqslant n-1)$$

holds for any real number $N \geqslant 2$, where the constant implied by the O symbol depends only on Λ and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}$.

Keywords: Lattice, basis, approximation, combinatorial sieve.

Bibliography: 17 titles.

For citation:
Ke Gong, 2021, "Note on a theorem of Davenport", *Chebyshevskii sbornik*, vol. 22, no. 2, pp. 484–489.

[1]Исследование выполнено за счет гранта Государственного фонда естественных наук Китая (проект № 11671119).

Introduction

We denote by $\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{c}$ the n-dimensional real vectors, and by

$$|\boldsymbol{x}| = |(x_1, \ldots, x_n)| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

the length of $\boldsymbol{x}$. Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$ be a given system of linearly independent vectors in n-dimensional real Euclidean space, the set of vectors

$$\Lambda = \{u_1\boldsymbol{\alpha}_1 + \cdots + u_n\boldsymbol{\alpha}_n \mid u_1, \ldots, u_n \in \mathbb{Z}\}$$

is called a n-dimensional lattice with basis $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$.

Let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}$ be an arbitrary set of $n-1$ linearly independent n-dimensional real vectors. A theorem of Davenport [2] says that, for large positive N, we can choose a basis of Λ such that the first $n-1$ vectors of it are not too far from $N\boldsymbol{c}_1, \ldots, N\boldsymbol{c}_{n-1}$ respectively. Using this theorem, Davenport [2] gave a simple and elegant proof of a generalization of Furtwängler's result on simultaneous Diophantine approximation. See also the monographs of Cassels [1], Gruber and Lekkerkerker [3], and Zhu [13].

In this note, we obtain

Theorem 1. *Let Λ be a n-dimensional lattice, and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}$ be any $n-1$ vectors in n-dimensional real Euclidean space. Then there exists a basis $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$ of Λ, such that for any real number $N \geqslant 2$, we have*

$$|\boldsymbol{\alpha}_i - N\boldsymbol{c}_i| = O(\log^2 N), \qquad (1 \leqslant i \leqslant n-1),$$

where the constant implied by the O symbol depends only on Λ and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}$.

If we replace the above error term with $O(N^{\varepsilon})$, where ε is any given positive number and the constant implied by the O symbol may depend on ε, Λ and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}$, this is the original result of H. Davenport. In 1975, using Brun's sieve, Y. Wang [9] obtained a refinement $O(\log^3 N)$ on the error term. In 1985, Z. H. Yang [10] and Q. Yao [11] refined the error term to $O(\log^{2+\varepsilon} N)$ and to $O(\log^2 N(\log\log N)^2)$ respectively. Here we obtain a still-further refinement of Davenport's theorem.

Proof

We need Iwaniec's shifted sieve [4], see also Martin [7].

Lemma 1. *Let $\mathcal{A}$ be a finite sequence, $U : \mathcal{A} \to \mathbb{Z}$ and $W : \mathcal{A} \to \mathbb{R}_{\geqslant 0}$. Let $p_1, \ldots, p_r$ be distinct primes and $Q = p_1 \cdots p_r$. Define*

$$T = \sum_{\substack{a \in \mathcal{A} \\ \gcd(U(a), Q) = 1}} W(a)$$

and for all $d \mid Q$, put

$$S_d = \sum_{\substack{a \in \mathcal{A} \\ U(a) \equiv 0 \,(\mathrm{mod}\, d)}} W(a).$$

Suppose there exist A and B such that

$$\left| S_d - \frac{A}{d} \right| \leqslant B$$

for all $d \mid Q$. Then there exists an absolute positive constant c such that

$$T \geqslant c\frac{A}{\log(\omega(Q))}\prod_{p \mid Q}\left(1-\frac{1}{p}\right)+O(B\omega^2(Q)),$$

where $\omega(Q)$ denote the number of distinct prime divisors of Q, and the constant implied by the O symbol is an absolute constant.

Proof. See Lemma 7 of Martin [7]. □

Lemma 2. *For $q \geqslant 3$, there holds*

$$\varphi(q) \geqslant c'\frac{q}{\log\log q}.$$

Proof. For a proof, see Theorem 5.1 of Prachar [8]. □

Lemma 3. *Let $q \geqslant 2$ be any positive integer, s and t be integers with $(t,q)=1$. Then there exists an absolute constant K such that each interval of length greater than $K\log^2 q$ contains a positive integer u such that $(tu+s,q)=1$.*

This lemma is essential for our refinement of Davenport's theorem. So we shall give a detailed proof by combining Iwaniec's shifted sieve with the arguments of Wang [9].

Proof. 1) Let G be an arbitrary real number,

$$H=K\log^2 q, \qquad Q=\prod_{\substack{p|q\\ p\leqslant\log^{\delta} q}} p,$$

where p runs through primes, $\delta=1+\varepsilon$ with ε being an arbitrarily small positive number, and $K\geqslant 1$ is a parameter to be determined. Then

$$\Sigma=\sum_{\substack{G<u\leqslant G+H\\ (tu+s,\,q)=1}} 1 \geqslant \Sigma_1-\Sigma_2,$$

where

$$\Sigma_1=\sum_{\substack{G<u\leqslant G+H\\ (tu+s,\,Q)=1}} 1, \qquad \Sigma_2=\sum_{\substack{p|q\\ p>\log^{\delta} q}}\ \sum_{\substack{G<u\leqslant G+H\\ tu+s\equiv 0\,(\mathrm{mod}\,p)}} 1.$$

For $(t,q)=1$, we have

$$\sum_{\substack{G<u\leqslant G+H\\ tu+s\equiv 0\,(\mathrm{mod}\,d)}} 1=\left[\frac{H}{d}\right]+\theta=\frac{H}{d}+\theta,$$

here and hereafter, we use θ to denote a number $|\theta|\leqslant 1$ but not always the same in each occurrence. Since the number of prime divisors of q is less than $2\log q$, we have

$$\Sigma_2=\sum_{\substack{p|q\\ p>\log^{\delta} q}}\left(\frac{H}{p}+\theta\right)\leqslant\frac{H}{\log^{\delta} q}\sum_{p|q}1+\sum_{p|q}1$$
$$\leqslant 2H\log^{1-\delta} q+2\log q\leqslant 4K\log^{3-\delta} q=4K\log^{2-\varepsilon} q.$$

We shall apply Lemma 2 with

$$\mathcal{A} = \{u \mid G < u \leqslant G + H\}, \qquad U(u) = tu + s, \qquad W(u) = 1,$$

and let $p_1 < \cdots < p_r$ be the set of primes satisfying $p_i \mid q$ and $p_i \leqslant \log^\delta q$ $(1 \leqslant i \leqslant r)$. Then, by Lemma 1 and Mertens's theorem, there exists an absolute positive constant c_1 such that

$$\begin{aligned}\Sigma_1 &\geqslant c_1 \frac{H}{\log(\omega(Q))} \prod_{\substack{p \mid q \\ p \leqslant \log^\delta q}} \left(1 - \frac{1}{p}\right) + O(\omega^2(Q)) \\ &\geqslant c_1 \frac{H}{\log(\omega(Q))} \frac{e^{-\gamma}}{\delta \log\log q} \left(1 + O\left(\frac{1}{\log\log q}\right)\right) + O(\omega^2(Q)).\end{aligned}$$

Since $\omega(Q) \leqslant \omega(q)$ and $\omega(q) \sim \frac{\log q}{\log\log q}$, thus

$$\Sigma_1 \geqslant c_2 K \frac{\log^2 q}{(\log\log q)^2}$$

if we choose K sufficiently large, where c_2 is an absolute positive constant.

Combing the above estimates for Σ_1 and Σ_2, we conclude that, when $q \geqslant q_0$ with q_0 being a sufficiently large constant,

$$\Sigma > c_3 K \frac{\log^2 q}{(\log\log q)^2} > 1.$$

2) When $q \leqslant q_0$, we use Eratosthenes sieve and Lemma 2 to obtain

$$\begin{aligned}\Sigma &= \sum_{\substack{G < u \leqslant G+H \\ (tu+s,\, q)=1}} 1 = \sum_{G < u \leqslant G+H} \sum_{d \mid (tu+s, q)} \mu(d) = \sum_{d \mid q} \mu(d) \sum_{\substack{G < u \leqslant G+H \\ tu+s \equiv 0 \pmod d}} 1 \\ &= H \sum_{d \mid q} \frac{\mu(d)}{d} + \theta \sum_{d \mid q} 1 = H \prod_{p \mid q} \left(1 - \frac{1}{p}\right) + \theta q > c_4 K \frac{\log^2 q}{\log\log q} + \theta q > 1,\end{aligned}$$

where $c_4 > 0$ is an absolute constant. Since $q \leqslant q_0$, we can choose K sufficiently large to guarantee that $\Sigma > 1$. □

Proof. [Theorem 1] Once we obtain Lemma 3, the proof of Theorem 1 follows the arguments of Y. Wang [9], or Cassels [1, § I.2.4]. □

Remarks

We give some remarks for further study.

i) Using the results we obtain here, we can also refine the corresponding results presented in Lekkerkerker [5, 6] and Zhu [12], where they considered some generalizations of Davenport's theorem.

ii) It is an interesting problem to reduce further the interval length in Lemma 3.

Acknowledgements

This work was supported by NSFC grant 11671119.

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